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OBSERVER DESIGN FOR A CLASS OF NONLINEAR DISCRETE-TIME SYSTEMS WITH TIME-DELAY

Yali Dong, Jinying Liu and Shengwei Mei

The problem of observer design for a class of nonlinear discrete-time systems with time-delay is considered. A new approach of nonlinear observer design is proposed for the class of systems. Based on differential mean value theory, the error dynamic is transformed into linear parameter variable system. By using Lyapunov stability theory and Schur complement lemma, the sufficient conditions expressed in terms of matrix inequalities are obtained to guarantee the observer error converges asymptotically to zero. Furthermore, the problem of observer design with affine gain is investigated. The computing method for observer gain matrix is given and it is also demonstrated that the observer error converges asymptotically to zero. Finally, an illustrative example is given to validate the effectiveness of the proposed method.

Keywords: observer design, stability, time-delay, differential mean value theory, Lyapunov–Krasovskii functional

Classification: 93C55, 93D05, 93D20, 93C83

1. INTRODUCTION

Time-delay frequently occurs in various practical systems, such as chemical engineering systems, neural networks and population dynamic model. The existence of the time-delay can significantly affect performances or even causes instability in dynamic systems. Thus, the stability problem of nonlinear systems with time-delay, especially the observer-based synthesis problem of nonlinear systems, has been paid much attention in the control community. In recent years, the observer design methods for nonlinear continuous-time systems with delays have been widely investigated [2, 5, 6, 14, 17]. However, there is relatively little research on observer design for nonlinear discrete-time systems with time-delay. In [3], a new state observer design methodology for linear time-varying multi-output systems was presented. The same methodology was also extended to a class of multi-output nonlinear systems and some sufficient conditions for the existence of the proposed observer were obtained. In [11], based on linear matrix inequality technique, the robust H_∞ observer design for a class of Lipsichtz nonlinear discrete-time systems with time-delay was investigated. In [7], nonlinear observers were designed for a class of dynamical discrete-time systems with both constant and time-varying delay nonlinearities by transforming the nonlinear system into a linear time-delay system. In [10], Lu addressed the issues of observer design for a class of Lipschitz nonlinear discrete-time systems with time-delay and disturbance input, where the Lipschitz vector was bounded in a component-wise manner rather than in an aggregated manner as in [11]. In [13], based on a time-scaled block triangular observer form, an observer design for uncontrolled nonlinear multi-output continuous-time systems was presented. In [9], the nonlinear discrete-time partial state observer design problem for rigid spacecraft systems was investigated. In [12], the observer design for a class of Lipschitz nonlinear dynamical systems was considered. In [4], via state transformation and the constructive use of a Lyapunov function, the new observer design approach was addressed by introducing a parameter ε in the observer. Some sufficient conditions which guarantee the estimation error to converge asymptotically to zero were given. In [1], a new approach to robust H_∞ filtering for a class of nonlinear systems with time-varying uncertainties was proposed in the LMI framework based on a general dynamical observer structure.

This paper investigates the problem of observer design for a class of nonlinear discrete-time systems with time-delay. By using differential mean value theory and constructing the Lyapunov–Krasovskii functional, a new approach of nonlinear observer design is proposed. The sufficient conditions that guarantee the observer error converges asymptotically to zero are given. Furthermore, under some reasonable assumptions, an approach of observer design with affine gain is presented, and it is demonstrated that the observer error converges asymptotically to zero.

References [10, 12] deal with the Lipschitz nonlinear systems. Compared with [10, 12], the observer design methods proposed in this paper includes a large variety of systems already studied in literature, namely the class of Lipschitz nonlinear systems. Thus, the methods proposed in this paper have wider applications in control theory and practice. Compared with [15], our systems include the system in [15], for which, the observer is the standard Luenberger form and the observer gain is a constant. By contrast, our observers have the additional term with delayed output. We need to design two observer gains and consider the function observer gain. Hence our results are less conservative than those in [15].

This paper is organized as follows. In Section 2, a class of nonlinear discrete-time systems with delayed state is studied and the corresponding observer is introduced. Based on differential mean value theory, the error dynamic is transformed into linear parameter variable system. The sufficient conditions expressed as inequalities that guarantees the observer error converges asymptotically to zero are established. In Section 3, the problem of the observer design with affine gain for a class of nonlinear discrete-time systems with delayed state is considered and the computing method for observer gain matrix is obtained. The sufficient conditions that guarantee the observer error converges asymptotically to zero are presented. In Section 4, a numerical example is given to show the performances of our method. Finally, some conclusions are drawn in Section 5.

Throughout this paper, I denotes an identity matrix of appropriate dimension. A^T stands for the transpose of A. $E_s = \{e_s(i)|e_s(i) = (0,\ldots,0,1,0,\ldots,0)^T, i = 1,\ldots,s\}$ represents the canonical basis of the vector space R^s for all $s \geq 1$. For a square matrix P, $P > 0\,(< 0, \leq 0, \geq 0)$ means that this matrix is positive (negative, semi-negative, semi-positive) definite. The set $Co(x,y) = \{\lambda x + (1-\lambda)y|0 \leq \lambda \leq 1)\}$ is the convex hull of$\{x,y\}$, and the symbol $\sum_{i,j=1}^{q,n}$ denotes $\sum_{i=1}^{q}\sum_{j=1}^{n}$.

2. NONLINEAR OBSERVER DESIGN

Consider a class of nonlinear discrete-time systems with delayed state as follows:

$$\begin{aligned} x(k+1) &= Ax(k) + A_d x(k-d) + Bf(x(k), x(k-d), y(k), y(k-d)), \\ y(k) &= Cx(k), \\ x(k) &= x_0(k), \quad k = 0, -1, \ldots, -d, \end{aligned} \tag{1}$$

where $x(k) \in R^n$ is the state vector and $y(k) \in R^p$ is the output vector. A, A_d, B and C are constant matrices of appropriate dimensions. $f : R^n \times R^n \times R^p \times R^p \to R^q$ is a nonlinear vector assumed to be differentiable. d is a positive integer for delay time, and $x_0(k)$ is an initial value at k.

A state observer for (1) is given by:

$$\begin{aligned} \hat{x}(k+1) &= A\hat{x}(k) + A_d\hat{x}(k-d) + Bf(\hat{x}(k), \hat{x}(k-d), y(k), y(k-d)) \\ &\quad + L(y(k) - \hat{y}(k)) + L_d(y(k-d) - \hat{y}(k-d)), \\ \hat{y}(k) &= C\hat{x}(k), \end{aligned} \tag{2}$$

where $\hat{x}(k) \in R^n$ is the estimate of the state $x(k), L, L_d \in R^{n\times p}$ are gain matrices.

Remark 2.1. The observer (2) is not in the standard Luenberger form. The amending term $L_d(y(k-d) - \hat{y}(k-d))$ is added, which improves the convergence. The design method comes from [16, 18].

