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ONE-POINT SOLUTIONS OBTAINED FROM BEST APPROXIMATION PROBLEMS FOR COOPERATIVE GAMES

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In this paper we focus on one-point (point-valued) solutions for transferable utility games (TU-games). Since each allocated profit vector is identified with an additive game, a solution can be regarded as a mapping which associates an additive game with each TU-game. Recently Kultti and Salonen proposed a minimum norm problem to find the best approximation in the set of efficient additive games for a given TU-game. They proved some interesting properties of the obtained solution. However, they did not show how to choose the inner product defining the norm to obtain a special class of solutions such as the Shapley value and more general random order values. In this paper, noting that there is a one-to-one correspondence between a game and a Harsanyi dividend vector, we propose a minimum norm problem in the dividend space, not in the game space. Since the dividends for any set with more than one elements are all zero for an additive game, our approach enables us to deal with simpler problems. We will make clear how to choose an inner product, i. e., a positive definite symmetric matrix, to obtain a Harsanyi payoff vector, a random order value and the Shapley value.

Keywords: cooperative games, one-point solutions, additive games, Harsanyi dividends

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1. INTRODUCTION

In this paper we deal with cooperative games with transferable utility, i. e., so-called TU-games. A main topic in the cooperative game theory is to provide a reasonable allocation scheme of profits obtained by the grand coalition among the players. Usually this scheme is called a solution of the game. A number of solutions have been proposed and they can be divided into two categories. One is the class of set-valued solutions: each solution associates a set of profit vectors with each cooperative game. Typical examples of solutions in this class are the core, the stable set, the kernel, and the Weber set. The other class consists of one-point (point-valued) solutions: each solution associates a profit vector with each game. Typical examples in this class are the Shapley value, the nucleolus and the random order value.

In this paper we focus on the second class, i. e., one-point solutions. Each solution provides an n dimensional vector for each TU-game, where n is the number of players. Here we should note that an additive game is also specified by n worths for individual players. Thus we may regard a one-point solution as a mapping from the set of all TU-games to the set of all additive games (see e. g. Monderer and Samet [6]). Studies of one-point solutions in the settings of best approximation problems have been made by several authors (Charnes et al. [1], Ruiz et al. [7, 8], Kultti and Salonen [5]). Kultti and Salonen formulated a minimum norm problem to find the best approximation in the set of efficient additive games for a given cooperative game, introduced norms from inner products in the space of cooperative games, and proved that any efficient linear solution that has the inessential game property can be obtained as a solution to the minimum norm problem with an appropriate inner product. However, they did not provide any information on how to choose the inner product to obtain a special class of solutions such as random order values and, in particular, the Shapley value.

In this paper we pay attention to the fact that each cooperative game can be completely characterized by its (Harsanyi) dividends (Harsanyi [4]), and formulate best approximation problems in the dividend space. Since all the dividends of coalitions with more than one players are zero for any additive game, this approach enables us to deal with simpler problems. A norm in the dividend space is derived from an inner product and we make clear how to choose the inner product to obtain a Harsanyi payoff vector, a random order value and in particular the Shapley value.

The paper is organized in the following way. In Section 2 we review cooperative games and their dividends. Section 3 is devoted to concise introduction of one-point solutions for games. In Section 4 we review one-point solutions as minimum norm solutions. In Section 5 we formulate minimum norm problems in the dividend space and show how to choose the inner product to obtain a Harsanyi payoff vector, a random order value and in particular the Shapley value. The last section concludes the paper.

2. COOPERATIVE GAMES AND HARSANYI DIVIDENDS

Let $N = \{1, 2, \ldots, n\}$ be a finite set of players. A subset $S \subseteq N$ is called a coalition. A *cooperative game* (transferable utility game, TU-game for short) on N is defined by a characteristic function $v : 2^N \to \mathbf{R}$ with $v(\emptyset) = 0$. The value $v(S)$ is called the worth of coalition S. We denote by $\mathcal{G}$ the set of all cooperative games on N, which is fixed throughout this paper. We use abbreviated notations such as

$$v(i) = v(\{i\}),\ S \cup i = S \cup \{i\},\ S \setminus i = S \setminus \{i\},$$

and so on. We distinguish between two inclusive relations $S \subset T$ and $S \subseteq T$. The former means a proper inclusive relation.

