GLOBAL FINITE-TIME OBSERVERS FOR A CLASS OF NONLINEAR SYSTEMS

Yunyan Li, Yanjun Shen and Xiaohua Xia

Global finite-time observers are designed for a class of nonlinear systems with bounded varying rational powers imposed on the increments of the nonlinearities whose solutions exist and are unique for all positive time. The global finite-time observers designed in this paper are with two homogeneous terms. The global finite-time convergence of the observation error system is achieved by combining global asymptotic stability and local finite-time stability.

Keywords: global finite-time observer, nonlinear system, homogeneity

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1. INTRODUCTION

Nonlinear observers have received a great deal of attraction since the formal introduction of the concept and the Lyapunov based approach of design as proposed in [27]. Quite a number of early works have been devoted to establishing link between nonlinear observability and linear observers [10, 12] by linearizing nonlinear systems through making change of coordinates [9, 10, 12]. In the past decades, a series of nonlinear observer design methods for various nonlinear systems are developed, for example, the extended Luenberger observer for nonlinear systems [30], the nonlinear observer proposed by observer error linearization [29], the observer design based on the Lyapunov based approach [20, 27], the observer canonical form approach [1, 10] and the high-gain approach [6, 7] and so on. For nonlinear systems with nonlinear terms satisfying Lipschitz conditions, over the years, a lot of works have investigated observer design for this kind of nonlinear systems. For example, necessary and sufficient conditions on the stability matrix that ensure asymptotic stability of the observer are presented in [21]. The observer synthesis for Lipschitz nonlinear systems is carried out using H_∞ optimization [18]. And [4] designs a robust nonlinear observer for Lipschitz nonlinear systems subject to disturbances and so on. Then, in [19], a globally asymptotically stable observer is designed for nonlinear systems with output dependent incremental rate while [11] develops a global high-gain-based observer for nonlinear systems with generalized output-feedback canonical form including output dependent diagonal terms.

Recently, since systems with finite-settling-time dynamics possess better disturbance rejection and robustness properties [28], finite-time convergent observers of nonlinear systems have become an active subject with the advance in finite-time stability and

stabilization [2, 15, 16]. Based on finite-time stability, a lot of finite-time observers [5, 14, 17, 23, 24] are proposed. In particular, [17] introduces a finite-time observer relying on the homogeneity properties of nonlinear systems [3]. Then, [14, 24] and [23] make considerable progress in finite-time high-gain observer design. [24] proposes a semi-global finite-time observer for single output nonlinear systems that are uniformly observable and globally Lipschitz. Then for the same class of nonlinear systems, two different kinds of global finite-time observers are proposed by [14] and [23], respectively. Later, semi-global finite-time observers are studied in [25] for the following nonlinear systems whose solutions exist for all positive time

$$
\begin{cases}
\dot{x}_1 &= x_2 + f_1(y, u), \\
\dot{x}_2 &= x_3 + f_2(y, x_2, u), \\
&\vdots \\
\dot{x}_n &= f_n(y, x_2, \ldots, x_n, u), \\
y &= x_1,
\end{cases} \tag{1}
$$

where $x \in \mathcal{R}^n$, $u \in \mathcal{R}^m$, $y \in \mathcal{R}$, with $f_i(\cdot)$ $(i = 2, \ldots, n)$ satisfying

$$
|f_i(y, x_2, \ldots, x_i, u) - f_i(y, \hat{x}_2, \ldots, \hat{x}_i, u)| \tag{2}
$$
$$
\leq \Gamma(u, y)\Big(1 + \sum_{j=2}^n |\hat{x}_j|^{v_j}\Big) \sum_{j=2}^i |x_j - \hat{x}_j| + l \sum_{j=2}^i |x_j - \hat{x}_j|^{\beta_{ij}},
$$

where $\Gamma(\cdot)$ is a continuous function, $l \geq 0$, $v_j \in [0, \frac{1}{j-1})$ $(j = 2, \ldots, n)$. There exist semi-global finite-time observers for nonlinear systems (1) when $\frac{q-i}{q-j+1} < \beta_{ij} < \frac{i}{j-1}$ $(2 \leq j \leq i \leq n)$ [25] (where $q > n$ is a positive number satisfying some conditions related to the homogeneity degree, refer to [25] for details). In [25], semi-global finite-time observers are also designed for systems (1) where the nonlinear terms have mixed and varying incremental rational powers

$$
|f_i(y, x_2, \ldots, x_i, u) - f_i(y, \hat{x}_2, \ldots, \hat{x}_i, u)| \tag{3}
$$
$$
\leq \Gamma(u, y)\Big(1 + \sum_{j=2}^n |\hat{x}_j|^{v_j}\Big) \sum_{j=2}^i |x_j - \hat{x}_j| + l_1 \sum_{j=2}^i |x_j - \hat{x}_j|^{\beta_{1,ij}} + l_2 \sum_{j=2}^i |x_j - \hat{x}_j|^{\beta_{2,ij}},
$$

where l_1, $l_2 > 0$ are two positive real numbers (where $l_1 = l_2$ in [25]), $\frac{q-i}{q-j+1} < \beta_{1,ij} < 1$, $1 < \beta_{2,ij} < \frac{i}{j-1}$ $(2 \leq j \leq i \leq n)$.

Then in [26], global asymptotic and finite-time stability are studied for a class of homogeneous nonlinear systems and the best possible lower bound $-\frac{1}{n}$ of the degree of the homogeneity is obtained. Motivated by the result in [26], for the rational and mixed rational powers with smaller lower bound satisfying $\frac{n-i}{n-j+1} < \beta_{ij} < \frac{i}{j-1}$ and $\frac{n-i}{n-j+1} < \beta_{1,ij} < 1$, $1 < \beta_{2,ij} < \frac{i}{j-1}$ $(2 \leq j \leq i \leq n)$ in conditions (2) and (3) of nonlinear systems (1) for $n \geq 3$, there are still no related results on asymptotic and finite-time observer design till now. In this paper, we aim to solve the problem of designing global finite-time observers. And we restrict our attention to estimating the states only for

those nonlinear systems (1) whose solutions globally exist and are unique for all positive time.

In order to solve the problem of designing global finite-time observer, we will employ homogeneity properties [3] and the argument method of [14] together. Under exactly the same gain update law as that in semi-global finite-time results [25], the global finite-time observers we will design are with two homogeneous terms, one of degree smaller than 1, the other of degree greater than 1. Moreover, the global finite-time convergence of the observation error system is derived based on two different homogeneous Lyapunov functions. The derivatives of the Lyapunov functions are calculated by splitting the whole space into three different sets to obtain global asymptotic stability and local finite-time stability.

The paper is organized as follows. The main results are presented in Section 2: the global finite-time observers for nonlinear systems (1) for $n \geq 3$ with conditions (2) and (3) where the rational and mixed rational powers satisfy $\frac{n-i}{n-j+1} < \beta_{ij} < \frac{i}{j-1}$ and $\frac{n-i}{n-j+1} < \beta_{1,ij} < 1$, $1 < \beta_{2,ij} < \frac{i}{j-1}$ ($2 \leq j \leq i \leq n$), respectively. In Section 3, two examples are given to illustrate the validity of the proposed design method. Finally, the paper is concluded in Section 4. Then, in the Appendix, an explicit proof of a useful lemma is included for the completeness of the paper.

2. GLOBAL FINITE-TIME OBSERVERS FOR A CLASS OF NONLINEAR SYSTEMS

In this section, we will design global finite-time converging observers for nonlinear system (1) for $n \geq 3$ with the conditions (2) and (3) where the rational and mixed rational powers satisfy $\frac{n-i}{n-j+1} < \beta_{ij} < \frac{i}{j-1}$ and $\frac{n-i}{n-j+1} < \beta_{1,ij} < 1$, $1 < \beta_{2,ij} < \frac{i}{j-1}$ ($2 \leq j \leq i \leq n$), respectively.

Before we give the explicit form of the global finite-time observers we will propose in the paper, let us review a semi-global finite-time observer designed in [25] for nonlinear system (1) with conditions (2) and (3) where the rational and mixed rational powers in the nonlinearities satisfy $\frac{q-i}{q-j+1} < \beta_{ij} < \frac{i}{j-1}$ and $\frac{q-i}{q-j+1} < \beta_{1,ij} < 1$, $1 < \beta_{2,ij} < \frac{i}{j-1}$ ($2 \leq j \leq i \leq n$) (where $q > n$ is a positive real number), respectively. The semi-global finite-time observer is shown in the following:

$$
\begin{cases}
\dot{\hat{x}}_1 & = \quad \hat{x}_2 + La_1\lceil e_1 \rfloor^{\alpha_1} + f_1(y, u), \\
\dot{\hat{x}}_2 & = \quad \hat{x}_3 + L^2 a_2 \lceil e_1 \rfloor^{\alpha_2} + f_2(y, \hat{x}_2, u), \\
& \vdots \\
\dot{\hat{x}}_n & = \quad L^n a_n \lceil e_1 \rfloor^{\alpha_n} + f_n(y, \hat{x}_2, \ldots, \hat{x}_n, u),
\end{cases}
$$

with the observer gain L being dynamically updated by

$$
\dot{L} = -L[\varphi_1(L^{1-\sigma} - \varphi_2) - \varphi_3 \Psi(u, y, \hat{x})], \ L(0) > \varphi_2, \tag{4}
$$

where φ_1, $\varphi_2 > 1$, φ_3 are three positive real numbers, $0 < \sigma < 1$ is chosen such that $\beta_{ij} < \frac{i-\sigma}{j-1+\sigma}$, $v_j < \frac{1-2\sigma}{j-1+\sigma}$ holds, $\Psi(u, y, \hat{x}) = \Gamma(u, y)(1 + \sum_{j=2}^{n} |\hat{x}_j|^{v_j})$, $\alpha_i = i\alpha - (i-1)$, $(i = 0, 1, \ldots, n)$, $\alpha \in (1 - \frac{1}{n}, 1)$ and $a_i > 0$ ($i = 1, \ldots, n$) are the coefficients

of Hurwitz polynomial

$$s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n. \tag{5}$$

The observer gain $L(t)$ in (4) satisfies the following properties.

Lemma 2.1. (Shen and Xia [25]) *For the observer gain $L(t)$ defined in (4), there exists an $M > 0$ such that $L(t) < M$, $t \in [0, T]$, $\forall T \in (0, \infty)$.*

In this paper, we are interested in designing global finite-time observers for system (1) for $n \geq 3$ with the rational power satisfying $\frac{n-i}{n-j+1} < \beta_{ij} < \frac{i}{j-1}$ $(2 \leq j \leq i \leq n)$ in condition (2) and with the mixed rational powers satisfying $\frac{n-i}{n-j+1} < \beta_{1,ij} < 1$, $1 < \beta_{2,ij} < \frac{i}{j-1}$ $(2 \leq j \leq i \leq n)$ in condition (3). Under the same gain update law (4), the global finite-time observers can be constructed as:

$$\begin{cases} \dot{\hat{x}}_1 &= \hat{x}_2 + La_1 \lceil e_1 \rceil^{\alpha_1} + L^{1-(\beta_1-1)(1-\eta)\sigma} a_1 \lceil e_1 \rceil^{\beta_1} + f_1(y, u), \\ \dot{\hat{x}}_2 &= \hat{x}_3 + L^2 a_2 \lceil e_1 \rceil^{\alpha_2} + L^{2-(\beta_2-1)(1-\eta)\sigma} a_2 \lceil e_1 \rceil^{\beta_2} + f_2(y, \hat{x}_2, u), \\ \vdots \\ \dot{\hat{x}}_n &= L^n a_n \lceil e_1 \rceil^{\alpha_n} + L^{n-(\beta_n-1)(1-\eta)\sigma} a_n \lceil e_1 \rceil^{\beta_n} + f_n(y, \hat{x}_2, \ldots, \hat{x}_n, u), \end{cases} \tag{6}$$

where $\beta_i = i\beta - (i-1)$ $(i = 0, 1, \ldots, n)$, $\beta > \frac{1+\sigma}{\sigma}$, $0 < \sigma < 1$, $0 < \eta < 1 - \alpha < 1$.

Definition 2.2. Denote the solutions of systems $(1), (6)$ with respect to the corresponding input functions and passing through x_0 and $\hat{x}_0$ as $x(t)$ and $\hat{x}(t)$, respectively. If there exists an open neighborhood $\mathcal{U} \subset \mathcal{R}^n$ of the origin such that $e_0 = x_0 - \hat{x}_0 \in \mathcal{U}$ implies $x(t) - \hat{x}(t) \in \mathcal{U}$ and a function $T : \mathcal{U} \setminus \{0\} \to (0, \infty)$, such that

$$\|x(t) - \hat{x}(t)\| \to 0, \text{ as } t \to T(e_0), \tag{7}$$

then, the system (6) with dynamic high gain (4) is called a finite-time observer of the system (1). In this case, all points $e_0 = x_0 - \hat{x}_0$ such that (7) holds constitute a domain of observer attraction. If the open set $\mathcal{U}$ can be chosen as the whole space $\mathcal{R}^n$, then system (6) with dynamic high gain (4) is called a global finite-time observer.