Our objective is to find the matrices L and L_d such that the corresponding estimation error

$$e(k) = x(k) - \hat{x}(k), \tag{3}$$

is asymptotically stable.

The following lemmas will play an important role in the paper.

Lemma 2.2. (Zemouche et al. [17]) Let $f : R^{2n} \to R^q$. Let $a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in R^{2n}$ with $a_i, b_i \in R^n$ for $i = 1, 2$. We assume that $f(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n)$ is differentiable on $Co(a, b)$. Then, there are constant vectors $c_1, c_2, \ldots, c_q \in Co(a, b)$, $c_i \neq a$, $c_i \neq b$, for $i = 1, 2, \ldots, q$, such that:

$$f(a) - f(b) = (\sum_{i,j=1}^{q,n} e_q(i)e_n^T(j)\frac{\partial f_i}{\partial x_j}(c_i))(a_1 - b_1) + (\sum_{i,j=1}^{q,n} e_q(i)e_n^T(j)\frac{\partial f_i}{\partial y_j}(c_i))(a_2 - b_2).$$

Lemma 2.3. (Schur complement lemma) (Kwon and Park [8]) Given constant symmetric matrices S_1, S_2, S_3, and $S_1 = S_1^T < 0$, $S_3 = S_3^T > 0$, then $S_1 + S_2S_3^{-1}S_2^T < 0$ if and only if

$$\begin{pmatrix} S_1 & S_2 \\ S_2^T & -S_3 \end{pmatrix} < 0.$$

From (1) and (2), the dynamics of the estimation error is given by

$$e(k+1) = (A - LC)e(k) + (A_d - L_dC)e(k-d) + B\delta f(k), \tag{4}$$

where $\delta f(k) = f(x(k), x(k-d), y(k), y(k-d)) - f(\hat{x}(k), \hat{x}(k-d), y(k), y(k-d))$.

Using the notations

$$X(k) = \begin{pmatrix} x^T(k) & x^T(k-d) \end{pmatrix}^T, \ \hat{X}(k) = \begin{pmatrix} \hat{x}^T(k) & \hat{x}^T(k-d) \end{pmatrix}^T,$$

and Lemma 2.2, there exist vectors $z_i(k) \in Co(X(k), \hat{X}(k))$ for all $i = 1, \dots, q$, such that:

$$\delta f(k) = S_{q,n}(k)e(k) + S_{q,n}^d(k)e(k-d), \tag{5}$$

where

$$S_{q,n}(k) = \sum_{i,j=1}^{q,n} e_q(i)e_n^T(j)\frac{\partial f_i}{\partial x_j(k)}(z_i(k), y(k), y(k-d)),$$

and

$$S_{q,n}^d(k) = \sum_{i,j=1}^{q,n} e_q(i)e_n^T(j)\frac{\partial f_i}{\partial x_j(k-d)}(z_i(k), y(k), y(k-d)).$$

By using of the following notations:

$$\begin{aligned}
&h_{ij}(k) = \tfrac{\partial f_i}{\partial x_j(k)}(z_i(k), y(k), y(k-d)), \ i = 1, \dots, q, \ j = 1, \dots, n, \\
&h(k) = (h_{11}(k), \dots, h_{1n}(k), \dots, h_{qn}(k)), \\
&h_{ij}^d(k) = \tfrac{\partial f_i}{\partial x_j(k-d)}(z_i(k), y(k), y(k-d)), \ i = 1, \dots, q, \ j = 1, \dots, n, \\
&h^d(k) = (h_{11}^d(k), \dots, h_{1n}^d(k), \dots, h_{qn}^d(k)), \\
&\bar{A}(h(k)) = A + BS_{q,n}(k), \text{ and } \bar{A}_d(h^d(k)) = A_d + BS_{q,n}^d(k)
\end{aligned}$$

the estimation error dynamics (4) can be rewritten as:

$$e(k+1) = (\bar{A}(h(k)) - LC)e(k) + (\bar{A}_d(h^d(k)) - L_dC)e(k-d). \tag{6}$$

Now, we introduce the following assumption.

Assumption 2.4. The functions $h_{ij}(k)$ and $h_{ij}^d(k)$ are bounded for $i = 1, \dots, q$, $j = 1, \dots, n$, i. e.,

$$\sup_k |h_{ij}(k)| < \infty, \quad \sup_k |h_{ij}^d(k)| < \infty.$$

Remark 2.5. Although some functions do not satisfy Assumption 2.4, the class of systems satisfying Assumption 2.4 includes a large variety of systems already studied in literature, namely the class of differential Lipschitz nonlinear systems. The Assumption 2.4 is equivalent to the Assumption 1 in [18].

Under Assumption 2.4, the parameter vector $h(k)$ remains in a bounded domain $H_{q,n}$ of which 2^{qn} vertices are defined by:

$$V_{H_{q,n}} = \{\alpha = (\alpha_{11}, \dots, \alpha_{1n}, \dots, \alpha_{qn}) | \alpha_{ij} \in \{\underline{h}_{ij}, \bar{h}_{ij}\}\},$$

where $\underline{h}_{ij} = \inf_k(h_{ij}(k))$ and $\bar{h}_{ij} = \sup_k(h_{ij}(k))$.

The parameter vector $h^d(k)$ evolves in a bounded domain $H^d_{q,n}$ of which 2^{qn} vertices are defined by:

$$V_{H^d_{q,n}} = \{\beta = (\beta_{11}, \ldots, \beta_{1n}, \ldots, \beta_{qn}) | \beta_{ij} \in \{\underline{h}^d_{ij}, \bar{h}^d_{ij}\}\},$$

where $\underline{h}^d_{ij} = \inf_k(h^d_{ij}(k))$ and $\bar{h}^d_{ij} = \sup_k(h^d_{ij}(k))$.

Now, we can state our main result about new matrix inequality conditions for the observer synthesis problem.

Theorem 2.6. Suppose that Assumption 2.4 is satisfied. Then, the estimation error (3) converges asymptotically towards zero if there exist matrices $0 < P \in R^{n\times n}, 0 \leq Q \in R^{n\times n}$, $M \in R^{(2n)\times n}$, $N \in R^{p\times n}$ and $R_d \in R^{p\times n}$ such that the following matrix inequality holds:

$$\begin{pmatrix} \zeta_1 + \zeta_2 + \zeta_2^T + \zeta_3 & \Sigma_1 \\ \Sigma_1^T & -\Sigma_2 \end{pmatrix} < 0, \ \forall \alpha \in V_{H_{q,n}}, \ \forall \beta \in V_{H^d_{q,n}}, \tag{7}$$

where

$$\zeta_1 = \begin{pmatrix} -P+Q & 0 \\ 0 & -Q \end{pmatrix}, \quad \zeta_2 = \begin{pmatrix} M & -M \end{pmatrix}, \quad \zeta_3 = dMP^{-1}M^T,$$

$$\Sigma_1 = \begin{pmatrix} \bar{A}^T(\alpha)P - C^TN & d\bar{A}^T(\alpha)P - dC^TN - dP \\ \bar{A}_d^T(\beta)P - C^TR_d & d\bar{A}_d^T(\beta)P - dC^TR_d \end{pmatrix}, \quad \Sigma_2 = \begin{pmatrix} P & 0 \\ 0 & dP \end{pmatrix}.$$

Moreover, the gain matrices L and L_d are given respectively by $L = P^{-1}N^T$, $L_d = P^{-1}R_d^T$.