The sum of two games $v, w \in \mathcal{G}$ is defined as

$$(v + w)(S) = v(S) + w(S),\ \ \forall S \subseteq N$$

and the scalar multiplication of $v \in \mathcal{G}$ by a scalar $\alpha \in \mathbf{R}$ is defined as

$$(\alpha v)(S) = \alpha v(S),\ \ \forall S \subseteq N.$$

Thus the space $\mathcal{G}$ of all games on N is a vector space and its dimension is clearly $2^n - 1$, since each game is specified by the worths $v(S)$ for all $S \subseteq N$ with $S \neq \emptyset$.

As a basis in $\mathcal{G}$ we generally consider *unanimity games* u_T for any $T \subseteq N$ with $T \neq \emptyset$ defined by

$$u_T(S) = \begin{cases} 1 & \text{if } S \supseteq T \\ 0 & \text{otherwise.} \end{cases}$$

Each game $v \in \mathcal{G}$ is a linear combination of unanimity games,

$$v = \sum_{T \subseteq N, T \neq \emptyset} d^v(T) u_T.$$

The coefficient $d^v(T)$ is called (Harsanyi) dividend (or Möbius transform) of T for the game v. Due to the Möbius inversion lemma, we have

$$d^v(T) = \sum_{S \subseteq T} (-1)^{|T|-|S|} v(S).$$

If we put $d^v(\emptyset) = 0$, $v = \sum_{T \subseteq N} d^v(T) u_T$ and the dividends can be obtained by the following recursive formula:

$$d^v(T) = \begin{cases} 0, & \text{if } T = \emptyset \\ v(T) - \sum\limits_{S \subset T} d^v(S), & \text{if } T \neq \emptyset. \end{cases}$$

The dividends satisfy the following relations.

$$d^{v+w}(T) = d^v(T) + d^w(T),\ d^{\alpha v}(T) = \alpha d^v(T),\ \forall T \subseteq N,$$

$$v(S) = \sum_{T \subseteq S} d^v(T),\ \ \forall S \subseteq N.$$

If we regard both v and d^v as $2^n - 1$ dimensional vector such as $(v(1), \ldots, v(n), v(1,2), \ldots, v(N))^\top$ and $(d^v(1), \ldots, d^v(n), d^v(1,2), \ldots, d^v(N))^\top$ respectively, they are related in terms of a matrix D as $v = Dd^v$. Here the (S,T) element of D is 1 if $S \supseteq T$ and 0 otherwise.

Now we define a fundamental class of TU-games.

Definition 2.1. A game $v \in \mathcal{G}$ is said to be additive if $v(S) + v(T) = v(S \cup T)$ for all $S, T \subseteq N$, such that $S \cap T = \emptyset$. The set of all additive cooperative games on N is denoted by $\mathcal{A}$.

An additive game is completely specified by n worths $v(1), v(2), \ldots,$ and $v(n)$, and therefore the set $\mathcal{A}$ of all additive games is a subspace of $\mathcal{G}$ and dim $\mathcal{A} = n$.

The following result can be proved easily, and the concept of additive games was extended to k-order additive games by Grabisch ([3]) in terms of the dividends.

Proposition 2.2. A cooperative game $v \in \mathcal{G}$ is additive if and only if

$$d^v(T) = \begin{cases} v(i) & \text{if } T = \{i\},\ i \in N, \\ 0 & \text{otherwise.} \end{cases}$$

3. ONE-POINT SOLUTIONS FOR COOPERATIVE GAMES

In a game $v \in \mathcal{G}$, the main issue is the distribution of the worth $v(N)$ among the players. A one-point solution, which is often called a value, of a game is a function $\phi : \mathcal{G} \to \mathbf{R}^n$, i. e., $\phi(v) = (\phi_1(v), \phi_2(v), \ldots, \phi_n(v))$ for each $v \in \mathcal{G}$. A function ϕ is usually assumed to be linear with respect to v. Hence the value is a linear combination of the values for unanimity games, i. e.,

$$\phi(v) = \phi(\sum_{T \subseteq N} d^v(T)u_T) = \sum_{T \subseteq N} d^v(T)\phi(u_T).$$