In paper [13], two homogeneous observers with different degrees are constructed for global output feedback stabilization problem of a class of nonlinear systems. The following remark summarizes the differences between the homogeneous observer (6) we designed and the homogeneous observers proposed in [13].

Remark 2.3. Note that in [13], a dual observer is employed to solve the problem of global output feedback stabilization for a class of nonlinear systems whose nonlinearities are bounded by both low-order and high-order terms. Compared the results in [13] with the global finite-time observer (6) we proposed in this paper, we have the following statements.

- The dual observer [13] is comprised of two seperate homogeneous observers, one estimating the low-order part of unmeasurable states and the other estimating the high-order components. However, here, two homogeneous terms one of degree less than 1 and the other greater than 1 are introduced in the design of the global finite-time observer simultaneously.

- In [13], either the low-order or the high-order observer, can only estimate those states in a limited region either close to or far away from the origin, but not all the states in the space. However, the observer we designed can estimate the states in the whole space.

- Both the low-order observer and high-order observer as well as the coefficients in the observers are derived by a recursive method in [13]. In this paper, we will see that the global finite-time stability of the proposed observer will be proved based on Lyapunov theory and all the coefficients in the observer are given explicitly.

For α_i $(1 \leq i \leq n)$ and β_{ij} $(2 \leq j \leq i \leq n)$ in (2), they satisfy the following properties.

Lemma 2.4. For β_{ij} $(2 \leq j \leq i \leq n)$ being given by (2), if $\frac{i}{j-1} > \beta_{ij} > \frac{n-i}{n-j+1}$, we have $-\alpha_{i-1} + \beta_{ij}\alpha_{j-1} - \alpha + 1 > 0$ $(2 \leq j \leq i \leq n)$. Moreover, select $0 < \sigma < 1$ such that $\beta > \frac{1+\sigma}{\sigma}$, then we have $\beta_{ij}\beta_{j-1} - \beta_{i-1} < \beta - 1$ $(2 \leq j \leq i \leq n)$.

P r o o f .   The proof of this lemma is simple, and thus it is omitted here.   □

In what follows, for $n \geq 3$, we will prove that system (6) is a global finite-time observer for nonlinear system (1) with conditions (2) and (3) where the rational and mixed rational powers satisfy $\frac{n-i}{n-j+1} < \beta_{ij} < \frac{i}{j-1}$ and $\frac{n-i}{n-j+1} < \beta_{1,ij} < 1$, $1 < \beta_{2,ij} < \frac{i}{j-1}$ $(2 \leq j \leq i \leq n)$. It is in three parts. First we will make change of coordinates of the error system and introduce a useful lemma. Then we will show that the observer (6) we proposed can render the error system globally finite-time stable for system (1) with condition (2) where the rational powers satisfy $\frac{n-i}{n-j+1} < \beta_{ij} < \frac{i}{j-1}$ $(2 \leq j \leq i \leq n)$. Finally, it will be verified that system (1) is also a global finite-time observer for nonlinear system with condition (3) where the mixed rational powers in its nonlinearities satisfy $\frac{n-i}{n-j+1} < \beta_{1,ij} < 1$, $1 < \beta_{2,ij} < \frac{i}{j-1}$ $(2 \leq j \leq i \leq n)$.

2.1. Pre-treatment of the system

The dynamics of the observation error $e = x - \hat{x}$ is given by

$$
\begin{cases}
\dot{e}_1 &= e_2 - La_1\lceil e_1 \rfloor^{\alpha_1} - L^{1-(\beta_1-1)(1-\eta)\sigma}a_1\lceil e_1 \rfloor^{\beta_1}, \\
\dot{e}_2 &= e_3 - L^2 a_2\lceil e_1 \rfloor^{\alpha_2} - L^{2-(\beta_2-1)(1-\eta)\sigma}a_2\lceil e_1 \rfloor^{\beta_2} + \tilde{f}_2, \\
&\vdots \\
\dot{e}_n &= -L^n a_n\lceil e_1 \rfloor^{\alpha_n} - L^{n-(\beta_n-1)(1-\eta)\sigma}a_n\lceil e_1 \rfloor^{\beta_n} + \tilde{f}_n,
\end{cases}
\tag{8}
$$

where $\tilde{f}_i = f_i(y, x_2, \ldots, x_i, u) - f_i(y, \hat{x}_2, \ldots, \hat{x}_i, u)$ $(2 \leq i \leq n)$. Consider the change of coordinates

$$
\varepsilon_i = \frac{e_i}{L^{i-1+\sigma}}.
$$

Then, (8) can be expressed as

$$
\begin{cases}
\dot{\varepsilon}_1 &= L\varepsilon_2 - L^{(\alpha_1-1)\sigma+1}a_1\lceil\varepsilon_1\rfloor^{\alpha_1} - \frac{\dot{L}}{L}\sigma\varepsilon_1 - L^{(\beta_1-1)\eta\sigma+1}a_1\lceil\varepsilon_1\rfloor^{\beta_1}, \\
\dot{\varepsilon}_2 &= L\varepsilon_3 - L^{(\alpha_2-1)\sigma+1}a_2\lceil\varepsilon_1\rfloor^{\alpha_2} - \frac{\dot{L}}{L}(\sigma+1)\varepsilon_2 - L^{(\beta_2-1)\eta\sigma+1}a_2\lceil\varepsilon_1\rfloor^{\beta_2} \\
&\quad + \frac{\tilde{f}_2}{L^{1+\sigma}}, \\
&\vdots \\
\dot{\varepsilon}_n &= -L^{(\alpha_n-1)\sigma+1}a_n\lceil\varepsilon_1\rfloor^{\alpha_n} - \frac{\dot{L}}{L}(n-1+\sigma)\varepsilon_n - L^{(\beta_n-1)\eta\sigma+1}a_n\lceil\varepsilon_1\rfloor^{\beta_n} \\
&\quad + \frac{\tilde{f}_n}{L^{n-1+\sigma}}.
\end{cases}
\tag{9}
$$

Before we investigate the global finite-time convergence of the observation error system (9), first let us consider the following homogeneous nonlinear system

$$
\begin{cases}
\dot{\varepsilon}_1 &= \rho\varepsilon_2 - \rho^{(\lambda_1-1)\sigma+1}a_1\lceil\varepsilon_1\rfloor^{\lambda_1}, \\
\dot{\varepsilon}_2 &= \rho\varepsilon_3 - \rho^{(\lambda_2-1)\sigma+1}a_2\lceil\varepsilon_1\rfloor^{\lambda_2}, \\
&\vdots \\
\dot{\varepsilon}_n &= -\rho^{(\lambda_n-1)\sigma+1}a_n\lceil\varepsilon_1\rfloor^{\lambda_n},
\end{cases}
\tag{10}
$$

where $\rho > 0$, $\lceil\varepsilon_1\rfloor^{\lambda_i} = |\varepsilon_1|^{\lambda_i}\text{sign}(\varepsilon_1)$, $\lambda_i = i\lambda - (i-1)$ $(i = 0, 1, \ldots, n)$, $\lambda > 1 - \frac{1}{n}$, $0 < \sigma < 1$, $a_i > 0$ $(1 \leq i \leq n)$ are given in (5).

In the following, we will see that under a new homogeneous Lyapunov function, nonlinear system (10) is finite-time stable for $\lambda \in (1 - \frac{1}{n}, 1)$ and asymptotically stable for $\lambda \geq 1$. Before we give this result for system (10), let us first list some conditions under which the result holds.

We suitably choose a_i $(1 \leq i \leq n)$ in (10) such that there exists a matrix $P \in \mathcal{R}^{n\times n}$, $P^T = P > 0$ satisfying

$$
A^TP + PA \leq -I, \quad h_1I \leq D_1P + PD_1 \leq h_2I,
\tag{11}
$$

where $A = \begin{bmatrix} -a_1 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & \ldots & 1 \\ -a_n & 0 & \ldots & 0 \end{bmatrix}$, $D_1 = \text{diag}\{\sigma, 1+\sigma, \ldots, n-1+\sigma\}$, h_1, $h_2 > 0$ are two real constants.

The following lemma summarizes some results for nonlinear system (10) where a new homogeneous Lyapunov function is proposed and some inequalities for system (10) are obtained based on this new Lyapunov function.

Lemma 2.5. Construct the following function as in [22]

$$
V(\varepsilon) = \begin{cases} \int_0^\infty \frac{1}{v^{q+1}}(\chi \circ \bar{V})(v\varepsilon_1, v^{\lambda_1}\varepsilon_2, \ldots, v^{\lambda_{n-1}}\varepsilon_n)\,\mathrm{d}v, & \varepsilon \in \mathcal{R}^n \setminus \{0\}, \\ 0, & \varepsilon = 0, \end{cases}
$$

where

$$
\chi(s) = \begin{cases} 0, & s \in (-\infty, 1] \\ 2(x-1)^2, & s \in (1, \frac{3}{2}) \\ 1 - 2(x-2)^2, & s \in [\frac{3}{2}, 2) \\ 1, & s \in [2, \infty) \end{cases}, \quad \chi(s) \in C'(\mathcal{R}, \mathcal{R}),
$$

$\bar{V}(\varepsilon) = \varepsilon^T P \varepsilon$, P satisfies condition (11), $q > 0$ is a positive integer. Then

(i) $V(\varepsilon)$ is a positive definite function homogeneous of degree q with respect to the weights $\{\lambda_{i-1}\}_{1 \le i \le n}$. $V(\varepsilon)$ is called a q h-Lyapunov function of $\bar{V}(\varepsilon)$ w.r.t. χ, ρ, $(\lambda_0, \lambda_1, \ldots, \lambda_{n-1})$.

(ii) If $a_i (1 \le i \le n)$ are chosen to satisfy condition (11), then there exist w_1, $w_2 > 0$ such that

$$w_1 V(\varepsilon) \le \frac{\partial V(\varepsilon)}{\partial \varepsilon}^T D_1 \varepsilon \le w_2 V(\varepsilon). \tag{12}$$

(iii) For $1 - \frac{1}{n} < \lambda < 1$, if $q > 1 + \max\{\lambda_i\}_{0 \le i \le n}$, $a_i (1 \le i \le n)$ and P satisfy condition (11), $\left. \frac{dV(\varepsilon)}{dt} \right|_{(10)}$ is C^1 on $\mathcal{R}^n$, then there exists a $w_3 > 0$ such that

$$\left. \frac{dV(\varepsilon)}{dt} \right|_{(10)} \le -w_3 \rho^{1-\sigma} V(\varepsilon)^\gamma, \tag{13}$$

where $\gamma = \frac{q+\lambda-1}{q}$.

(iv) For $\lambda \ge 1$, $n \ge 3$, if $q > 1 + \max\{\lambda_i\}_{0 \le i \le n}$, $a_i (1 \le i \le n)$ and P satisfy condition (11), $a_n P_{1n} > 0$ (where P_{1n} is the element of P at the first line and nth column), then $\left. \frac{dV(\varepsilon)}{dt} \right|_{(10)}$ is C^1 on $\mathcal{R}^n$, and there exists a $w_4 > 0$ such that

$$\left. \frac{dV(\varepsilon)}{dt} \right|_{(10)} \le -w_4 \rho^{1-\sigma} V(\varepsilon)^\gamma. \tag{14}$$

The p r o o f s of (i) and (ii) are quite easy. And the proofs of (iii) and (iv) are very similar. The main ideas of proofs (iii) and (iv) are to construct a compact set containing the origin on which the derivative of the constructed homogeneous Lyapunov function satisfies some key inequalities. Then inequality (13) and (14) are derived by use of the homogeneity properties of both the Lyapunov function and the system (10). The detailed proof is given in the Appendix.