P r o o f. To prove this, we use the following Lyapunov–Krasovskii functional:

$$V(k) = V_1(k) + V_2(k) + V_3(k), \tag{8}$$

where

$$V_1(k) = e^T(k)Pe(k), V_2(k) = \sum_{l=k-d}^{k-1} e^T(l)Qe(l), V_3(k) = \sum_{i=-d}^{-1}\sum_{m=k+i}^{k-1} \eta^T(m)P\eta(m),$$

$$\eta(k) = e(k+1) - e(k) = (\bar{A}(h(k)) - LC - I)e(k) + (\bar{A}_d(h^d(k)) - L_dC)e(k-d).$$

Let $\lambda(k) = \begin{pmatrix} e^T(k) & e^T(k-d) \end{pmatrix}^T$. Then, we have

$$\begin{aligned} \Delta V_1(k) &= V_1(k+1) - V_1(k) \\ &= e^T(k+1)Pe(k+1) - e^T(k)Pe(k) \\ &= \{(\bar{A}(h(k)) - LC)e(k) + (\bar{A}_d(h^d(k)) - L_dC)e(k-d)\}^T P \\ &\quad \times\{(\bar{A}(h(k)) - LC)e(k) + (\bar{A}_d(h^d(k)) - L_dC)e(k-d)\} \\ &\quad - e^T(k)Pe(k) \\ &= \lambda^T(k) \begin{pmatrix} \bar{A}(h(k)) - LC & \bar{A}_d(h^d(k)) - L_dC \end{pmatrix}^T P \\ &\quad \times \begin{pmatrix} \bar{A}(h(k)) - LC & \bar{A}_d(h^d(k)) - L_dC \end{pmatrix} \lambda(k) - e^T(k)Pe(k), \end{aligned} \tag{9}$$

$$\begin{aligned}\Delta V_2(k) &= V_2(k+1) - V_2(k) = \sum_{l=k+1-d}^{k} e^T(l)Qe(l) - \sum_{l=k-d}^{k-1} e^T(l)Qe(l) \\ &= e^T(k)Qe(k) - e^T(k-d)Qe(k-d),\end{aligned} \tag{10}$$

$$\begin{aligned}\Delta V_3(k) &= V_3(k+1) - V_3(k) \\ &= \sum_{i=-d}^{-1} \sum_{m=k+1+i}^{k} \eta^T(m)P\eta(m) - \sum_{i=-d}^{-1} \sum_{m=k+i}^{k-1} \eta^T(m)P\eta(m) \\ &= \sum_{i=-d}^{-1} \{\eta^T(k)P\eta(k) - \eta^T(k+i)P\eta(k+i)\} \\ &= d\eta^T(k)P\eta(k) - \sum_{i=-d}^{-1} \eta^T(k+i)P\eta(k+i) \\ &= d\lambda^T(k) \begin{pmatrix} \bar{A}(h(k)) - LC - I & \bar{A}_d(h^d(k)) - L_dC \end{pmatrix}^T P \\ &\quad \times \begin{pmatrix} \bar{A}(h(k)) - LC - I & \bar{A}_d(h^d(k)) - L_dC \end{pmatrix} \lambda(k) \\ &\quad - \sum_{l=k-d}^{k-1} \eta^T(l)P\eta(l).\end{aligned} \tag{11}$$

Note that

$$\sum_{l=k-d}^{k-1} \eta(l) = \sum_{l=k-d}^{k-1} [e(l+1) - e(l)] = e(k) - e(k-d), \tag{12}$$

then

$$e(k) - e(k-d) - \sum_{l=k-d}^{k-1} \eta(l) = 0.$$

Hence, the following equation holds for the arbitrary parameter matrix M of appropriate dimension:

$$2\lambda^T(k)M[e(k) - e(k-d) - \sum_{l=k-d}^{k-1} \eta(l)] = 0. \tag{13}$$

From (9), (10), (11) and (13), we get

$$\begin{aligned}\Delta V(k) &= V(k+1) - V(k) \\ &= \Delta V_1(k) + \Delta V_2(k) + \Delta V_3(k) + 2\lambda^T(k)M\Big[e(k) - e(k-d) - \sum_{l=k-d}^{k-1} \eta(l)\Big] \\ &= \Delta V_1(k) + \Delta V_2(k) + \Delta V_3(k) + 2\lambda^T(k) \begin{pmatrix} M & -M \end{pmatrix} \lambda(k) \\ &\quad - \sum_{l=k-d}^{k-1} 2\lambda^T(k)M\eta(l) \\ &= \Delta V_1(k) + \Delta V_2(k) + \Delta V_3(k) + 2\lambda^T(k) \begin{pmatrix} M & -M \end{pmatrix} \lambda(k) \\ &\quad + d\lambda^T(k)MP^{-1}M^T\lambda(k) - \sum_{l=k-d}^{k-1} [2\lambda^T(k)M\eta(l) + \lambda^T(k)MP^{-1}M^T\lambda(k)]\end{aligned}$$

$$
\begin{aligned}
&= \lambda^T(k)\Big\{\begin{pmatrix} Q-P & 0 \\ 0 & -Q \end{pmatrix} + \begin{pmatrix} M & -M \end{pmatrix} + \begin{pmatrix} M & -M \end{pmatrix}^T + dMP^{-1}M^T \\
&+ \begin{pmatrix} (\bar{A}(h(k)) - LC)^T & (\bar{A}(h(k)) - LC)^T - I \\ (\bar{A}_d(h^d(k)) - L_dC)^T & (\bar{A}_d(h^d(k)) - L_dC)^T \end{pmatrix} \begin{pmatrix} P & 0 \\ 0 & dP \end{pmatrix} \\
&\times \begin{pmatrix} \bar{A}(h(k)) - LC & \bar{A}_d(h^d(k)) - LC \\ \bar{A}(h(k)) - LC - I & \bar{A}_d(h^d(k)) - L_dC \end{pmatrix} \Big\}\lambda(k) \\
&- \sum_{l=k-d}^{k-1} \{\eta^T(l)P\eta(l) + 2\lambda^T(k)M\eta(l) + \lambda^T(k)MP^{-1}M^T\lambda(k)\}.
\end{aligned}
$$