Typical examples of values is the Shapley value φ given by

$$\varphi_i(u_T) = \begin{cases} \dfrac{1}{|T|}, & \text{if } i \in T \\ 0, & \text{otherwise.} \end{cases}$$

As is known well, by using the dividends, the Shapley value can be rewritten as follows

$$\varphi_i(v) = \sum_{T \subseteq N, T \ni i} \frac{d^v(T)}{|T|}.$$

More general value can be introduced by considering a sharing system

$$p = (p_i^T)_{T \subseteq N, i \in T} \text{ satisfying } p_i^T \geq 0, \ \sum_{i \in T} p_i^T = 1, \forall T \subseteq N, T \neq \emptyset.$$

The set of all sharing systems is denoted by P.

Definition 3.1. (Derks, van der Laan and Vasil'ev [2]) Given a sharing system $p \in P$, the Harsanyi payoff vector for a game $v \in \mathcal{G}$ is defined by

$$\phi_i^p(v) = \sum_{T \subseteq N, T \ni i} p_i^T d^v(T), \quad i \in N.$$

We should note that the Harsanyi payoff vector is efficient, i. e., $\sum_{i \in N} \phi_i^p(v) = v(N)$. The Shapley value is the Harsanyi payoff vector with $p_i^T = \frac{1}{|T|}$ for all $i \in T$.

Now we introduce another type of one-point solutions. Let π be a permutation on N, which assigns rank number $\pi(i) \in N$ to player $i \in N$. Let

$$\pi^i = \{j \in N \mid \pi(j) \leq \pi(i)\}.$$

Definition 3.2. The marginal contribution vector $m^\pi(v) \in \mathbf{R}^n$ of v and π is given by

$$m_i^\pi(v) = v(\pi^i) - v(\pi^i \setminus i), \quad i \in N.$$

The Weber set $W(v)$ is the convex hull of all marginal contribution vectors $m^\pi(v)$. Each element of $W(v)$ is called a random order value.

It is clear that the marginal contribution vector satisfies the efficiency and therefore so does any random order value.

Proposition 3.3. (Derks et al. [2]) If we define the sharing system p^π for a permutation π on N by

$$(p^\pi)_i^T = \begin{cases} 1 & \text{if } i \in T \text{ and } T \subseteq \pi^i, \\ 0 & \text{otherwise,} \end{cases}$$

then $\phi^{p^\pi}(v) = m^\pi(v)$.

Proposition 3.4. (Derks et al. [2]) A Harsanyi payoff vector ϕ^p is a random order value if and only if the sharing system p belongs to the following set P^*, i. e.,

$$p \in P^* = \{p \in P \mid \sum_{S \supseteq T} (-1)^{|S|-|T|} p_i^S \geq 0, \ \forall T \subseteq N, \ T \ni i\}.$$

4. ONE-POINT SOLUTIONS AS MINIMUM NORM SOLUTIONS

In this section we consider the minimum norm problem introduced by Kultti and Salonen [5].

We may identify a value which is an n dimensional vector with an additive game (see e. g. Monderer and Samet [6]). In other words, the value $\phi : \mathcal{G} \to \mathbf{R}^n$ is identified with the function $f : \mathcal{G} \to \mathcal{A}$ with $\phi_i(v) = f(v)(i)$. Hence it is quite natural to consider the best approximation of a given game $v \in \mathcal{G}$ by an additive game.