Then, for the following two systems with $n \ge 3$

$$\begin{cases} \dot{\varepsilon}_1 &= L\varepsilon_2 - L^{(\beta_1-1)\eta\sigma+1} a_1 \lceil \varepsilon_1 \rfloor^{\beta_1}, \\ \dot{\varepsilon}_2 &= L\varepsilon_3 - L^{(\beta_2-1)\eta\sigma+1} a_2 \lceil \varepsilon_1 \rfloor^{\beta_2}, \\ &\vdots \\ \dot{\varepsilon}_n &= -L^{(\beta_n-1)\eta\sigma+1} a_n \lceil \varepsilon_1 \rfloor^{\beta_n}, \end{cases} \tag{15}$$

and

$$\begin{cases} \dot{\varepsilon}_1 &= L\varepsilon_2 - L^{(\alpha_1-1)\sigma+1} a_1 \lceil \varepsilon_1 \rfloor^{\alpha_1}, \\ \dot{\varepsilon}_2 &= L\varepsilon_3 - L^{(\alpha_2-1)\sigma+1} a_2 \lceil \varepsilon_1 \rfloor^{\alpha_2}, \\ &\vdots \\ \dot{\varepsilon}_1 &= -L^{(\alpha_n-1)\sigma+1} a_n \lceil \varepsilon_1 \rfloor^{\alpha_n}, \end{cases} \tag{16}$$

by Lemma 2.5, there exist $\underline{c}_1$, $\bar{c}_1$, $c_1 > 0$ and $\underline{c}_2$, $\bar{c}_2$, $c_2 > 0$ such that

$$\underline{c}_1 V_\beta(\varepsilon) \leq \frac{\partial V_\beta(\varepsilon)}{\partial \varepsilon} D_1 \varepsilon \leq \bar{c}_1 V_\beta(\varepsilon), \quad \left.\frac{dV_\beta(\varepsilon)}{dt}\right|_{(15)} < -c_1 L^{1-\eta\sigma} V_\beta(\varepsilon)^{\gamma_1}, \qquad (17)$$

and

$$\underline{c}_2 V_\alpha(\varepsilon) \leq \frac{\partial V_\alpha(\varepsilon)}{\partial \varepsilon} D_1 \varepsilon \leq \bar{c}_2 V_\alpha(\varepsilon), \quad \left.\frac{dV_\alpha(\varepsilon)}{dt}\right|_{(16)} \leq -c_2 L^{1-\sigma} V_\alpha(\varepsilon)^{\gamma_2}, \qquad (18)$$

hold, where $V_\beta(\varepsilon)$ is an q_1 h-Lyapunov function of $\bar{V}_\beta(\varepsilon)$ w.r.t. $\chi, L, (\beta_0, \beta_1, \ldots, \beta_{n-1})$, $V_\alpha(\varepsilon)$ is an q_2 h-Lyapunov function of $\bar{V}_\alpha(\varepsilon)$ w.r.t. $\chi, L, (\alpha_0, \alpha_1, \ldots, \alpha_{n-1})$, q_1, $q_2 > 0$ are two positive real numbers, $\bar{V}_\beta(\varepsilon) = \bar{V}_\alpha(\varepsilon) = \varepsilon^T P \varepsilon$, P satisfies condition (11), $\gamma_1 = \frac{q_1 + \beta - 1}{q_1}$, $\gamma_2 = \frac{q_2 + \alpha - 1}{q_2}$.

2.2. Global finite-time observers for nonlinear system (1) for $n \geq 3$ with condition (2) where the rational powers in the nonlinearities satisfy $\frac{n-i}{n-j+1} < \beta_{ij} < \frac{i}{j-1}$ $(2 \leq j \leq i \leq n)$

In this subsection, the global finite-time convergence of the error system (8) between the observer (6) we designed and the nonlinear system (1) for $n \geq 3$ with condition (2) (where the rational powers satisfy $\frac{n-i}{n-j+1} < \beta_{ij} < \frac{i}{j-1}$ $(2 \leq j \leq i \leq n)$) is proved.

Theorem 2.6. If $\frac{n-i}{n-j+1} < \beta_{ij} < \frac{i}{j-1}$ $(2 \leq j \leq i \leq n)$, then for $n \geq 3$, any $\alpha \in (1 - \frac{1}{n}, 1)$, there exist $\varphi_i > 0$ $(i = 1, 2, 3)$, $0 < \sigma < 1$, $\beta > \frac{1+\sigma}{\sigma}$ and $0 < \eta < 1 - \alpha$ such that the system (6) with the observer gain (4) is a global finite-time observer for the nonlinear system (1) under the condition (2).

P r o o f. From [8] and [14], we know that global asymptotic stability and local finite-time stability mean global finite-time stability. Here, in this paper, we will employ this principle and divide the proof of the global finite-time convergence of the observation error system into global asymptotic stability and local finite-time stability.

First of all, for $n \geq 3$, by suitably choosing a_i $(1 \leq i \leq n)$ such that there exists $P^T = P > 0$ satisfying condition (11) and $a_n P_{1n} > 0$, which is always possible. For $\delta > 0$, define $\mathcal{B}_{V_\alpha,\delta} \triangleq \{\varepsilon : V_\alpha(\varepsilon) < \delta\}$, $\mathcal{B}_{V_\beta,\delta} \triangleq \{\varepsilon : V_\beta(\varepsilon) < \delta\}$. As shown in the following figure, we have $\mathcal{B}_{V_\beta,\delta_3} \subset \mathcal{B}_{V_\beta,\delta_1} \subset \mathcal{B}_{V_\beta,1}$ by choosing $1 > \delta_1 > \delta_3 > 0$ (where δ_1, δ_3 will be given in the proof). The proof is in three parts. First, we use $V_\beta(\varepsilon)$ to derive $\frac{dV_\beta(\varepsilon)}{dt} < 0$ for $\varepsilon \in \mathcal{R}^n \setminus \mathcal{B}_{V_\beta,1}$ and $\varepsilon \in \mathcal{B}_{V_\beta,1} \setminus \mathcal{B}_{V_\beta,\delta_1}$, respectively. When $\varepsilon \in \mathcal{B}_{V_\beta,\delta_3}$, $V_\alpha(\varepsilon)$ is employed to prove the finite-time stability of the system (9). Finally, when $\varepsilon \in \mathcal{B}_{V_\beta,\delta_1} \setminus \mathcal{B}_{V_\beta,\delta_3}$, for $\forall \epsilon > 0$, there exist $\varphi_i > 0$ $(i = 1, 2, 3)$ such that $\delta_1 - \delta_3 < \epsilon$, then by continuity of $\frac{dV_\alpha(\varepsilon)}{dt}$, we obtain $\frac{dV_\alpha(\varepsilon)}{dt} < 0$.
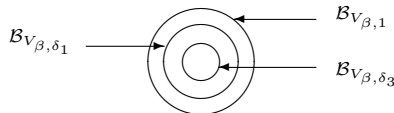
Part I: When $\varepsilon \in \mathcal{P} = \mathcal{R}^n \setminus \mathcal{B}_{V_\beta, 1}$, let us consider the q_1 h-Lyapunov function $V_\beta(\varepsilon)$. Based on (17), calculating the derivative of $V_\beta(\varepsilon)$ along the solution of the system (9), we have

$$\left. \frac{\mathrm{d}V_\beta(\varepsilon)}{\mathrm{d}t} \right|_{(9)} = \left. \frac{\mathrm{d}V_\beta(\varepsilon)}{\mathrm{d}t} \right|_{(15)} + \varphi_1(L^{1-\sigma} - \varphi_2) \frac{\partial V_\beta(\varepsilon)}{\partial \varepsilon}^T D_1 \varepsilon - \varphi_3 \Psi(u, y, \hat{x}) \frac{\partial V_\beta(\varepsilon)}{\partial \varepsilon}^T$$

$$\times D_1 \varepsilon + \frac{\partial V_\beta(\varepsilon)}{\partial \varepsilon}^T \tilde{G}_1 + \frac{\partial V_\beta(\varepsilon)}{\partial \varepsilon}^T \tilde{F} \leq -c_1 L^{1-\eta\sigma} V_\beta(\varepsilon)^{\gamma_1} + \bar{c}_1 \varphi_1 (L^{1-\sigma} - \varphi_2) V_\beta(\varepsilon)$$

$$-\underline{c}_1 \varphi_3 \Psi(u, y, \hat{x}) V_\beta(\varepsilon) + \frac{\partial V_\beta(\varepsilon)}{\partial \varepsilon}^T \tilde{G}_1 + \frac{\partial V_\beta(\varepsilon)}{\partial \varepsilon}^T \tilde{F}, \tag{19}$$

where $\tilde{G}_1 = (-L^{(\alpha_1 - 1)\sigma + 1} a_1 \lceil \varepsilon_1 \rceil^{\alpha_1}, \ \ldots, \ -L^{(\alpha_n - 1)\sigma + 1} a_n \lceil \varepsilon_1 \rceil^{\alpha_n})^T$, $\tilde{F} = (0, \frac{\tilde{f}_2}{L^{1+\sigma}}, \ \ldots, \frac{\tilde{f}_n}{L^{n-1+\sigma}})^T$.

For $\frac{\partial V_\beta(\varepsilon)}{\partial \varepsilon}^T \tilde{G}_1$, by Lemma 4.2 in [3], we have

$$\frac{\partial V_\beta(\varepsilon)}{\partial \varepsilon}^T \tilde{G}_1 \leq L^{1-(1-\alpha)\sigma} a^* \sum_{i=1}^n \left| \frac{\partial V_\beta(\varepsilon)}{\partial \varepsilon_i} \right| |\varepsilon_1|^{\alpha_i} \leq L^{1-(1-\alpha)\sigma} a^* k_1$$

$$\times \sum_{i=1}^n V_\beta(\varepsilon)^{\frac{q_1 - \beta_{i-1} + \alpha_i}{q_1}} \leq L^{1-(1-\alpha)\sigma} a^* k_1 n V_\beta(\varepsilon), \tag{20}$$

where $k_1 = \max_{\{z: V_\beta(z) = 1\}} \left| \frac{\partial V_\beta(z)}{\partial z_i} \right| |z_1|^{\alpha_i}$, $a^* = \max_{1 \leq i \leq n} \{a_i\}$.

For $\frac{\partial V_\beta(\varepsilon)}{\partial \varepsilon}^T \tilde{F}$, we can obtain that $\frac{\partial V_\beta(\varepsilon)}{\partial \varepsilon}^T \tilde{F} \leq \Psi(u, y, \hat{x}) \sum_{i=2}^n \sum_{j=2}^i \left| \frac{\partial V_\beta(\varepsilon)}{\partial \varepsilon_i} \right| \frac{|e_j|}{L^{i-1+\sigma}} + l \sum_{i=2}^n \sum_{j=2}^i \left| \frac{\partial V_\beta(\varepsilon)}{\partial \varepsilon_i} \right| \frac{|e_j|^{\beta_{ij}}}{L^{i-1+\sigma}}$. Note that under the condition $\beta_{ij} < \frac{i}{j-1}$, there exists a $\sigma_1 > 0$ such that $\beta_{ij} < \frac{i}{j-1+\sigma_1}$, $v_j < \frac{1-\sigma_1}{j-1+\sigma_1}$ $(2 \leq j \leq i \leq n)$, and let $0 < \sigma < \sigma_1$. Because $L(t) > \varphi_2 > 1$, we have $L^{(j-1+\sigma)\beta_{ij} - (i-1+\sigma)} < L^{1-\sigma}$. Then, similarly by Lemma 4.2 in [3], we have

$$\frac{\partial V_\beta(\varepsilon)}{\partial \varepsilon}^T \tilde{F} \leq \Psi(u, y, \hat{x}) \sum_{i=2}^n \sum_{j=2}^i \left| \frac{\partial V_\beta(\varepsilon)}{\partial \varepsilon_i} \right| |\varepsilon_j| + l L^{1-\sigma} \sum_{i=2}^n \sum_{j=2}^i \left| \frac{\partial V_\beta(\varepsilon)}{\partial \varepsilon_i} \right| |\varepsilon_j|^{\beta_{ij}}$$

$$\leq k_2 \Psi(u, y, \hat{x}) \sum_{i=2}^n \sum_{j=2}^i V_\beta(\varepsilon)^{\frac{q_1 - \beta_{i-1} + \beta_{j-1}}{q_1}} + l k_3 L^{1-\sigma} \sum_{i=2}^n \sum_{j=2}^i V_\beta(\varepsilon)^{\frac{q_1 - \beta_{i-1} + \beta_{ij}\beta_{j-1}}{q_1}}$$

$$\leq k_2 n^2 \Psi(u, y, \hat{x}) V_\beta(\varepsilon) + l k_3 n^2 L^{1-\sigma} V_\beta(\varepsilon)^{\frac{q_1 + \bar{\beta}}{q_1}}, \tag{21}$$

where $\bar{\beta} = \max_{2 \leq j \leq i \leq n} \{\beta_{ij}\beta_{j-1} - \beta_{i-1}\}$, $k_2 = \max_{\{z: V_\beta(z) = 1\}} \left| \frac{\partial V_\beta(z)}{\partial z_i} \right| |z_j|$, $k_3 = \max_{\{z: V_\beta(z) = 1\}} \left| \frac{\partial V_\beta(z)}{\partial z_i} \right| |z_j|^{\beta_{ij}}$.