Let $\Xi_1 = \bar{A}(h(k)) - LC, \Xi_2 = \bar{A}_d(h^d(k)) - L_dC$, then

$$
\begin{aligned}
\Delta V(k) &= \lambda^T(k)\Big\{\zeta_1 + \zeta_2 + \zeta_2^T + \zeta_3 + \begin{pmatrix} \Xi_1^T P & d\Xi_1^T P - dP \\ \Xi_2^T P & d\Xi_2^T P \end{pmatrix} \begin{pmatrix} P & 0 \\ 0 & dP \end{pmatrix}^{-1} \\
&\times \begin{pmatrix} P\Xi_1 & P\Xi_2 \\ dP\Xi_1 - dP & dP\Xi_2 \end{pmatrix} \Big\}\lambda(k) \\
&- \sum_{l=k-d}^{k-1} \{\lambda^T(k)M + \eta^T(l)P\}P^{-1}\{M^T\lambda(k) + P\eta(l)\} \\
&\leq \lambda^T(k)H_1(h(k), h^d(k))\lambda(k),
\end{aligned}
$$

where

$$
\begin{aligned}
H_1(h(k), h^d(k)) &= \zeta_1 + \zeta_2 + \zeta_2^T + \zeta_3 + \begin{pmatrix} \Xi_1^T P & d\Xi_1^T P - dP \\ \Xi_2^T P & d\Xi_2^T P \end{pmatrix} \begin{pmatrix} P & 0 \\ 0 & dP \end{pmatrix}^{-1} \\
&\times \begin{pmatrix} P\Xi_1 & P\Xi_2 \\ dP\Xi_1 - dP & dP\Xi_2 \end{pmatrix}.
\end{aligned}
$$

Using the convexity principle and Lemma 2.3, if the condition (7) is satisfied, then we have

$$H_1(h(k), h^d(k)) < 0,\ \forall\, h(k) \in H_{q,n},\ \forall\, h^d(k) \in H_{q,n}^d.$$

It follows that $\Delta V(k) < 0$. This completes the proof. □

Note that the matrix inequality (7) is not linear inequality due to the existence of the nonlinear term ζ_3. Thus it is inconvenient to be used in practice. The following theorem is an improved result.

Theorem 2.7. Suppose that Assumption 2.4 is satisfied. Then the estimation error (3) converges asymptotically towards zero if there exist matrices $0 < P \in R^{n\times n},\ 0 \leq Q \in$

$R^{n\times n}$, $M \in R^{(2n)\times n}$, $N \in R^{p\times n}$ and $R_d \in R^{p\times n}$ such that the following linear matrix inequality holds:

$$\begin{pmatrix} \Pi_1 & \Pi_2 \\ \Pi_2^T & -\Pi_3 \end{pmatrix} < 0, \ \forall\, \alpha \in V_{H_{q,n}}, \ \forall\, \beta \in V_{H^d_{q,n}}, \tag{14}$$

where

$$\Pi_1 = \begin{pmatrix} \zeta_1 & 0 \\ 0 & \zeta_2 + \zeta_2^T \end{pmatrix}, \quad \zeta_1 = \begin{pmatrix} -P + Q & 0 \\ 0 & -Q \end{pmatrix}, \quad \zeta_2 = \begin{pmatrix} M & -M \end{pmatrix},$$

$$\Pi_2 = \begin{pmatrix} \bar{A}(\alpha)P - C^T N & d\bar{A}^T(\alpha)P - dC^T N - dP & 0 \\ \bar{A}_d^T(\beta)P - C^T R_d & d\bar{A}_d^T(\beta)P - dC^T R_d & 0 \\ 0 & 0 & M \end{pmatrix},$$

$$\Pi_3 = \begin{pmatrix} P & 0 & 0 \\ 0 & dP & 0 \\ 0 & 0 & (1/d)P \end{pmatrix}.$$

Moreover, the gain matrices L and L_d are given respectively by $L = P^{-1}N^T$, $L_d = P^{-1}R_d^T$.

Proof. Consider the candidate Lyapunov–Krasovskii functional (8). Let

$$\mu(k) = \begin{pmatrix} e^T(k) & e^T(k-d) & \lambda^T(k) \end{pmatrix}^T.$$

From the proof of Theorem 2.6, we get

$$\begin{aligned} \Delta V(k) \ \le\ & \lambda^T(k)\Big\{ \begin{pmatrix} Q-P & 0 \\ 0 & -Q \end{pmatrix} + \begin{pmatrix} M & -M \end{pmatrix} + \begin{pmatrix} M & -M \end{pmatrix}^T \\ & + dMP^{-1}M^T + \begin{pmatrix} (\bar{A}(h(k)) - LC)^T & (\bar{A}(h(k)) - LC)^T - I \\ (\bar{A}_d(h^d(k)) - L_dC)^T & (\bar{A}_d(h^d(k)) - L_dC)^T \end{pmatrix} \\ & \times \begin{pmatrix} P & 0 \\ 0 & dP \end{pmatrix} \begin{pmatrix} \bar{A}(h(k)) - LC & \bar{A}_d(h^d(k)) - L_dC \\ \bar{A}(h(k)) - LC - I & \bar{A}_d(h^d(k)) - L_dC \end{pmatrix} \Big\}\lambda(k). \end{aligned}$$

Let $\Xi_1 = \bar{A}(h(k)) - LC, \Xi_2 = \bar{A}_d(h^d(k)) - L_dC$ then

$$\begin{aligned} \Delta V(k) \ \le\ & \mu^T(k)\Big\{ \begin{pmatrix} Q-P & 0 & 0 \\ 0 & -Q & 0 \\ 0 & 0 & \begin{pmatrix} M & -M \end{pmatrix} + \begin{pmatrix} M & -M \end{pmatrix}^T \end{pmatrix} \\ & + \begin{pmatrix} \Xi_1^T & \Xi_1^T - I & 0 \\ \Xi_2^T & \Xi_2^T & 0 \\ 0 & 0 & M \end{pmatrix} \begin{pmatrix} P & 0 & 0 \\ 0 & dP & 0 \\ 0 & 0 & I \end{pmatrix} \\ & \times \begin{pmatrix} \Xi_1 & \Xi_2 & 0 \\ \Xi_1 - I & \Xi_2 & 0 \\ 0 & 0 & dP^{-1}M^T \end{pmatrix} \Big\}\mu(k) \\ = \ & \mu^T(k) H_2(h(k), h^d(k))\mu(k), \end{aligned} \tag{15}$$

where

$$H_2(h(k),h^d(k)) = \begin{pmatrix} \zeta_1 & 0 \\ 0 & \zeta_2+\zeta_2^T \end{pmatrix} + \begin{pmatrix} \Xi_1^T P & d\Xi_1^T P - dP & 0 \\ \Xi_2^T P & d\Xi_2^T P & 0 \\ 0 & 0 & M \end{pmatrix}$$

$$\times \begin{pmatrix} P & 0 & 0 \\ 0 & dP & 0 \\ 0 & 0 & (1/d)P \end{pmatrix}^{-1} \begin{pmatrix} P\Xi_1 & P\Xi_2 & 0 \\ dP\Xi_1 - dP & dP\Xi_2 & 0 \\ 0 & 0 & M^T \end{pmatrix}.$$

By the convexity principle and Lemma 2.3, if the condition (14) is satisfied, then we have

$$H_2(h(k),h^d(k)) < 0, \ \forall\, h(k) \in H_{q,n}, \ \forall\, h^d(k) \in H_{q,n}^d.$$

It follows that $\Delta V(k) < 0$. This completes the proof. □

Remark 2.8. The free weighting matrix M is used to describe the relationships between terms $e(k)$, $e(k-d)$ and $\sum_{l=k-d}^{k-1}\eta(l)$, and the condition (7) is delay-dependent, which reduces their conservatism. When the observer gain matrix L_d in (2) vanishes, we can obtain the following stable result immediately from Theorem 2.6.