Kultti and Salonen [5] proposed the approach by efficient minimum norm solutions, which is a generalization of the results by Ruiz et al. [7, 8]. They defined an optimization problem (minimum norm problem) for each cooperative game $\bar{v} \in \mathcal{G}$

$$\begin{array}{ll} \text{minimize} & \langle v - \bar{v}, v - \bar{v} \rangle \\ \text{subject to} & v \in \mathcal{A}, \ v(N) = \bar{v}(N). \end{array} \tag{1}$$

Here $\langle \cdot, \cdot \rangle$ is an inner product on $\mathcal{G}$, i. e., the $2^n - 1$ dimensional real space. In this case a one-point solution f is a function $f : \bar{v} \mapsto v^*$, where v^* is the unique optimal solution to the above problem, i. e., the minimum norm solution.

They proved some interesting properties of minimum norm solutions.

- *Efficiency.* $f(v)(N) = v(N) \ \forall v \in \mathcal{G}$.
- *Linearity.* $f(\alpha v + \beta w) = \alpha f(v) + \beta f(w), \ \ \forall \alpha, \beta \in \mathbf{R}, \ v, w \in \mathcal{G}$.
- *Inessential game property.* If $v \in \mathcal{A}$, then $f(v) = v$.

Theorem 4.1. (Kultti and Salonen [5]) For each $\bar{v} \in \mathcal{G}$, the solution $f(\bar{v})$ to the above minimum norm problem exists uniquely. The function $f : \mathcal{G} \to \mathcal{A}$ is efficient, linear and has the inessential game property.

Theorem 4.2. (Kultti and Salonen [5]) Let $f : \mathcal{G} \to \mathcal{A}$ be any efficient linear solution that has the inessential game property. Then there is an inner product such that $f(\bar{v})$ solves the above minimum norm problem for all $\bar{v} \in \mathcal{G}$.

However there remains an open problem: How to choose the inner product?

5. MINIMUM NORM SOLUTIONS IN THE DIVIDEND SPACE

We consider a minimum norm problem by Kultti and Salonen again.

$$\begin{array}{ll} \text{minimize} & \langle v-\bar{v}, v-\bar{v}\rangle \\ \text{subject to} & v \in \mathcal{A},\ v(N)=\bar{v}(N). \end{array}$$

We may eliminate the variables $v(S)$ with $|S|>1$ by the equality constraints $v(S)=\sum_{i\in S} v(i)$ for any $S\subseteq N$, $|S|>1$ for $v\in\mathcal{A}$. Then the remaining essential variables are only $v(1), v(2), \ldots$, and $v(n)$. However the objective function, i. e.,the inner product (norm) of the deviation vector $v-\bar{v}$, of the above problem is represented by these variables in a rather complicated manner. It causes a difficulty in choosing an inner product.

Since we can obtain the dividends $\{d^v(S)|S\subseteq N, S\neq\emptyset\}$ of v by the linear transformation from the worths $\{v(S)|S\subseteq N, S\neq\emptyset\}$ and vice versa, the above problem can be rewritten as the optimization problem with respect to the dividends as in the following.

$$\begin{array}{ll} \text{minimize} & \langle d-\bar{d}, d-\bar{d}\rangle \\ \text{subject to} & d(S)=0 \text{ if } |S|>1,\ \displaystyle\sum_{i\in N} d(i)=\bar{v}(N)=\sum_{S\subseteq N}\bar{d}(S). \end{array} \tag{2}$$

Here $\bar{d}=d^{\bar{v}}$ is the 2^n-1 dimensional dividend vector of $\bar{v}$, and $\langle\cdot,\cdot\rangle$ is an appropriate inner product in the dividend space.

The essential variables in the above optimization problem are $d(1), d(2), \ldots,\ d(n)$. The objective function is explicitly written by these variables and the constraint is only one: $\sum_{i\in N} d(i)=\sum_{S\subseteq N}\bar{d}(S)$. Hence the problem is quite simple. If we denote the optimal solution of the above problem by d^*, then for each game $\bar{v}\in\mathcal{G}$, the solution $f(\bar{v})\in\mathcal{A}$ can be obtained by $f(\bar{v})(i)=d^*(i)$ for all $i\in N$.