Then, by substituting (20) and (21) into (19), we have

$$\left. \frac{\mathrm{d}V_\beta(\varepsilon)}{\mathrm{d}t} \right|_{(9)} \leq -c_1 L^{1-\eta\sigma} V_\beta(\varepsilon)^{\gamma_1} + \bar{c}_1 \varphi_1 L^{1-\sigma} V_\beta(\varepsilon) - \bar{c}_1 \varphi_1 \varphi_2 V_\beta(\varepsilon) - \underline{c}_1 \varphi_3 \Psi(u, y, \hat{x}) V_\beta(\varepsilon)$$

$$+ L^{1-(1-\alpha)\sigma} a^* k_1 n V_\beta(\varepsilon) + k_2 n^2 \Psi(u, y, \hat{x}) V_\beta(\varepsilon) + l k_3 n^2 L^{1-\sigma} V_\beta(\varepsilon)^{\frac{q_1 + \bar{\beta}}{q_1}}. \tag{22}$$

From Lemma 2.4, we know that $\gamma_1 > \frac{q_1 + \bar{\beta}}{q_1}$. Then, for all $\varepsilon \in \mathcal{P}$, there exist $d_{11} > 0$, $d_{21} > 1$, $d_{31} > 0$ such that when $0 < \varphi_1 < d_{11}$, $\varphi_2 > d_{21}$, $\varphi_3 > d_{31}$, we have

$$\left. \frac{\mathrm{d}V_\beta(\varepsilon)}{\mathrm{d}t} \right|_{(9)} \leq -\bar{c}_1 \varphi_1 \varphi_2 V_\beta(\varepsilon), \ \varepsilon \in \mathcal{P}, \tag{23}$$

where $d_{11} = \frac{c_1}{3\bar{c}_1}$, $d_{21} = \max\{(\frac{3a^* k_1 n}{c_1})^{\frac{1}{(1-\alpha-\eta)\sigma}}, (\frac{3lk_3 n^2}{c_1})^{\frac{1}{(1-\eta)\sigma}}\}$, $d_{31} = \frac{k_2 n^2}{\underline{c}_1}$.

When $\varepsilon \in \mathcal{B}_{V_\beta,1}$, we again use the q_1 h-Lyapunov function $V_\beta(\varepsilon)$. First, we have

$$\frac{\partial V_\beta(\varepsilon)^T}{\partial \varepsilon} \tilde{G}_1 \leq L^{1-(1-\alpha)\sigma} a^* k_1 n V_\beta(\varepsilon)^{\frac{q_1-\beta_{n-1}+\alpha_n}{q_1}},$$

$$\frac{\partial V_\beta(\varepsilon)^T}{\partial \varepsilon} \tilde{F} \leq k_2 n^2 \Psi(u,y,\hat{x}) V_\beta(\varepsilon)^{\frac{q_1-\beta_{n-1}+\beta}{q_1}} + lk_3 n^2 L^{1-\sigma} V_\beta(\varepsilon)^{\frac{q_1+\beta}{q_1}}. \tag{24}$$

Then from (19) and (24), we obtain

$\left. \frac{\mathrm{d}V_\beta(\varepsilon)}{\mathrm{d}t} \right|_{(9)} \leq -c_1 L^{1-\eta\sigma} V_\beta(\varepsilon)^{\gamma_1} + \bar{c}_1 \varphi_1 (L^{1-\sigma} - \varphi_2) V_\beta(\varepsilon) - \underline{c}_1 \varphi_3 \Psi(u,y,\hat{x}) V_\beta(\varepsilon) + a^* k_1 n$

$\times L^{1-(1-\alpha)\sigma} V_\beta(\varepsilon)^{\frac{q_1-\beta_{n-1}+\alpha_n}{q_1}} + k_2 n^2 \Psi(u,y,\hat{x}) V_\beta(\varepsilon)^{\frac{q_1-\beta_{n-1}+\beta}{q_1}} + lk_3 n^2 L^{1-\sigma} V_\beta(\varepsilon)^{\frac{q_1+\beta}{q_1}}$,

where $\underline{\beta} = \min_{2 \leq j \leq i \leq n}\{\beta_{ij}\beta_{j-1} - \beta_{i-1}\}$. There exists a $d_{22} > 1$ such that $0 < g_{11} < g_{13} < 1$, $0 < g_{12}$, $g_{14} < 1$ when $0 < \varphi_1 < d_{11}$, $\varphi_2 > d_{22}$, $\varphi_3 > d_{31}$. Then we have

$$\left. \frac{\mathrm{d}V_\beta(\varepsilon)}{\mathrm{d}t} \right|_{(9)} \leq -\bar{c}_1 \varphi_1 \varphi_2 V_\beta(\varepsilon), \ \varepsilon \in \mathcal{B}_{V_\beta,1} \setminus \mathcal{B}_{V_\beta,\delta_1}, \tag{25}$$

where $\delta_1 = \max\{g_{12}, g_{13}, g_{14}\}$, $g_{11} = (\frac{3\bar{c}_1 \varphi_1}{c_1})^{\frac{q_1}{\beta-1}} \varphi_2^{-\frac{(1-\eta)\sigma q_1}{\beta-1}}$, $g_{12} = (\frac{3lk_3 n^2}{c_1})^{\frac{q_1}{\beta-\underline{\beta}-1}}$ $\varphi_2^{-\frac{(1-\eta)\sigma q_1}{\beta-\underline{\beta}-1}}$, $g_{13} = (\frac{3a^* k_1 n}{c_1})^{\frac{q_1}{\beta_n-\alpha_n}} \varphi_2^{-\frac{(1-\alpha-\eta)\sigma q_1}{\beta_n-\alpha_n}}$, $g_{14} = (\frac{k_2 n^2}{\underline{c}_1})^{\frac{q_1}{\beta_{n-1}-\beta}} \varphi_3^{-\frac{q_1}{\beta_{n-1}-\beta}}$.

Thus, from (23) and (25), we can derive

$$\left. \frac{\mathrm{d}dV_\beta(\varepsilon)}{\mathrm{d}t} \right|_{(9)} \leq -\bar{c}_1 \varphi_1 \varphi_2 V_\beta(\varepsilon), \ \varepsilon \in \mathcal{R}^n \setminus \mathcal{B}_{V_\beta,\delta_1}. \tag{26}$$

Part II: In this part, we will consider $\varepsilon \in \mathcal{B}_{V_\beta,\delta_1}$. Here, we use the q_2 h-Lyapunov function $V_\alpha(\varepsilon)$. Because $V_\beta(\varepsilon)$, $V_\alpha(\varepsilon)$ are homogeneous of degrees q_1 and q_2, respectively, we have $V_\alpha(\varepsilon) \leq k^* V_\beta(\varepsilon)^{\frac{q_2}{q_1}}$, where $k^* = \max_{\{z:V_\beta(z)=1\}} V_\alpha(z)$. Then there exist $d_{23} > 1$, $d_{32} > 0$ such that $k^* \delta_1^{\frac{q_2}{q_1}} \leq 1$, i.e., $V_\alpha(\varepsilon) \leq 1$ when $\varphi_2 > d_{23}$, $\varphi_3 > d_{32}$. Under this condition, based on (18), calculating the derivative of $V_\alpha(\varepsilon)$ along the solution of system (9), using the same method as that in part I, we have

$$\left. \frac{\mathrm{d}V_\alpha(\varepsilon)}{\mathrm{d}t} \right|_{(9)} \leq -c_2 L^{1-\sigma} V_\alpha(\varepsilon)^{\gamma_2} + \bar{c}_2 \varphi_1 (L^{1-\sigma} - \varphi_2) V_\alpha(\varepsilon) - \underline{c}_2 \varphi_3 \Psi(u,y,\hat{x}) V_\alpha(\varepsilon)$$

$$+ \frac{\partial V_\alpha(\varepsilon)^T}{\partial \varepsilon} \tilde{G}_2 + \frac{\partial V_\alpha(\varepsilon)^T}{\partial \varepsilon} \tilde{F}, \tag{27}$$

where $\tilde{G}_2 = (-L^{(\beta_1-1)\eta\sigma+1} a_1 \lceil \varepsilon_1 \rfloor^{\beta_1}, \ldots, -L^{(\beta_n-1)\eta\sigma+1} a_n \lceil \varepsilon_1 \rfloor^{\beta_n})^T$.

For $\frac{\partial V_\alpha(\varepsilon)}{\partial \varepsilon}^T \tilde{G}_2$ and $\frac{\partial V_\alpha(\varepsilon)}{\partial \varepsilon}^T \tilde{F}$, similarly, we have

$$
\begin{aligned}
\frac{\partial V_\alpha(\varepsilon)}{\partial \varepsilon}^T \tilde{G}_2 &\leq L^{(\beta_n-1)\eta\sigma+1}a^*k_4 n V_\alpha(\varepsilon), \\
\frac{\partial V_\alpha(\varepsilon)}{\partial \varepsilon}^T \tilde{F} &\leq k_5 n^2 \Psi(u,y,\hat{x})V_\alpha(\varepsilon) + lk_6 n^2 L^{1-\sigma} V_\alpha(\varepsilon)^{\frac{q_2+\alpha}{q_2}},
\end{aligned}
\tag{28}
$$

where $\underline{\alpha} = \min_{2 \leq j \leq i \leq n}\{\beta_{ij}\alpha_{j-1} - \alpha_{i-1}\}$, $k_4 = \max_{\{z:V_\alpha(z)=1\}}\left|\frac{\partial V_\alpha(z)}{\partial z_i}\right||z_1|^{\beta_i}$, $k_5 = \max_{\{z:V_\alpha(z)=1\}}|\frac{\partial V_\alpha(z)}{\partial z_i}||z_j|$, $k_6 = \max_{\{z:V_\alpha(z)=1\}}|\frac{\partial V_\alpha(z)}{\partial z_i}||z_j|^{\beta_{ij}}$.

Then by substituting (28) into (27), we have

$$
\begin{aligned}
\frac{dV_\alpha(\varepsilon)}{dt}\bigg|_{(9)} &\leq -c_2 L^{1-\sigma}V_\alpha(\varepsilon)^{\gamma_2} + \bar{c}_2\varphi_1(L^{1-\sigma} - \varphi_2)V_\alpha(\varepsilon) - \underline{c}_2\varphi_3\Psi(u,y,\hat{x})V_\alpha(\varepsilon) \\
&\quad + L^{(\beta_n-1)\eta\sigma+1}a^*k_4 n V_\alpha(\varepsilon) + k_5 n^2 \Psi(u,y,\hat{x})V_\alpha(\varepsilon) + lk_6 n^2 L^{1-\sigma}V_\alpha(\varepsilon)^{\frac{q_2+\alpha}{q_2}}.
\end{aligned}
$$

From Lemma 2.1 and Lemma 2.4, we know $\gamma_2 < \frac{q_2+\alpha}{q_2}$, $\varphi_2 < L(t) < M$. And because $0 < \varphi_1 < d_{11}$, there exists a $d_{24} > 0$ such that $g_{22} < g_{21}$, $g_{22} < g_{23}$ when $L(t) > \varphi_2 > d_{24}$. Moreover, there exists a $d_{33} > 0$ such that when $\varphi_3 > d_{33}$, $\varphi_2 > d_{24}$, we have

$$
\frac{dV_\alpha(\varepsilon)}{dt}\bigg|_{(9)} \leq -\frac{1}{4}c_2 L^{1-\sigma}V_\alpha(\varepsilon)^{\gamma_2}, \ \varepsilon \in \mathcal{B}_{V_\alpha,\delta_2} \setminus \{0\},
$$

where $\delta_2 = g_{22}$, $d_{33} = \frac{k_5 n^2}{c_2}$, $g_{21} = (\frac{c_2}{4\bar{c}_2\varphi_1})^{\frac{q_2}{1-\alpha}}$, $g_{22} = (\frac{c_2}{4a^*k_4 n})^{\frac{q_2}{1-\alpha}}\varphi_2^{-\frac{\sigma(1+(\beta_n-1)\eta)q_2}{1-\alpha}}$, $g_{23} = (\frac{c_2}{4lk_6 n^2})^{\frac{q_2}{\alpha-\alpha+1}}$.

Then, by Theorem 4.2 in [2], the system (9) is locally finite-time stable on $\mathcal{B}_{V_\alpha,\delta_2}$.