Corollary 2.9. Consider the observer (2) with $L_d = 0$. Suppose that Assumption 2.4 is satisfied. Then, the estimation error (3) converges asymptotically towards zero if there exist matrices $0 < P \in R^{n\times n}, 0 \le Q \in R^{n\times n}, M \in R^{(2n)\times n}$ and $N \in R^{p\times n}$ such that the following matrix inequality holds:

$$\begin{pmatrix} \zeta_1+\zeta_2+\zeta_2^T+\zeta_3 & \bar{\Sigma}_1 \\ \bar{\Sigma}_1^T & -\Sigma_2 \end{pmatrix} < 0, \ \forall\, \alpha \in V_{H_{q,n}}, \ \forall\, \beta \in V_{H_{q,n}^d}, \tag{16}$$

where

$$\zeta_1 = \begin{pmatrix} -P+Q & 0 \\ 0 & -Q \end{pmatrix}, \quad \zeta_2 = \begin{pmatrix} M & -M \end{pmatrix}, \quad \zeta_3 = dMP^{-1}M^T,$$

$$\bar{\Sigma}_1 = \begin{pmatrix} \bar{A}^T(\alpha)P - C^T N & d\bar{A}^T(\alpha)P - dC^T N - dP \\ \bar{A}_d^T(\beta)P & d\bar{A}_d^T(\beta)P \end{pmatrix}, \quad \Sigma_2 = \begin{pmatrix} P & 0 \\ 0 & dP \end{pmatrix}.$$

Moreover, the matrix L is given by $L = P^{-1}N^T$.

Remark 2.10. When the observer gain L in (2) vanishes, we can obtain the following stable result immediately according to Theorem 2.6.

Corollary 2.11. Consider the observer (2) with $L = 0$. Suppose that Assumption 2.4 is satisfied. Then, the estimation error (3) converges asymptotically towards zero if there exist matrices $0 < P \in R^{n\times n}, 0 \le Q \in R^{n\times n}, M \in R^{(2n)\times n}$ and $R_d \in R^{p\times n}$ such that the following matrix inequality holds:

$$\begin{pmatrix} \zeta_1+\zeta_2+\zeta_2^T+\zeta_3 & \tilde{\Sigma}_1 \\ \tilde{\Sigma}_1^T & -\Sigma_2 \end{pmatrix} < 0, \ \forall\, \alpha \in V_{H_{q,n}}, \ \forall\, \beta \in V_{H_{q,n}^d}, \tag{17}$$

where

$$\zeta_1 = \begin{pmatrix} -P+Q & 0 \\ 0 & -Q \end{pmatrix}, \quad \zeta_2 = \begin{pmatrix} M & -M \end{pmatrix}, \quad \zeta_3 = dMP^{-1}M^T,$$

$$\tilde{\Sigma}_1 = \begin{pmatrix} \bar{A}^T(\alpha)P & d\bar{A}^T(\alpha)P - dP \\ \bar{A}_d^T(\beta)P - C^T R_d & d\bar{A}_d^T(\beta)P - dC^T R_d \end{pmatrix}, \quad \Sigma_2 = \begin{pmatrix} P & 0 \\ 0 & dP \end{pmatrix}.$$

Moreover, the matrix L_d is given by $L_d = P^{-1}R_d^T$.

Remark 2.12. Compared with Corollary 2.9 and Corollary 2.11, Theorem 2.6 is less conservative.

Remark 2.13. Using LMI Toolbox for (14), we can solve matrices P, Q, M, N, R_d. If the linear matrix inequality (14) is feasible, using P, N, R_d, we can get the observer gain matrices L and L_d.

3. OBSERVER DESIGN WITH AFFINE GAIN

In previous sections, the observer gain matrices L and L_d are constant matrices. In this section, we use affine observer gains matrices L and L_d under an additional assumption, which is given as follows.

Assumption 3.1. There exist subsets $S \subset \{1,\ldots,q\} \times \{1,\ldots,n\}$ and $S^d \subset \{1,\ldots, q\} \times \{1,\ldots,n\}$ such that

$$\frac{\partial f_i}{\partial x_j(k)}(x(k), x(k-d), y(k), y(k-d)) = g_{ij}(y(k), y(k-d)), \ \forall\, (i,j) \in S, \tag{18}$$

$$\frac{\partial f_l}{\partial x_m(k-d)}(x(k), x(k-d), y(k), y(k-d)) = g^d_{lm}(y(k), y(k-d)), \ \forall\, (l,m) \in S^d. \tag{19}$$

This Assumption 3.1 means that

$$\frac{\partial f_i}{\partial x_j(k)}(x(k), x(k-d), y(k), y(k-d)), \ \forall\, (i,j) \in S,$$

and

$$\frac{\partial f_l}{\partial x_m(k-d)}(x(k), x(k-d), y(k), y(k-d)), \ \forall\, (l,m) \in S^d,$$

are independent in $x(k)$ and $x(k-d)$.

Consider the following state observer for system (1)

$$\begin{aligned} \hat{x}(k+1) &= A\hat{x}(k) + A_d\hat{x}(k-d) + Bf(\hat{x}(k), \hat{x}(k-d), y(k), y(k-d)) \\ &\quad + L(k)(y(k) - \hat{y}(k)) + L_d(k)(y(k-d) - \hat{y}(k-d)), \\ \hat{y}(k) &= C\hat{x}(k), \end{aligned} \tag{20}$$

where $\hat{x}(k)$ is the estimate of the state $x(k)$, $L(k) = L_0 + \sum_{(i,j)\in S} g_{ij}(y(k), y(k-d))L_{ij}, \ L_d(k) = L_0^d + \sum_{(l,m)\in S^d} g^d_{lm}(y(k), y(k-d))L^d_{lm}$.

In this section we try to find the matrices L_0, L_{ij} for all $(i,j) \in S$ and L_0^d, L_{ij}^d for all $(l,m) \in S^d$ such that the estimation error

$$e(k) = x(k) - \hat{x}(k), \tag{21}$$

converges asymptotically toward to zero.