We may describe an inner product $\langle\cdot,\cdot\rangle$ in terms of $(2^n-1)\times(2^n-1)$ positive definite symmetric matrix Q whose (S,T) element is q_{ST}. Thus

$$\langle d, d'\rangle=\sum_{S,T\subseteq N, S,T\neq\emptyset} q_{ST}\, d(S)d'(T).$$

Since $v=Dd^v$, the minimum norm problem (2) in the dividend space with the inner product specified by the positive definite matrix Q is obviously equivalent to the minimum norm problem (1) in the game space with the inner product specified by the positive definite matrix $D^\top QD$. However the number of constraints is actually only one in the problem (2) and selection of the matrix is much easier as is shown below.

Noting that $d(S)=0$ for $|S|>1$, but that $\bar{d}(S)\neq 0$ generally, the above optimization problem can be essentially rewritten (by deleting the constant term) as

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{i\in N}\sum_{j\in N} q_{ij}d(i)d(j)-2\sum_{i\in N}\sum_{S\subseteq N, S\neq\emptyset} q_{iS}d(i)\bar{d}(S) \\ \text{subject to} & \displaystyle\sum_{i\in N} d(i)=\sum_{S\subseteq N}\bar{d}(S), \end{array} \tag{3}$$

where $q_{ij}=q_{\{i\}\{j\}}$ and $q_{iS}=q_{\{i\}S}$.

The theorems by Kultti and Salonen are obviously valid in this case.

Theorem 5.1. For each $\bar{v} \in \mathcal{G}$, the solution $f(\bar{v})$ obtained through the above minimum norm problem exists uniquely. The function $f : \mathcal{G} \to \mathcal{A}$ is efficient, linear and has the inessential game property.

Theorem 5.2. Let $f : \mathcal{G} \to \mathcal{A}$ be any efficient linear solution that has the inessential game property. Then there is an inner product such that $f(\bar{v})$ can be obtained by the solution to the above minimum norm problem with $\bar{d} = d^{\bar{v}}$ for all $\bar{v} \in \mathcal{G}$.

Now we consider special cases of the inner products.

Lemma 5.3. Given a sharing system $p = (p_i^T)_{T \subseteq N, i \in T} \in P$, let

$$q_{iS} = \begin{cases} p_i^S & \text{if } i \in S, \\ 0 & \text{if } i \notin S, \end{cases}$$

and, when $|S| > 1$ and $|T| > 1$,

$$q_{ST} = \begin{cases} \text{sufficiently large} & \text{if } S = T, \\ 0 & \text{otherwise.} \end{cases}$$

Then the matrix Q is positive definite.

P r o o f. We decompose the matrix Q as in the following:

$$Q = \begin{pmatrix} I & R^\top \\ R & D \end{pmatrix}$$

where I is the $n \times n$ identity matrix, R is a $(2^n - n - 1) \times n$ matrix whose elements are q_{Si} and D is a diagonal matrix with sufficiently large elements. We also decompose a $2^n - 1$ dimensional vector x as $\begin{pmatrix} y \\ z \end{pmatrix}$ with $y \in \mathbf{R}^n$. We prove that $x^\top Qx = y^\top y + 2z^\top Ry + z^\top Dz > 0$ for all $x \neq 0$. If $y = 0$, then $x^\top Qx = z^\top Dz > 0$ for any $z \neq 0$ because D is a diagonal matrix with positive elements. Hence we suppose that $y \neq 0$. It is sufficient to prove the case where $\max_{i=1,\ldots,n} y_i = 1$, since $x^\top Qx = \alpha^2 \left(\frac{x}{\alpha}\right)^\top Q \left(\frac{x}{\alpha}\right)$ if $\max_{i=1,\ldots,n} y_i = \alpha > 0$. We should note that the row corresponding to S in R consists of the elements p_i^S $(i \in S)$ and 0 $(i \notin S)$ whose sum is equal to 1. Hence

$$x^\top Qx \geq y^\top y - 2z^\top \mathbf{1} + z^\top Dz$$

where $\mathbf{1}$ is the $2^n - n - 1$ dimensional vector whose components are all 1. The minimum of $y^\top y$ under the condition $\max_{i=1,\ldots,n} y_i = 1$ is equal to 1. On the other hand the minimum of $-2z^\top \mathbf{1} + z^\top Dz$ is $-\mathbf{1}^\top D^{-1} \mathbf{1}$ and this value can be made larger than -1 by taking the diagonal elements of D^{-1} sufficiently small, i. e., by taking the diagonal elements of D sufficiently large. Therefore $x^\top Qx > 0$ for any $x \neq 0$. $\square$