From $V_\alpha(\varepsilon) \leq k^* V_\beta(\varepsilon)^{\frac{q_2}{q_1}}$, we can obtain $\mathcal{B}_{V_\beta,\delta_3} \subset \mathcal{B}_{V_\alpha,\delta_2}$, where $\delta_3 = (\frac{g_{22}}{k^*})^{\frac{q_1}{q_2}} = \varphi_2^{-\frac{\sigma(1+(\beta_n-1)\eta)q_1}{1-\alpha}}(\frac{1}{k^*})^{\frac{q_1}{q_2}}(\frac{c_2}{4a^*k_4 n})^{\frac{q_1}{1-\alpha}}$. Then, $\mathcal{B}_{V_\beta,\delta_3}$ is a domain of observer attraction, i.e.,

$$
\frac{dV_\alpha(\varepsilon)}{dt}\bigg|_{(9)} \leq -\frac{1}{4}c_2 L^{1-\sigma}V_\alpha(\varepsilon)^{\gamma_2}, \ \varepsilon \in \mathcal{B}_{V_\beta,\delta_3} \setminus \{0\}.
\tag{29}
$$

Part III: For any $\epsilon > 0$, there exist sufficiently large φ_2, φ_3 and $0 < \varphi_1 < d_{11}$, $\varphi_2 > d_{2i}$ $(1 \leq i \leq 4)$, $\varphi_3 > d_{3j}$ $(1 \leq j \leq 3)$ such that $0 < \delta_1 - \delta_3 < \epsilon$. Because $\frac{dV_\alpha(\varepsilon)}{dt}\big|_{(9)}$ is continuous on $\mathcal{R}^n$, we have

$$
\frac{dV_\alpha(\varepsilon)}{dt}\bigg|_{(9)} < 0, \ \varepsilon \in \mathcal{B}_{V_\beta,\delta_1} \setminus \mathcal{B}_{V_\beta,\delta_3}.
\tag{30}
$$

Thus, from (26), (29) and (30), by combining global asymptotic stability and local finite-time stability, we get that the system (9) is globally finite-time stable, i.e., there exists a $T_1 > 0$ such that $\varepsilon_i(t) = 0$ when $t > T_1$.

From Lemma 2.1, there exists an $M^* > 0$ such that $L^{i-1+\sigma} \leq M^*$ $(i = 1, \ldots, n)$. Then, we have $\frac{e_i(t)}{M^*} \leq \frac{e_i(t)}{L^{i-1+\sigma}} = \varepsilon_i(t) = 0$ $(t > T_1)$, i.e., $e_i(t) = 0$ $(t > T_1)$ $(i = 1, \ldots, n)$, which means system (6) is a global finite-time observer for system (1) under the condition (2).

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

2.3. Global finite-time observers for nonlinear system (1) for $n \geq 3$ with condition (3) where the mixed rational powers in the nonlinearities satisfy $\frac{n-i}{n-j+1} < \beta_{1,ij} < 1$, $1 < \beta_{2,ij} < \frac{i}{j-1}$ $(2 \leq j \leq i \leq n)$

Similarly to what is done in [25], we can extend the results to the system (1) with condition (3) for $n \geq 3$ which is with mixed rational powers in the nonlinearities: system (6) is a global finite-time observer for this kind of nonlinear system.

Theorem 2.7. If $\frac{n-i}{n-j+1} < \beta_{1,ij} < 1$, $1 < \beta_{2,ij} < \frac{i}{j-1}$ $(2 \leq j \leq i \leq n)$, then for $n \geq 3$, any $\alpha \in (1 - \frac{1}{n}, 1)$, there exist $0 < \sigma < 1$, $\beta > \frac{1+\sigma}{\sigma}$ and $0 < \eta < 1 - \alpha$ such that global finite-time observers in the form (6) with the observer gain (4) can be designed for the nonlinear systems (1) with the condition (3).

P r o o f.   The proof is similar to Theorem 2.6 and thus is omitted here. $\qquad\qquad$ $\square$

3. EXAMPLE

In this section, two examples are given to illustrate the effectiveness of the results as proposed in Theorem 2.6 and Theorem 2.7, respectively.

Example 3.1. Consider nonlinear system

$$\begin{cases} \dot{x}_1 & = & x_2, \\ \dot{x}_2 & = & x_3, \\ \dot{x}_3 & = & x_3^{\frac{3}{2}} - x_3, \\ y & = & x_1. \end{cases} \tag{31}$$

It can be verified that the following type of nonlinear condition holds: $|(x_3^{\frac{3}{2}} - x_3) - (\hat{x}_3^{\frac{3}{2}} - \hat{x}_3)| \leq (1 + \frac{3}{2}|\hat{x}_3|^{\frac{1}{2}})|x_3 - \hat{x}_3| + |x_3 - \hat{x}_3|^{\frac{3}{2}}$. Following the result in this paper, an observer can be designed as follows:

$$\begin{cases} \dot{\hat{x}}_1 & = & \hat{x}_2 + 3L\lceil y - \hat{x}_1 \rceil^{\alpha} + 3L^{1-(\beta-1)(1-\eta)\sigma}\lceil y - \hat{x}_1 \rceil^{\beta}, \\ \dot{\hat{x}}_2 & = & \hat{x}_3 + 3L^2\lceil y - \hat{x}_1 \rceil^{2\alpha-1} + 3L^{2-2(\beta-1)(1-\eta)\sigma}\lceil y - \hat{x}_1 \rceil^{2\beta-1}, \\ \dot{\hat{x}}_3 & = & \hat{x}_3^{\frac{3}{2}} - \hat{x}_3 + L^3\lceil y - \hat{x}_1 \rceil^{3\alpha-2} + L^{3-3(\beta-1)(1-\eta)\sigma}\lceil y - \hat{x}_1 \rceil^{3\beta-2}, \\ \dot{L} & = & -L[\varphi_1(L^{1-\sigma} - \varphi_2) - \varphi_3(1 + \frac{3}{2}|\hat{x}_2|^{\frac{1}{2}})]. \end{cases}$$

Condition I

Parameters: $\alpha = 0.95$, $\beta = 10^5$, $\sigma = 0.01$, $\eta = 0.01$, $\varphi_1 = 0.1$, $\varphi_2 = 1.2$, $\varphi_3 = 0.2$. The initial values: $x_1(0) = 0.6$, $x_2(0) = 0.1$, $x_3(0) = 0.2$, $\hat{x}_1(0) = 0.2$, $\hat{x}_2(0) = 0.4$, $\hat{x}_3(0) = 0.1$, $L(0) = 1.5$.
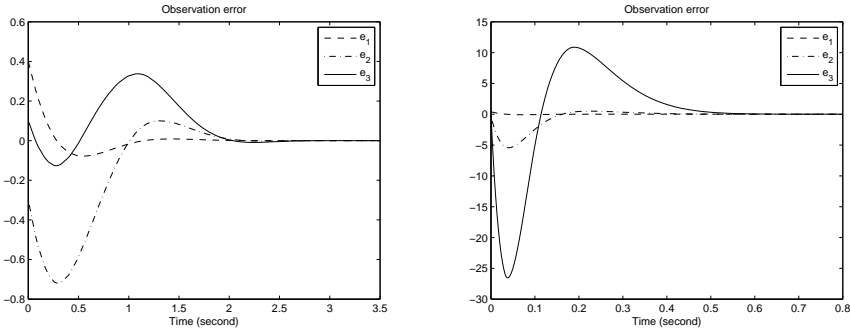
Condition II

Parameters: $\alpha = 0.95$, $\beta = 10^5$, $\sigma = 0.01$, $\eta = 0.01$, $\varphi_1 = 0.1$, $\varphi_2 = 1.2$, $\varphi_3 = 0.2$. The initial values: $x_1(0) = 0.6$, $x_2(0) = 0.1$, $x_3(0) = 0.2$, $\hat{x}_1(0) = 0.2$, $\hat{x}_2(0) = 0.4$, $\hat{x}_3(0) = 0.1$, $L(0) = 15$.

Fig. 1. Trajectories of the observation error of system (31) under condition I and II without noise.

Figure 1 shows the simulation results.

Example 3.2. For the following nonlinear system

$$\begin{cases} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_3, \\ \dot{x}_3 &= -x_3^{\frac{3}{5}} + x_3^{\frac{5}{3}}, \\ y &= x_1, \end{cases} \tag{32}$$

from Lemma A.4 in [13], we have that nonlinear condition with mixed rational powers $|(-x_3^{\frac{3}{5}} + x_3^{\frac{5}{3}}) - (-\hat{x}_3^{\frac{3}{5}} + \hat{x}_3^{\frac{5}{3}})| \leq (|x_3^{\frac{3}{5}} - \hat{x}_3^{\frac{3}{5}}| + |x_3^{\frac{5}{3}} - \hat{x}_3^{\frac{5}{3}}|) \leq \frac{5}{3}|\hat{x}_3|^{\frac{2}{3}}|x_3 - \hat{x}_3| + 2^{\frac{2}{5}}|x_3 - \hat{x}_3|^{\frac{3}{5}} + |x_3 - \hat{x}_3|^{\frac{5}{3}}$ holds.

From Theorem 2.7, the observer dynamics is designed as follows

$$\begin{cases} \dot{\hat{x}}_1 &= \hat{x}_2 + 3L\lceil y - \hat{x}_1 \rfloor^{\alpha} + 3L^{1-(\beta-1)(1-\eta)\sigma}\lceil y - \hat{x}_1 \rfloor^{\beta}, \\ \dot{\hat{x}}_2 &= \hat{x}_3 + 3L^2\lceil y - \hat{x}_1 \rfloor^{2\alpha-1} + 3L^{2-2(\beta-1)(1-\eta)\sigma}\lceil y - \hat{x}_1 \rfloor^{2\beta-1}, \\ \dot{\hat{x}}_3 &= -\hat{x}_3^{\frac{3}{5}} + \hat{x}_3^{\frac{5}{3}} + L^3\lceil y - \hat{x}_1 \rfloor^{3\alpha-2} + L^{3-3(\beta-1)(1-\eta)\sigma}\lceil y - \hat{x}_1 \rfloor^{3\beta-2}, \\ \dot{L} &= -L[\varphi_1(L^{1-\sigma} - \varphi_2) - \frac{5}{3}\varphi_3|\hat{x}_3|^{\frac{2}{3}}]. \end{cases}$$

Condition I

Parameters: $\alpha = 0.9$, $\sigma = 0.1$, $\eta = 0.01$, $\beta = 10^4$, $\varphi_1 = 0.2$, $\varphi_2 = 1.5$, $\varphi_3 = 2$. The initial values: $x_1(0) = 1$, $x_2(0) = 0.1$, $x_3(0) = 0.2$, $\hat{x}_1(0) = 0.5$, $\hat{x}_2(0) = 0.2$, $\hat{x}_3(0) = 0.1$, $L(0) = 2$.

Condition II

Parameters: $\alpha = 0.8$, $\sigma = 0.2$, $\eta = 0.1$, $\beta = 10^3$, $\varphi_1 = 0.1$, $\varphi_2 = 5$, $\varphi_3 = 4$. The initial values: $x_1(0) = 0.5$, $x_2(0) = 0.4$, $x_3(0) = 0.3$, $\hat{x}_1(0) = 0.6$, $\hat{x}_2(0) = 0.1$, $\hat{x}_3(0) = 0.5$, $L(0) = 20$.
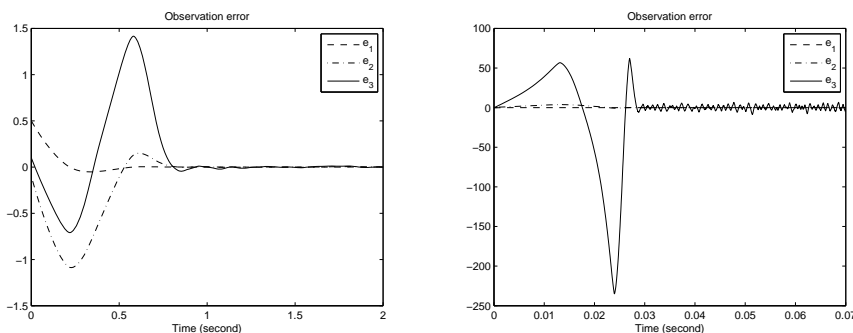

Fig. 2. Trajectories of the observation error of system (32) under condition I and II with noise added on $\hat{x}_1$, $\hat{x}_2$ and L.

In both example 3.1 and example 3.2, we choose $a_1 = a_2 = 3$, $a_3 = 1$, i.e.,

$$A = \begin{pmatrix} -3 & 1 & 0 \\ -3 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \text{ and } P = \begin{pmatrix} 5 & -3 & 0.2 \\ -3 & 4 & -3 \\ 0.2 & -3 & 7 \end{pmatrix} > 0. \text{ It can be verified that } A \text{ and } P$$

satisfy $A^T P + PA \leq -I$ and $a_3 P_{13} = 0.2 > 0$.

The simulations (without noise in Example 3.1 and with uniform random number noise imposed on $\hat{x}_1$, $\hat{x}_2$ in Example 3.2) in Figure 1 and Figure 2 show the dynamics of the observation errors of Example 3.1 and Example 3.2, respectively. The simulation results show the effectiveness of the proposed observers which can render the error systems converge in finite time. And we can see that although the observation errors converge faster with a bigger high gain, but they are a bit more noise-sensitive. Thus, in future work, the design of finite-time adaptive observer can be an interesting topic.