From (1) and (20), the dynamics of the estimation error is expressed as follows:

$$\begin{aligned} e(k+1) \;=\; & (A - L(k)C)e(k) + (A_d - L_d(k)C)e(k-d) + B[f(x(k), x(k-d), y(k), \\ & y(k-d)) - f(\hat{x}(k), \hat{x}(k-d), y(k), y(k-d))]. \end{aligned} \tag{22}$$

By taking again the notations of section 2, the equation (22) can be written as:

$$e(k+1) = (\bar{A}(h(k)) - L(k)C)e(k) + (\bar{A}_d(h^d(k)) - L_d(k)C)e(k-d). \tag{23}$$

Note that here we have

$$h_{ij}(k) = g_{ij}(y(k), y(k-d)), (i,j) \in S,$$

and

$$h_{lm}^d(k) = g_{lm}^d(y(k), y(k-d)), (l,m) \in S^d.$$

Then, for $\alpha \in V_{H_{q,n}}$ we have $\alpha_{ij} \in \{\underline{g}_{ij}, \bar{g}_{ij}\}$ for all $(i,j) \in S$, and for $\beta \in V_{H_{q,n}^d}$ we have $\beta_{lm} \in \{\underline{g}_{lm}^d, \bar{g}_{lm}^d\}$ for all $(l,m) \in S^d$, where

$$\underline{g}_{ij} = \min_k(g_{ij}(k)), \bar{g}_{ij} = \max_k(g_{ij}(k)), \underline{g}_{ij}^d = \min_k(g_{ij}^d(k)), \bar{g}_{ij}^d = \max_k(g_{ij}^d(k)).$$

Now, we state the following theorem.

Theorem 3.2. Suppose that Assumption 2.4 and Assumption 3.1 are satisfied. Then the estimation error (21) converges asymptotically towards zero if there exist matrices $P > 0, Q \geq 0, M, R_0, R_0^d, R_{ij}$ and R_{lm}^d of appropriate dimensions for $(i,j) \in S, (l,m) \in S^d$ such that the following linear matrix inequality holds:

$$\begin{pmatrix} \Pi_1 & \Pi_4 \\ \Pi_4^T & -\Pi_3 \end{pmatrix} < 0, \ \forall \alpha \in V_{H_{q,n}}, \ \forall \beta \in V_{H_{q,n}^d}, \tag{24}$$

where

$$\Pi_1 = \begin{pmatrix} \zeta_1 & 0 \\ 0 & \zeta_2 + \zeta_2^T \end{pmatrix}, \quad \Pi_3 = \begin{pmatrix} P & 0 & 0 \\ 0 & dP & 0 \\ 0 & 0 & (1/d)P \end{pmatrix},$$

$$\zeta_1 = \begin{pmatrix} -P + Q & 0 \\ 0 & -Q \end{pmatrix}, \quad \zeta_2 = \begin{pmatrix} M & -M \end{pmatrix},$$

$$\Pi_4 = \begin{pmatrix} \bar{A}^T(\alpha)P - C^T w(R_0, R_{ij}) & d\bar{A}^T(\alpha)P - dC^T w(R_0, R_{ij}) - dP & 0 \\ \bar{A}_d^T(\beta)P - C^T w_d(R_0^d, R_{lm}^d) & d\bar{A}_d^T(\beta)P - dC^T w_d(R_0^d, R_{lm}^d) & 0 \\ 0 & 0 & M \end{pmatrix}.$$

$$w(R_0,R_{ij})=R_0+\sum_{(i,j)\in S}\alpha_{ij}R_{ij}, w_d(R_0^d,R_{lm}^d)=R_0^d+\sum_{(l,m)\in S^d}\beta_{lm}R_{lm}^d.$$

Moreover, the observer gain matrices are given by:

$$L_0=P^{-1}R_0^T, L_{ij}=P^{-1}R_{ij}^T,\ \forall\,(i,j)\in S,$$

$$L_0^d=P^{-1}(R_0^d)^T,\quad L_{lm}^d=P^{-1}(R_{lm}^d)^T,\ \forall\,(l,m)\in S^d.$$

Remark 3.3. Note the affine observer gains in Section 3 are functions, but the observer gain matrices in Section 2 are constant matrices. Comparing Theorem 3.2 with Theorem 2.7, we can see that the number of variables in Theorem 3.2 is greater than that in Theorem 2.7. Thus Theorem 3.2 is less conservative than Theorem 2.7. If Assumption 3.1 holds only when $S=\Phi$ and $S^d=\Phi$, where Φ denotes an empty set, then Theorem 3.2 degenerates to Theorem 2.7.

4. NUMERICAL EXAMPLE

In this subsection, a numerical example is provided to show the high performance of the proposed approach.

Consider the following nonlinear discrete-time system:

$$\begin{array}{rcl} x(k+1) & = & Ax(k)+A_dx(k-d)+Bf(x(k),x(k-d)),\\ y(k) & = & Cx(k),\\ x(k) & = & 1,\ \ k=0,-1, \end{array} \tag{25}$$

where

$$A=\begin{pmatrix} -0.5 & 0\\ 1 & 0.5\end{pmatrix},\quad A_d=\begin{pmatrix} 1 & 0\\ 1 & 0\end{pmatrix},\quad B=\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix},$$

$$f(x(k),x(k-d))=\begin{pmatrix} 0.25\arctan x_2(k)\\ 0.2\sin x_2(k-1)\end{pmatrix},\quad C=\begin{pmatrix} 1 & 0\end{pmatrix},\ d=1.$$

Then, using our approach, we obtain:

$$h_{12}(k)=\frac{0.25}{1+x_2^2(k)},\quad h_{22}^d(k)=0.2\cos x_2(k-1),$$

$$h_{11}(k)=h_{21}(k)=h_{22}(k)=h_{11}^d(k)=h_{12}^d(k)=h_{21}^d(k)=0.$$

Therefore, the vertex sets can be obtained as follows:

$$V_{H_{q,n}}=\{(0,0,0,0),(0,0.25,0,0)\}, V_{H_{q,n}^d}=\{(0,0,0,0.2),(0,0.25,0,-0.2)\}.$$

Using LMI Toolbox to solve the inequality (14), we get:

$$P=\begin{pmatrix} 454430 & 0\\ 0 & 469860\end{pmatrix},\ N=\begin{pmatrix} -454430 & 469860\end{pmatrix},\ R_d=\begin{pmatrix} 454430 & 469860\end{pmatrix}.$$

Thus, the observer gain matrices L and L_d can be taken as:

$$L = P^{-1}N^T = \begin{pmatrix} -1 & 1 \end{pmatrix}^T \textit{ and } L_d = P^{-1}R_d^T = \begin{pmatrix} 1 & 1 \end{pmatrix}^T. \tag{26}$$