As an example, in the case of $n = 3$ we may take the following matrix:

$$Q = \begin{pmatrix} 1 & 0 & 0 & p_1^{\{1,2\}} & p_1^{\{1,3\}} & 0 & p_1^{\{1,2,3\}} \\ 0 & 1 & 0 & p_2^{\{1,2\}} & 0 & p_2^{\{2,3\}} & p_2^{\{1,2,3\}} \\ 0 & 0 & 1 & 0 & p_3^{\{1,3\}} & p_3^{\{2,3\}} & p_3^{\{1,2,3\}} \\ p_1^{\{1,2\}} & p_2^{\{1,2\}} & 0 & 3 & 0 & 0 & 0 \\ p_1^{\{1,3\}} & 0 & p_3^{\{1,3\}} & 0 & 3 & 0 & 0 \\ 0 & p_2^{\{2,3\}} & p_3^{\{2,3\}} & 0 & 0 & 3 & 0 \\ p_1^{\{1,2,3\}} & p_2^{\{1,2,3\}} & p_3^{\{1,2,3\}} & 0 & 0 & 0 & 5 \end{pmatrix}.$$

Theorem 5.4. Given a game $\bar{v} \in \mathcal{G}$, the solution $f(\bar{v})$ obtained from the minimum norm problem in the dividend space with the inner product specified by the matrix Q defined in the above lemma coincides with the Harsanyi payoff vector $\phi^p(\bar{v})$.

P r o o f. Let us consider the unanimity game u_T for each nonempty $T \subseteq N$. Then $d^{u_T}(T) = 1$ and $d^{u_T}(S) = 0$ for $S \neq T$. The necessary and sufficient optimality conditions for the minimum norm problem with u_T and the inner product by the matrix induced from the sharing system p are the following:

$$\begin{array}{ll} 2d(i) - 2p_i^T + \lambda = 0 & \text{if } i \in T \\ 2d(i) + \lambda = 0 & \text{if } i \notin T \\ \displaystyle\sum_{i \in N} d(i) = 1, & \end{array}$$

where λ is the Lagrange multiplier for the constraint $\sum_{i \in N} d(i) = 1$. Therefore the optimal solution is exactly $d^*(i) = p_i^T$ for $i \in T$ and $d^*(i) = 0$ for $i \notin T$. Thus

$$f(\bar{v})(i) = \sum_{T \subseteq N} d^{\bar{v}}(T) f(u_T)(i) = \sum_{T \subseteq N, T \ni i} p_i^T d^{\bar{v}}(T)$$

and therefore we obtain the Harsanyi payoff vector $\phi^p(\bar{v})$. □

Corollary 5.5. If the sharing system p is in P^*, then the solution $f(\bar{v})$ is a random order value, i. e., $f(\bar{v}) \in W(\bar{v})$.

Corollary 5.6. If the sharing system is given by $p = p^\pi$ for a permutation π on N, then the solution obtained by the minimum norm solution in the dividend space is the marginal contribution vector $m^\pi(\bar{v})$.

Corollary 5.7. If the sharing system is given by $p_i^T = \frac{1}{|T|}$ for all $i \in T$, the solution obtained by the minimum norm solution in the dividend space coincides with the Shapley value $\varphi(\bar{v})$.

6. CONCLUSION

Each one-point solution for a cooperative game can be regarded as the minimum norm solution and hence we have considered the best approximation of a cooperative game to an additive game as in Kultti and Salonen. However, the difference is that we have defined approximation problems in the dividend space. It has enabled us to deal with simpler problems. Therefore it has been made clear how to choose the inner product in terms of sharing systems to obtain a Harsanyi payoff vector, a random order value, and the Shapley value.

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