4. CONCLUSION

This paper has addressed the problem of global finite-time observer design for a class of nonlinear systems for $n \geq 3$ with the rational powers in the increments of nonlinearities satisfying $\frac{n-i}{n-j+1} < \beta_{ij} < \frac{i}{j-1}$ $(2 \leq j \leq i \leq n)$ and the mixed rational powers satisfying $\frac{n-i}{n-j+1} < \beta_{1,ij} < 1, 1 < \beta_{2,ij} < \frac{i}{j-1}$ $(2 \leq j \leq i \leq n)$ where semi-global finite-time observers exist for this kind of nonlinear systems with the rational and mixed rational powers satisfying $\frac{q-i}{q-j+1} < \beta_{ij} < \frac{i}{j-1}$ $(2 \leq j \leq i \leq n)$ and $\frac{q-i}{q-j+1} < \beta_{1,ij} < 1, 1 < \beta_{2,ij} < \frac{i}{j-1}$ $(2 \leq j \leq i \leq n)$ (where $q > n$ is a positive real number). We have shown that, under the same gain update law, by introducing two different homogeneous terms of degrees $\alpha - 1 < 0$ and $\beta - 1 > 0$ with respect to the weights $\{\alpha_i\}_{1 \leq i \leq n}$ and $\{\beta_i\}_{1 \leq i \leq n}$, we can design global finite-time observers by combining global asymptotical stability and local

finite-time stability. Moreover, through two examples, the validity of the observers we designed was shown.

A. APPENDIX

In this section, the detailed proof of Lemma 2.5 is included. Before we give the explicit proof of Lemma 2.5, let us introduce a useful result first.

Lemma A.1. If a_i $(1 \leq i \leq n)$ in (5) are chosen such that condition (11) holds, then, for any $x = (0, x_2, \ldots, x_n)^T$, $y = (x_2, \ldots, x_n, 0)^T \in \mathcal{R}^n$, we have $x^T P y + y^T P x \leq -\sum_{i=2}^{n} x_i^2$.

The following is the detailed proof of Lemma 2.5.

P r o o f. First, let us introduce some definitions. For $\pi > 0$, $0 < \sigma < 1$, define

$$\mathcal{F}_\pi \triangleq \{\varepsilon : |\varepsilon_1| = \pi\},$$

$$\overline{\mathcal{B}}_{1,\pi} \triangleq \{\varepsilon : \varepsilon^T \varepsilon \leq \pi\},$$

$$\mathcal{B}_{1,\pi} \triangleq \{\varepsilon : \varepsilon^T \varepsilon < \pi\},$$

$$\overline{\mathcal{B}}_{2,\pi} \triangleq \left\{ (\varepsilon_1, \rho^{-(\lambda_n - 1)\lambda_1 \sigma} \varepsilon_2, \ldots, \rho^{-(\lambda_n - 1)\lambda_{n-1}\sigma} \varepsilon_n)^T : \sum_{i=2}^{n} \varepsilon_i^2 \leq \pi^2 \right\},$$

$$\overline{\mathcal{B}}_{3,\pi} \triangleq \left\{ (\varepsilon_1, \rho^{-\lambda_n \lambda_1 \sigma} \varepsilon_2, \ldots, \rho^{-\lambda_n \lambda_{n-1}\sigma} \varepsilon_n)^T : \sum_{i=2}^{n} \varepsilon_i^2 \leq \pi^2 \right\},$$

$$\mathcal{B}_{3,\pi} \triangleq \left\{ (\varepsilon_1, \ \rho^{-\lambda_n \lambda_1 \sigma} \varepsilon_2, \ldots, \ \rho^{-\lambda_n \lambda_{n-1}\sigma} \varepsilon_n)^T : \sum_{i=2}^{n} \varepsilon_i^2 < \pi^2 \right\}, \ \overline{\mathcal{P}}_\pi \triangleq \left\{ \varepsilon : |\varepsilon_1| \leq \pi \right\},$$

$$\overline{\mathcal{B}}_{4,\pi} \triangleq \left\{ (\varepsilon_1, \rho^{-2\lambda_1 \sigma} \varepsilon_2, \ldots, \rho^{-2\lambda_{n-1}\sigma} \varepsilon_n)^T : \sum_{i=2}^{n} \varepsilon_i^2 \leq \pi^2 \right\},$$

$$\mathcal{B}_{4,\pi} \triangleq \left\{ (\varepsilon_1, \rho^{-2\lambda_1 \sigma} \varepsilon_2, \ldots, \rho^{-2\lambda_{n-1}\sigma} \varepsilon_n)^T : \sum_{i=2}^{n} \varepsilon_i^2 < \pi^2 \right\}, \ \mathcal{P}_\pi \triangleq \left\{ \varepsilon : |\varepsilon_1| < \pi \right\}$$

and

$$\mathcal{S}_\pi \triangleq \{\varepsilon : \varepsilon^T \varepsilon = \pi\}.$$

It is not difficult to get that $V(\varepsilon)$ is C^1 for $\varepsilon \in \mathcal{R}^n$.

The proofs of (i) and (ii) are quite easy. For (i), by change of integration, it is very easy to verify that $V(\varepsilon)$ is homogeneous of degree q with respect to the weights $\{\lambda_i\}_{0 \leq i \leq n-1}$. From condition (11), it is also not difficult to derive the inequality (12) in (ii).

The proofs of (iii) and (iv) are a bit complicated, but the main ideas are the same. Thus, in the following, we only give the proof of (iv), but the main difference between the proofs of (iii) and (iv) will also be stated.

First, it is not difficult to verify that for $n = 2$, there does not exist such a_1, $a_2 > 0$ and $P > 0$ which satisfy the condition (11) and $a_2 P_{12} > 0$. And for $n \geq 3$, it is always

possible to find $a_i > 0$ ($1 \leq i \leq n$) such that there exists $P^T = P > 0$ satisfying the condition (11) and $a_n P_{1n} > 0$.

The proof is divided into two parts. The first part is to construct a compact set $\overline{\mathcal{A}}$ (where $\overline{\mathcal{A}}$ will be given later) encircling the origin where some inequalities are obtained. Actually, the compact set is constructed in four parts. In each part, $\left. \frac{dV(\varepsilon)}{dt} \right|_{(10)}$ and $V(\varepsilon)$ satisfy some inequalities on a certain set. Then, the compact set $\overline{\mathcal{A}}$ is derived by combination of the four sets. In the second part, for any $\varepsilon \in \mathcal{R}^n \setminus \{0\}$, the relationship between $\left. \frac{dV(\varepsilon)}{dt} \right|_{(10)}$ and $\left. \frac{dV(\varepsilon_0)}{dt} \right|_{(10)}$, $\varepsilon_0 \in \overline{\mathcal{A}}$ is established by use of the homogeneity theory. Then, we get the inequality (14) in (iv).

Part I: This part is divided into six parts. In the first four parts, we will show that $\left. \frac{dV(\varepsilon)}{dt} \right|_{(10)}$ satisfies some inequalities on the following sets $\mathcal{S}_1 \cap \overline{\mathcal{P}}_{\rho^{-\sigma}}$, $(\overline{\mathcal{P}}_{(1+\pi_1)\rho^{-\sigma}} \setminus \mathcal{P}_{(1-\pi_1)\rho^{-\sigma}}) \cap \overline{\mathcal{B}}_{3,\pi_1}$, $\mathcal{F}_{\rho^{-h\sigma}} \cap (\overline{\mathcal{B}}_{1,1} \setminus \mathcal{B}_{3,\pi_1})$ and $(\overline{\mathcal{P}}_{\rho^{-\sigma}} \setminus \mathcal{P}_{\rho^{-h\sigma}}) \cap (\overline{\mathcal{B}}_{3,\pi_1} \setminus \mathcal{B}_{3,\pi_1})$, separately, where $\pi_1 > 0$, $h > \{\bar{h}_1, \bar{h}_2\}$, $\rho > \{\rho_1, \rho_2\}$ will be given later. Then in the fifth part, $V(\varepsilon)$ admits some inequalities for ε belonging to each of these four sets. Finally, in the sixth part, by combination of these four sets, we derive the compact set $\overline{\mathcal{A}}$.

(1) Let l_1 be the largest $l > 0$ such that $\max_{\{v \leq l\}} \max_{\{\varepsilon \in \overline{\mathcal{B}}_{1,2} \setminus \mathcal{B}_{1,\frac{1}{2}}\}} \bar{V}(v\varepsilon_1, \ldots, v^{\lambda_{n-1}}\varepsilon_n)$ ≤ 1. Let l_2 be the smallest $l > 0$ such that $\min_{\{v \geq l\}} \min_{\{\varepsilon \in \overline{\mathcal{B}}_{1,2} \setminus \mathcal{B}_{1,\frac{1}{2}}\}} \bar{V}(v\varepsilon_1, \ldots, v^{\lambda_{n-1}}\varepsilon_n)$ ≥ 2. Then we have $V(\varepsilon) = \int_{l_1}^{l_2} \frac{1}{v^{q+1}} (\chi \circ \bar{V}(v\varepsilon_1, \ldots, v^{\lambda_{n-1}}\varepsilon_n)) \, dv + \frac{1}{q l_2^q}$, $\varepsilon \in \overline{\mathcal{B}}_{1,2} \setminus \mathcal{B}_{1,\frac{1}{2}}$. And

$$\left. \frac{dV(\varepsilon)}{dt} \right|_{(10)} = 2\rho \int_{l_1}^{l_2} \frac{\chi'(\bar{V}(v\varepsilon_1, \ldots, v^{\lambda_{n-1}}\varepsilon_n))}{v^{q+\lambda}} K(v, \varepsilon_1, \ldots, \varepsilon_n) \, dv, \ \varepsilon \in \overline{\mathcal{B}}_{1,2} \setminus \mathcal{B}_{1,\frac{1}{2}}, \ (A.1)$$

where

$$K(v, \varepsilon_1, \ldots, \varepsilon_n) = \begin{bmatrix} 0 \\ v^{\lambda_1}\varepsilon_2 \\ \vdots \\ v^{\lambda_{n-1}}\varepsilon_n \end{bmatrix}^T P \begin{bmatrix} v^{\lambda_1}\varepsilon_2 \\ \vdots \\ v^{\lambda_{n-1}}\varepsilon_n \\ 0 \end{bmatrix} + \begin{bmatrix} v\varepsilon_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}^T P \begin{bmatrix} -a_1\rho^{(\lambda_1-1)\sigma}\lceil v\varepsilon_1 \rfloor^{\lambda_1} \\ \vdots \\ -a_n\rho^{(\lambda_n-1)\sigma}\lceil v\varepsilon_1 \rfloor^{\lambda_n} \end{bmatrix}$$

$$+ \begin{bmatrix} v\varepsilon_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}^T P \begin{bmatrix} v^{\lambda_1}\varepsilon_2 \\ \vdots \\ v^{\lambda_{n-1}}\varepsilon_n \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ v^{\lambda_1}\varepsilon_2 \\ \vdots \\ v^{\lambda_{n-1}}\varepsilon_n \end{bmatrix}^T P \begin{bmatrix} -a_1\rho^{(\lambda_1-1)\sigma}\lceil v\varepsilon_1 \rfloor^{\lambda_1} \\ \vdots \\ -a_n\rho^{(\lambda_n-1)\sigma}\lceil v\varepsilon_1 \rfloor^{\lambda_n} \end{bmatrix}. \quad (A.2)$$

When $\varepsilon \in \mathcal{S}_1 \cap \overline{\mathcal{P}}_{\rho^{-\sigma}}$, from Lemma A.1, equations (A.1) and (A.2), there exists $\rho_1 > 2$ such that when $\rho > \rho_1$, we have $\left. \frac{dV(\varepsilon)}{dt} \right|_{(10)} < -\frac{\rho}{2} \int_{l_1}^{l_2} \frac{1}{v^{q+\lambda}} \sum_{i=2}^{n} v^{2\lambda_i-1}\varepsilon_i^2 \chi'(\bar{V}(v\varepsilon_1, \ldots$ $\ldots, v^{\lambda_{n-1}}\varepsilon_n)) \, dv, \varepsilon \in \mathcal{S}_1 \cap \overline{\mathcal{P}}_{\rho^{-\sigma}}$, where $a^* = \max_{\{1 \leq i \leq n\}} a_i$, $\bar{p} = \max_{\{1 \leq i, \ j \leq n\}} |P_{ij}|$.