The state observer for (25) is given by:

$$\begin{aligned} \hat{x}(k+1) &= \begin{pmatrix} -0.5 & 0 \\ 1 & 0.5 \end{pmatrix} \hat{x}(k) + \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \hat{x}(k-1) \\ &+ \begin{pmatrix} 0.25 \arctan \hat{x}_2(k) \\ 0.2 \sin \hat{x}_2(k-1) \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} (x_1(k) - \hat{x}_1(k)) \\ &+ \begin{pmatrix} 1 \\ 1 \end{pmatrix} (x_1(k-1) - \hat{x}_1(k-1)), \\ \hat{y}(k) &= \hat{x}_1(k). \end{aligned} \tag{27}$$

The dynamics of the estimation error is:

$$e(k+1) = 0.5e(k) + \begin{pmatrix} 0.25 \arctan x_2(k) - 0.25 \arctan \hat{x}_2(k) \\ 0.2 \sin x_2(k-1) - 0.2 \sin \hat{x}_2(k-1) \end{pmatrix},$$

where

$$e(k) = \begin{pmatrix} e_1(k) \\ e_2(k) \end{pmatrix} = x(k) - \hat{x}(k) = \begin{pmatrix} x_1(k) - \hat{x}_1(k) \\ x_2(k) - \hat{x}_2(k) \end{pmatrix}. \tag{28}$$

It is seen from Figure 1 that the estimation error (28) converges asymptotically towards zero.

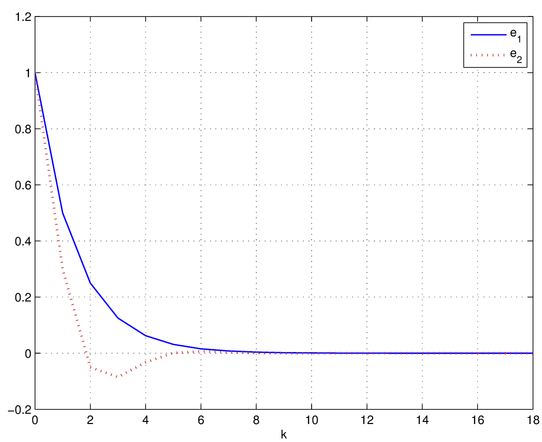

Fig. 1. The estimation error behavior.

5. CONCLUSIONS

This paper investigates the problem of observer design for a class of nonlinear discrete-time systems with time-delay. Based on differential mean value theory, the error dynamic is transformed into linear parameter variable system. By using Lyapunov stability theory and Schur complement lemma, a new approach of nonlinear observer design is proposed. Sufficient conditions that guarantee the observer error converges asymptotically to zero are given. The corresponding computing method for observer gain matrix is presented. Moreover, the method obtained is generalized to the observer design with affine gain. The approach of observer design with affine gain is proposed and it is also demonstrated that the observer error converges asymptotically to zero under appropriate conditions.

APPENDIX

Proof of Theorem 3.2. Similar to the proof in Theorem 1, we choose the following Lyapunov–Krasovskii functional:

$$\begin{aligned} &V(k) = V_1(k) + V_2(k) + V_3(k),\\ &V_1(k) = e^T(k)Pe(k),\ V_2(k) = \sum_{l=k-d}^{k-1} e^T(l)Qe(l),\ V_3(k) = \sum_{i=-d}^{-1}\sum_{m=k+i}^{k-1} \eta^T(m)P\eta(m),\\ &\eta(k) = e(k+1) - e(k) = (\bar{A}(h(k)) - L(k)C - I)e(k) + (\bar{A}_d(h^d(k)) - L_d(k)C)e(k-d). \end{aligned} \tag{29}$$

The difference of $V_i(k), i = 1, 2, 3$, along the trajectories of system (1) is calculated as follows:

$$\begin{aligned} \Delta V_1(k) &= V_1(k+1) - V_1(k)\\ &= \{(\bar{A}(h(k)) - L(k)C)e(k) + (\bar{A}_d(h^d(k)) - L_d(k)C)e(k-d)\}^T P\\ &\quad \times\{(\bar{A}(h(k)) - L(k)C)e(k) + (\bar{A}_d(h^d(k)) - L_d(k)C)e(k-d)\}\\ &\quad - e^T(k)Pe(k)\\ &= \lambda^T(k)\begin{pmatrix} \bar{A}(h(k)) - L(k)C & \bar{A}_d(h^d(k)) - L_d(k)C \end{pmatrix}^T P\\ &\quad \times \begin{pmatrix} \bar{A}(h(k)) - L(k)C & \bar{A}_d(h^d(k)) - L_d(k)C \end{pmatrix}\lambda(k) - e^T(k)Pe(k), \end{aligned} \tag{30}$$

$$\begin{aligned} \Delta V_2(k) &= V_2(k+1) - V_2(k) = \sum_{l=k+1-d}^{k} e^T(l)Qe(l) - \sum_{l=k-d}^{k-1} e^T(l)Qe(l)\\ &= e^T(k)Qe(k) - e^T(k-d)Qe(k-d), \end{aligned} \tag{31}$$

$$\begin{aligned} \Delta V_3(k) &= V_3(k+1) - V_3(k)\\ &= \sum_{i=-d}^{-1}\sum_{m=k+1+i}^{k} \eta^T(m)P\eta(m) - \sum_{i=-d}^{-1}\sum_{m=k+i}^{k-1} \eta^T(m)P\eta(m)\\ &= \sum_{i=-d}^{-1}\{\eta^T(k)P\eta(k) - \eta^T(k+i)P\eta(k+i)\}\\ &= d\eta^T(k)P\eta(k) - \sum_{i=-d}^{-1}\eta^T(k+i)P\eta(k+i)\\ &= d\lambda^T(k)\begin{pmatrix} \bar{A}(h(k)) - L(k)C - I & \bar{A}_d(h^d(k)) - L_d(k)C \end{pmatrix}^T P\\ &\quad \times \begin{pmatrix} \bar{A}(h(k)) - L(k)C - I & \bar{A}_d(h^d(k)) - L_d(k)C \end{pmatrix}\lambda(k)\\ &\quad - \sum_{l=k-d}^{k-1}\eta^T(l)P\eta(l). \end{aligned} \tag{32}$$

Note that

$$\sum_{l=k-d}^{k-1} \eta(l) = \sum_{l=k-d}^{k-1} [e(l+1) - e(l)] = e(k) - e(k-d), \tag{33}$$

then

$$e(k) - e(k-d) - \sum_{l=k-d}^{k-1} \eta(l) = 0.$$

Hence, the following equation holds for the arbitrary parameter matrix M of appropriate dimension

$$\lambda^T(k)M[e(k) - e(k-d) - \sum_{l=k-d}^{k-1} \eta(l)] = 0. \tag{34}$$

Let $\mu(k) = \begin{pmatrix} e^T(k) & e^T(k-d) & \lambda^T(k) \end{pmatrix}^T$. From $(30), (31), (32)$ and (34), we get

$$\begin{aligned}
\Delta V(k) &= V(k+1) - V(k) \\
&= \Delta V_1(k) + \Delta V_2(k) + \Delta V_3(k) + 2\lambda^T(k)M[e(k) - e(k-d) - \sum_{l=k-d}^{k-1} \eta(l)] \\
&= \Delta V_1(k) + \Delta V_2(k) + \Delta V_3(k) + 2\lambda^T(k)\begin{pmatrix} M & -M \end{pmatrix}\lambda(k) \\
&\quad - \sum_{l=k-d}^{k-1} 2\lambda^T(k)M\eta(l) \\
&= \Delta V_1(k) + \Delta V_2(k) + \Delta V_3(k) + 2\lambda^T(k)\begin{pmatrix} M & -M \end{pmatrix}\lambda(k) \\
&\quad + d\lambda^T(k)MP^{-1}M^T\lambda(k) - \sum_{l=k-d}^{k-1} [2\lambda^T(k)M\eta(l) + \lambda^T(k)MP^{-1}M^T\lambda(k)] \\
&= \mu^T(k)\Big\{ \begin{pmatrix} Q-P & 0 & 0 \\ 0 & -Q & 0 \\ 0 & 0 & \begin{pmatrix} M & M \end{pmatrix} + \begin{pmatrix} M & M \end{pmatrix}^T \end{pmatrix} \\
&\quad + \begin{pmatrix} (\bar{A}(h(k)) - L(k)C)^T & (\bar{A}(h(k)) - L(k)C)^T - I & 0 \\ (\bar{A}_d(h^d(k)) - L_d(k)C)^T & (\bar{A}_d(h^d(k)) - L_d(k)C)^T & 0 \\ 0 & 0 & M \end{pmatrix} \\
&\quad \times \begin{pmatrix} P & 0 & 0 \\ 0 & dP & 0 \\ 0 & 0 & I \end{pmatrix} \\
&\quad \times \begin{pmatrix} \bar{A}(h(k)) - L(k)C & \bar{A}_d(h^d(k)) - L_d(k)C & 0 \\ \bar{A}(h(k)) - L(k)C - I & \bar{A}_d(h^d(k)) - L_d(k)C & 0 \\ 0 & 0 & dP^{-1}M^T \end{pmatrix} \Big\}\mu(k) \\
&\quad - \sum_{l=k-d}^{k-1} [\eta^T(l)P\eta(l) + 2\lambda^T(k)M\eta(l) + \lambda^T(k)MP^{-1}M^T\lambda(k)] \\
&= \mu^T(k)\Big\{ \begin{pmatrix} \zeta_1 & 0 \\ 0 & \zeta_2 + \zeta_2^T \end{pmatrix}
\end{aligned}$$

$$
+\begin{pmatrix} (\bar{A}(h(k))-L(k)C)^TP & d\bar{A}(h(k))-L(k)C)^TP-dP & 0 \\ (\bar{A}_d(h^d(k))-L_d(k)C)^TP & d\bar{A}_d(h^d(k))-L_d(k)C)^TP & 0 \\ 0 & 0 & M \end{pmatrix}
$$

$$
\times\begin{pmatrix} P & 0 & 0 \\ 0 & dP & 0 \\ 0 & 0 & (1/d)P \end{pmatrix}^{-1}
$$

$$
\times\begin{pmatrix} P(\bar{A}(h(k))-L(k)C) & P(\bar{A}_d(h^d(k))-L_d(k)C) & 0 \\ dP(\bar{A}(h(k))-L(k)C)-dP & dP(\bar{A}_d(h^d(k))-L_d(k)C) & 0 \\ 0 & 0 & M^T \end{pmatrix}\Bigg\}\mu(k)
$$

$$
-\sum_{l=k-d}^{k-1}[\lambda^T(k)M+\eta^T(l)P]P^{-1}[M^T\lambda(k)+P\eta(l)].
$$

Let

$$
\Xi_3=\bar{A}(h(k))-(L_0+\sum_{(i,j)\in S}h_{ij}(k)L_{ij})C,
$$

$$
\Xi_4=\bar{A}_d(h^d(k))-(L_0^d+\sum_{(l,m)\in S^d}h^d_{lm}(k)L^d_{lm})C,
$$

then we have

$$
\Delta V(k)\le\mu^T(k)H_3(h(k),h^d(k))\mu(k),\ \forall\,(i,j)\in S,\ \forall\,(l,m)\in S^d, \tag{35}
$$

where

$$
\begin{aligned}
H_3(h(k),h^d(k)) \quad = \quad & \begin{pmatrix} \zeta_1 & 0 \\ 0 & \zeta_2+\zeta_2^T \end{pmatrix}+\begin{pmatrix} \Xi_3^TP & d\Xi_3^TP-dP & 0 \\ \Xi_4^TP & d\Xi_4^TP & 0 \\ 0 & 0 & M \end{pmatrix} \\
& \times\begin{pmatrix} P & 0 & 0 \\ 0 & dP & 0 \\ 0 & 0 & (1/d)P \end{pmatrix}^{-1}\begin{pmatrix} P\Xi_3 & P\Xi_4 & 0 \\ dP\Xi_3-dP & dP\Xi_4 & 0 \\ 0 & 0 & M^T \end{pmatrix}.
\end{aligned}
$$

Note that the condition $\Delta V(k)<0$ is satisfied if we have

$$
H_3(h(k),h^d(k))<0,\ \forall\,h(k)\in H_{q,n},\ \forall\,h^d(k)\in H^d_{q,n}.
$$

Since the matrix function $H_3(h(k),h^d(k))$ is affine in $h(k)$ and $h^d(k)$, using convexity principle, we deduce that $\Delta V(k)<0$ if the following condition is satisfied

$$
H_3(\alpha,\beta)<0,\ \forall\,\alpha\in H_{q,n},\ \forall\,\beta\in H^d_{q,n}. \tag{36}
$$

Using the notation

$$
L_0=P^{-1}R_0^T,\ L_{ij}=P^{-1}R_{ij}^T,\ \forall\,(i,j)\in S,
$$

$$
L_0^d=P^{-1}(R_0^d)^T,\ L^d_{lm}=P^{-1}(R^d_{lm})^T,\ \forall\,(l,m)\in S^d,
$$

the condition (36) is equivalent to (24) based on Lemma 2.3. It follows that $\Delta V(k)<0$. This completes the proof of theorem. □

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Yali Dong, School of Science, Tianjin Polytechnic University, Tianjin 300387. China.
e-mail: dongyl@vip.sina.com

Jinying Liu, Binhai School of Foreign Affairs, Tianjin Foreign Studies University, Tianjin 300270. China.
e-mail: liujinying_815@yahoo.com.cn

Shengwei Mei, Department of Electrical Engineering, Tsinghua University, Beijing 100084. China.
e-mail: meishengwei@mail.tsinghua.edu.cn