And clearly, we have $(\mathcal{S}_1 \cap \overline{\mathcal{P}}_0) \subset (\mathcal{S}_1 \cap \overline{\mathcal{P}}_{\rho^{-\sigma}}) \subset (\mathcal{S}_1 \cap \overline{\mathcal{P}}_{2-\sigma})$. Let l_3 be the largest $l > 0$ such that $\max_{\{v \leq l\}} \max_{\{\varepsilon \in \mathcal{S}_1 \cap \overline{\mathcal{P}}_0\}} \bar{V}(v\varepsilon, \ldots, v^{\lambda_{n-1}}\varepsilon_n) \leq 1$. Let l_4 be the smallest

$l > 0$ such that $\min_{\{v \geq l\}} \min_{\{\varepsilon \in \mathcal{S}_1 \cap \overline{\mathcal{P}}_0\}} \bar{V}(v\varepsilon, \ldots, v^{\lambda_{n-1}}\varepsilon_n) \geq 2$. It is not difficult to get $l_3 \geq l_1$, $l_4 \leq l_2$. Then we have

$$\left.\frac{\mathrm{d}V(\varepsilon)}{\mathrm{d}t}\right|_{(10)} < -\rho d_1, \ \varepsilon \in \mathcal{S}_1 \cap \overline{\mathcal{P}}_{\rho - \sigma}, \tag{A.3}$$

where $d_1 = \frac{1}{2}\min_{\{\varepsilon \in \mathcal{S}_1 \cap \overline{\mathcal{P}}_{2-\sigma}\}} \int_{l_3}^{l_4} \frac{1}{v^{q+\lambda}} \sum_{i=2}^{n} v^{2\lambda_{i-1}}\varepsilon_i^2 \chi'(\bar{V}(v\varepsilon_1, \ldots, v^{\lambda_{n-1}}\varepsilon_n))\,\mathrm{d}v$.

(2) For $\varepsilon = (\pm 1, 0, \ldots, 0)^T$, from Lemma A.1, (A.1) and (A.2), we have $\left.\frac{\mathrm{d}V(\varepsilon)}{\mathrm{d}t}\right|_{(10)} = -2\rho \int_{l_1}^{l_2} \frac{\chi'(\bar{V}(\pm v, \ldots, 0))}{v^{q+\lambda}} \sum_{i=1}^{n} a_i P_{1i} \rho^{(\lambda_i - 1)\sigma}|v|^{1+\lambda_i}\,\mathrm{d}v$. Because $a_1 P_{11} > 0$, $a_n P_{1n} > 0$, $\lambda_n > \lambda_i$ $(1 \leq i \leq n)$ when $\lambda > 1$, there exist $\pi_1 \in (0, 1)$ and $\rho_2 > 1$ such that when $\rho > \rho_2$, we have $\left.\frac{\mathrm{d}V(\varepsilon)}{\mathrm{d}t}\right|_{(10)} < -\rho^{1-\sigma} \int_{l_1}^{l_2} \frac{a_n P_{1n}|v|^{1+\lambda_n}}{v^{q+\lambda}} \chi'(\bar{V}(\pm v, 0, \ldots, 0))\,\mathrm{d}v, \ \varepsilon \in (\overline{\mathcal{P}}_{1+\pi_1} \setminus \mathcal{P}_{1-\pi_1}) \cap \overline{\mathcal{B}}_{2,\pi_1}$.

Because $\left.\frac{\mathrm{d}V(\varepsilon)}{\mathrm{d}t}\right|_{(10)}$ is homogeneous of degree $q + \lambda - 1$ with respect to the weights $\{\lambda_i\}_{0 \leq i \leq n-1}$, we get

$$\left.\frac{\mathrm{d}V(\varepsilon)}{\mathrm{d}t}\right|_{(10)} < -d_2 \rho^{1-(q+\lambda)\sigma}, \ \varepsilon \in (\overline{\mathcal{P}}_{(1+\pi_1)\rho^{-\sigma}} \setminus \mathcal{P}_{(1-\pi_1)\rho^{-\sigma}}) \cap \overline{\mathcal{B}}_{3,\pi_1}, \tag{A.4}$$

where $d_2 = \int_{l_1}^{l_2} \frac{a_n P_{1n} v^{1+\lambda_n}}{v^{q+\lambda}} \chi'(\bar{V}(\pm v, 0, \ldots, 0))\,\mathrm{d}v$.

(3) Let l_5 be the largest $l > 0$ such that $\max_{\{v \leq l\}} \max_{\{\varepsilon \in \overline{\mathcal{P}}_{(1+\pi_1)\rho^{-\sigma}} \cap (\overline{\mathcal{B}}_{1,1} \setminus \mathcal{B}_{3,\pi_1})\}} \bar{V}(v\varepsilon_1, \ldots, v^{\lambda_{n-1}}\varepsilon_n) \leq 1$. And let l_6 be the smallest $l > 0$ such that

$$\min_{\{v \geq l\}} \min_{\{\varepsilon \in \overline{\mathcal{P}}_{(1+\pi_1)\rho^{-\sigma}} \cap (\overline{\mathcal{B}}_{1,1} \setminus \mathcal{B}_{3,\pi_1})\}} \bar{V}(v\varepsilon_1, \ldots, v^{\lambda_{n-1}}\varepsilon_n) \geq 2.$$

Then for $\varepsilon \in \overline{\mathcal{P}}_{(1+\pi_1)\rho^{-\sigma}} \cap (\overline{\mathcal{B}}_{1,1} \setminus \mathcal{B}_{3,\pi_1})$, we have

$$V(\varepsilon) = \int_{l_5}^{l_6} \frac{1}{v^{q+\lambda}}(\chi \circ \bar{V}(v\varepsilon_1, \ldots, v^{\lambda_{n-1}}\varepsilon_n))\,\mathrm{d}v + \frac{1}{ql_6^q}$$

and

$$\left.\frac{\mathrm{d}V(\varepsilon)}{\mathrm{d}t}\right|_{(10)} = 2\rho \int_{l_5}^{l_6} \frac{1}{v^{q+\lambda}} \chi'(\bar{V}(v\varepsilon_1, \ldots, v^{\lambda_{n-1}}\varepsilon_n))K(v, \varepsilon_1, \ldots, \varepsilon_n)\,\mathrm{d}v.$$

And for any $\varepsilon \in \overline{\mathcal{P}}_{(1+\pi_1)\rho^{-\sigma}} \cap (\overline{\mathcal{B}}_{1,1} \setminus \mathcal{B}_{3,\pi_1})$, there exists $\tilde{\rho} \geq 1$ such that $\varepsilon = (\tilde{\rho}^\sigma(\tilde{\rho}^{-\sigma}\rho^{-\sigma}\varepsilon_1), \tilde{\rho}^{\lambda_1\sigma}\rho^{-\lambda_n\lambda_1\sigma}\varepsilon_2, \ldots, \tilde{\rho}^{\lambda_{n-1}\sigma}\rho^{-\lambda_n\lambda_{n-1}\sigma}\varepsilon_n)^T$, $|\varepsilon_1| \leq 1 + \pi_1$, $\sum_{i=2}^{n} \varepsilon_i^2 = \pi_1^2$. By use of the boundedness of the compact set $\varepsilon \in \overline{\mathcal{P}}_{(1+\pi_1)\rho^{-\sigma}} \cap (\overline{\mathcal{B}}_{1,1} \setminus \mathcal{B}_{3,\pi_1})$, we can get that $\tilde{\rho}$ is upper bounded with respect to ρ.

For any $\varepsilon \in \mathcal{F}_{\rho^{-h\sigma}} \cap (\overline{\mathcal{B}}_{1,1} \setminus \mathcal{B}_{3,\pi_1})$, there exists $\bar{h}_1 > \lambda_n \lambda_{n-1}$ such that when $h \geq \bar{h}_1$, we have

$$\left. \frac{\mathrm{d}V(\varepsilon)}{ddt} \right|_{(10)} < -\frac{\rho}{2} \int_{l_5}^{l_6} \frac{\chi'(\bar{V}(v\rho^{-h\sigma}, \ldots, v^{\lambda_{n-1}} \tilde{\rho}^{\lambda_{n-1}\sigma} \rho^{-\lambda_n \lambda_{n-1}\sigma} \varepsilon_n))}{v^{q+\lambda}}$$

$$\sum_{i=2}^{n} \tilde{\rho}^{2\lambda_{i-1}\sigma} \rho^{-2\lambda_n \lambda_{i-1}\sigma} v^{2\lambda_{i-1}} \varepsilon_i^2 \, \mathrm{d}v.$$

And for any $\varepsilon \in \mathcal{F}_{\rho^{-h\sigma}} \cap (\overline{\mathcal{B}}_{1,1} \setminus \mathcal{B}_{3,\pi_1})$, let $l_7(\varepsilon)$ and $l_8(\varepsilon)$ be such that $\frac{5}{4} \leq \bar{V}(v\varepsilon_1, \ldots, v^{\lambda_{n-1}}\varepsilon_n) \leq \frac{7}{4}$ when $l_7(\varepsilon) \leq l \leq l_8(\varepsilon)$ (without loss of generality, it is assumed that $0 \leq l_7(\varepsilon) \leq l_8(\varepsilon)$). Note that from the definition of $\chi(s)$, $1 \leq \chi'(s) \leq 2$ for $\frac{5}{4} \leq s \leq \frac{7}{4}$. Then, there exists $\bar{h}_2 > \lambda_n \lambda_{n-1}$ such that when $h > \bar{h}_2$ we can have

$$\left. \frac{\mathrm{d}V(\varepsilon)}{\mathrm{d}t} \right|_{(10)} < -\frac{\rho}{2} \int_{l_7(\varepsilon)}^{l_8(\varepsilon)} \frac{\sum_{i=2}^{n} \tilde{\rho}^{2\lambda_{i-1}\sigma} \rho^{-2\lambda_n \lambda_{i-1}\sigma} v^{2\lambda_{i-1}} \varepsilon_i^2}{v^{q+\lambda}} \, \mathrm{d}v$$

$$< -\frac{5\rho}{16\bar{\lambda}(q+\lambda-1)} \frac{l_8(\varepsilon)^{q+\lambda-1} - l_7(\varepsilon)^{q+\lambda-1}}{l_7(\varepsilon)^{q+\lambda-1} l_8(\varepsilon)^{q+\lambda-1}},$$

where $\bar{\lambda} = \lambda_{\max}(P)$.

It is clear that $\{z : z^T P z = \frac{5}{4}\} \cap \{z : z^T P z = \frac{7}{4}\} = \emptyset$, thus, we can derive the following inequality $M_1 < \sum_{i=1}^{n} (z_i^1 \frac{q+\lambda-1}{\lambda_{i-1}} - z_i^2 \frac{q+\lambda-1}{\lambda_{i-1}})^2$, where $M_1 > 0$ is a positive real number, $z^1 = (z_1^1, \ldots, z_n^1)^T \in \{z : z^T P z = \frac{7}{4}\}$ and $z^2 = (z_1^2, \ldots, z_n^2)^T \in \{z : z^T P z = \frac{5}{4}\}$. Because

$$(l_8(\varepsilon)\tilde{\rho}^\sigma(\tilde{\rho}^{-\sigma}\rho^{-h\sigma}\varepsilon_1), l_8(\varepsilon)^{\lambda_1}\tilde{\rho}^{\lambda_1\sigma}\rho^{-\lambda_n\lambda_1\sigma}\varepsilon_2, \ldots, l_8(\varepsilon)^{\lambda_{n-1}}\tilde{\rho}^{\lambda_{n-1}\sigma}\rho^{-\lambda_n\lambda_{n-1}\sigma}\varepsilon_n)^T$$

$$\in \{z : z^T P z = \frac{7}{4}\},$$

$$(l_7(\varepsilon)\tilde{\rho}^\sigma(\tilde{\rho}^{-\sigma}\rho^{-h\sigma}\varepsilon_1), l_7(\varepsilon)^{\lambda_1}\tilde{\rho}^{\lambda_1\sigma}\rho^{-\lambda_n\lambda_1\sigma}\varepsilon_2, \ldots, l_7(\varepsilon)^{\lambda_{n-1}}\tilde{\rho}^{\lambda_{n-1}\sigma}\rho^{-\lambda_n\lambda_{n-1}\sigma}\varepsilon_n)^T$$

$$\in \{z : z^T P z = \frac{5}{4}\},$$

we can get $M_1 \leq \tilde{\rho}^{2(q+\lambda-1)\sigma} \rho^{-2\lambda_n(q+\lambda-1)\sigma} (l_8(\varepsilon)^{q+\lambda-1} - l_7(\varepsilon)^{q+\lambda-1})^2 (1 + \sum_{i=2}^{n} \varepsilon_i^{\frac{2(q+\lambda-1)}{\lambda_{i-1}}})$, $\sum_{i=2}^{n} \varepsilon_i^2 = \pi_1^2$.

Note that $\{z : 1 \leq z^T P z \leq 2\}$ is a bounded compact set. Then, there exist M_2, $M_3 > 0$ such that $M_2 < \sum_{i=2}^{n} z_i^{\frac{2(q+\lambda-1)}{\lambda_{i-1}}} < M_3$, $z \in \{z : 1 \leq z^T P z \leq 2\}$. It is clear to get that there exist $\varepsilon^j \in \overline{\mathcal{P}}_{(1+\pi_1)\rho^{-\sigma}} \cap (\overline{\mathcal{B}}_{1,1} \setminus \mathcal{B}_{3,\pi_1})$ such that

$$(l_j(\varepsilon)\tilde{\rho}^\sigma(\tilde{\rho}^{-\sigma}\rho^{-h\sigma}\varepsilon_1^j), l_j(\varepsilon)^{\lambda_1}\tilde{\rho}^{\lambda_1\sigma}\rho^{-\lambda_n\lambda_1\sigma}\varepsilon_2^j, \ldots, l_j(\varepsilon)^{\lambda_{n-1}}$$
$$\tilde{\rho}^{\lambda_{n-1}\sigma}\rho^{-\lambda_n\lambda_{n-1}\sigma}\varepsilon_n^j)^T \in \{z : 1 \leq z^T P z \leq 2\}, \ j = 7, 8.$$

And,

$$M_3 > \tilde{\rho}^{2(q+\lambda-1)\sigma} \ \rho^{-2\lambda_n(q+\lambda-1)\sigma} l_j(\varepsilon)^{2(q+\lambda-1)} \sum_{i=2}^{n} \varepsilon_i^{j \frac{2(q+\lambda-1)}{\lambda_{i-1}}}, \ j = 7, 8, \quad \sum_{i=2}^{n} \varepsilon_i^{j^2} = \pi_1^2.$$

Thus, we get

$$l_8(\varepsilon)^{q+\lambda-1} - l_7(\varepsilon)^{q+\lambda-1} > \min_{\{\varepsilon:\sum_{i=2}^n \varepsilon_i^2 = \pi_1^2\}} \sqrt{\frac{\rho^{2\lambda_n(q+\lambda-1)\sigma} M_1}{\tilde{\rho}^{2(q+\lambda-1)\sigma}(\sum_{i=2}^n \varepsilon_i^{\frac{2(q+\lambda-1)}{\lambda_{i-1}}} + 1)}}$$

and

$$\frac{1}{l_j(\varepsilon)^{q+\lambda-1}} > \min_{\{\varepsilon:\sum_{i=2}^n \varepsilon_i^2 = \pi_1^2\}} \sqrt{\frac{\tilde{\rho}^{2(q+\lambda-1)\sigma} \sum_{i=2}^n \varepsilon_i^{\frac{2(q+\lambda-1)}{\lambda_{i-1}}}}{\rho^{2\lambda_n(q+\lambda-1)\sigma} M_3}}, \ j = 7, 8.$$

Therefore, we have

$$\left.\frac{\mathrm{d}V(\varepsilon)}{\mathrm{d}t}\right|_{(10)} < -\rho^{1-\lambda_n(q+\lambda-1)\sigma} \tilde{\rho}^{(q+\lambda-1)\sigma} d_3, \ \varepsilon \in \mathcal{F}_{\rho^{-h\sigma}} \cap (\overline{\mathcal{B}}_{1,1} \setminus \mathcal{B}_{3,\pi_1}), \qquad (\text{A}.5)$$

where $d_3 = \min_{\{\varepsilon:\sum_{i=2}^n \varepsilon_i^2 = \pi_1^2\}} \dfrac{5\sqrt{M_1} \sum_{i=2}^n \varepsilon_i^{\frac{2(q+\lambda-1)}{\lambda_{i-1}}}}{16\bar{\lambda}(q+\lambda-1)M_3 \sqrt{\sum_{i=2}^n \varepsilon_i^{\frac{2(q+\lambda-1)}{\lambda_{i-1}}} + 1}}.$

(4) Fourthly, when $\varepsilon \in (\overline{\mathcal{P}}_{\rho^{-\sigma}} \setminus \mathcal{P}_{\rho^{-h\sigma}}) \cap (\overline{\mathcal{B}}_{3,\pi_1} \setminus \mathcal{B}_{3,\pi_1})$, because for any $\varepsilon^1 = (\varepsilon_1^1, \varepsilon_2^1, \ldots, \varepsilon_n^1)^T \in (\overline{\mathcal{P}}_{\rho^{-\sigma}} \setminus \mathcal{P}_{\rho^{-h\sigma}}) \cap (\overline{\mathcal{B}}_{3,\pi_1} \setminus \mathcal{B}_{3,\pi_1})$ and any $\varepsilon^2 = (\pm\rho^{-\sigma}, \varepsilon_2^1, \ldots, \varepsilon_n^1)^T \in \mathcal{F}_{\rho^{-\sigma}} \cap (\overline{\mathcal{B}}_{3,\pi_1} \setminus \mathcal{B}_{3,\pi_1})$, we have $\|\varepsilon^1 - \varepsilon^2\|_2^2 \leq 4\rho^{-2\sigma}$. Because of the continuity of $\left.\frac{\mathrm{d}V(\varepsilon)}{\mathrm{d}t}\right|_{(10)}$ on $\varepsilon \in \mathcal{R}^n$, we have

$$\left.\frac{\mathrm{d}V(\varepsilon)}{\mathrm{d}t}\right|_{(10)} < -\frac{d_2}{2}\rho^{1-(q+\lambda)\sigma} < 0, \ \varepsilon \in (\overline{\mathcal{P}}_{\rho^{-\sigma}} \setminus \mathcal{P}_{\rho^{-h\sigma}}) \cap (\overline{\mathcal{B}}_{3,\pi_1} \setminus \mathcal{B}_{3,\pi_1}). \qquad (\text{A}.6)$$

(5) From (A.3), we can select $\rho > \max_{\{1 \leq i \leq 2\}}\{2, \rho_i\}$ such that

$$V(\varepsilon)^{-\gamma} \geq d_4^{-\gamma}, \ \varepsilon \in \mathcal{S}_1 \cap \mathcal{P}_{\rho^{-\sigma}}, \qquad (\text{A}.7)$$

where $d_4 = \max_{\sum_{i=2}^n \varepsilon_i^2 = 1} V(\varepsilon)$.

When $\varepsilon \in \mathcal{F}_{\rho^{-\sigma}} \cap \overline{\mathcal{B}}_{3,\pi_1}$, we can have $V(\pm\rho^{-\sigma}, \rho^{-\lambda_n\lambda_1\sigma}\varepsilon_2, \ldots, \rho^{-\lambda_n\lambda_{n-1}\sigma}\varepsilon_n) =$

$$= \rho^{-q\sigma} V(\pm 1, \rho^{-(\lambda_n-1)\lambda_1\sigma}\varepsilon_2, \ldots, \rho^{-(\lambda_n-1)\lambda_{n-1}\sigma}\varepsilon_n) \leq d_5\rho^{-q\sigma},$$

where $d_5 = \max_{\sum_{i=2}^n \varepsilon_i^2 \leq \pi_1^2} V(\pm 1, \varepsilon_2, \ldots, \ldots \varepsilon_n)$. Then, we have

$$V(\varepsilon)^{-\gamma} > d_5^{-\gamma}\rho^{\sigma(q+\lambda-1)}, \ \varepsilon \in \mathcal{F}_{\rho^{-\sigma}} \cap \overline{\mathcal{B}}_{3,\pi_1}. \qquad (\text{A}.8)$$

When $\varepsilon \in \mathcal{F}_{\rho^{-h\sigma}} \cap (\overline{\mathcal{B}}_{1,1} \setminus \mathcal{B}_{3,\pi_1})$,

$$V(\pm\tilde{\rho}^\sigma \tilde{\rho}^{-\sigma}\rho^{-h\sigma}, \tilde{\rho}^{\lambda_1\sigma}\rho^{-\lambda_n\lambda_1\sigma}\varepsilon_2, \ldots, \tilde{\rho}^{\lambda_{n-1}\sigma}\rho^{-\lambda_n\lambda_{n-1}\sigma}\varepsilon_n) =$$
$$= \tilde{\rho}^{q\sigma}\rho^{-\lambda_n q\sigma} V(\pm\tilde{\rho}^{-\sigma}\rho^{-(h-\lambda_n)\sigma}, \varepsilon_2, \ldots, \varepsilon_n) \leq d_6\tilde{\rho}^{q\sigma}\rho^{-\lambda_n q\sigma},$$

where $d_6 = \max_{|\varepsilon_1| \leq 1, \sum_{i=2}^n \varepsilon_i^2 \leq \pi_1^2} V(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$. Then the following inequality holds:

$$V(\varepsilon)^{-\gamma} > d_6^{-\gamma}\rho^{\lambda_n(q+\lambda-1)\sigma} \tilde{\rho}^{-(q+\lambda-1)\sigma}, \ \varepsilon \in \mathcal{F}_{\rho^{-h\sigma}} \cap (\overline{\mathcal{B}}_{1,1} \setminus \mathcal{B}_{3,\pi_1}). \qquad (\text{A}.9)$$

When $\varepsilon \in (\overline{\mathcal{P}}_{\rho^{-\sigma}} \setminus \mathcal{P}_{\rho^{-h\sigma}}) \cap (\overline{\mathcal{B}}_{3,\pi_1} \setminus \mathcal{B}_{3,\pi_1})$,

$$V(\pm \rho^{-(1+(h-1)s)\sigma}, \rho^{-\lambda_n \lambda_1 \sigma} \varepsilon_2, \ldots, \rho^{-\lambda_n \lambda_{n-1} \sigma} \varepsilon_n) =$$
$$= \rho^{-q\sigma} V(\pm \rho^{-(h-1)s\sigma}, \rho^{-(\lambda_n-1)\lambda_1 \sigma} \varepsilon_2, \ldots, \rho^{-(\lambda_n-1)\lambda_{n-1}\sigma} \varepsilon_n) \leq d_6 \rho^{-q\sigma},$$

where $0 < s < 1$. Therefore, we have

$$V(\varepsilon)^{-\gamma} > d_6^{-\gamma} \rho^{(q+\lambda-1)\sigma}, \ \varepsilon \in (\overline{\mathcal{P}}_{\rho^{-\sigma}} \setminus \mathcal{P}_{\rho^{-h\sigma}}) \cap (\overline{\mathcal{B}}_{3,\pi_1} \setminus \mathcal{B}_{3,\pi_1}). \tag{A.10}$$

(6) Thus, selecting $h > \{\bar{h}_1, \bar{h}_2\}$ and $\rho > \{\rho_1, \rho_2\}$, from the above inequalities (A.3), (A.7); (A.4), (A.8); (A.5), (A.9) and (A.6), (A.10), we can obtain a compact set which encircles the origin $\overline{\mathcal{A}} \triangleq (\mathcal{S}_1 \cap \overline{\mathcal{P}}_{\rho^{-h\sigma}}) \cup (\mathcal{F}_{\rho^{-\sigma}} \cap \overline{\mathcal{B}}_{3,\pi_1}) \cup (\mathcal{F}_{\rho^{-h\sigma}} \cap (\overline{\mathcal{B}}_{1,1} \setminus \mathcal{B}_{3,\pi_1})) \cup ((\overline{\mathcal{P}}_{\rho^{-\sigma}} \setminus \mathcal{P}_{\rho^{-h\sigma}}) \cap (\overline{\mathcal{B}}_{3,\pi_1} \setminus \mathcal{B}_{3,\pi_1}))$. And

$$\left. \frac{dV(\varepsilon)}{dt} \right|_{(10)} V(\varepsilon)^{-\gamma} \leq -w_4 \rho^{1-\sigma}, \ \varepsilon \in \mathcal{A}, \tag{A.11}$$

where $w_4 = \min\{d_1 d_4^{-\gamma}, d_2 d_5^{-\gamma}, d_3 d_6^{-\gamma}, \frac{d_2 d_6^{-\gamma}}{2}\} > 0$.

Part II: It is clear that $V(\varepsilon)$ and $\left. \frac{dV(\varepsilon)}{dt} \right|_{(10)}$ are homogeneous of degrees q and $q+\lambda-1$ with respect to the weights $\{\lambda_i\}_{0 \leq i \leq n-1}$. For any $\varepsilon \in \mathcal{R}^n \setminus \{0\}$, there exist $v_0 > 0$ and $\varepsilon^0 \in \mathcal{A}$ such that $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)^T = (v_0 \varepsilon_1^0, \ldots, v_0^{\lambda_{n-1}} \varepsilon_n^0)^T$. Then we have $\left. \frac{dV(\varepsilon)}{dt} \right|_{(10)} = v_0^{q+\lambda-1} \left. \frac{dV(\varepsilon^0)}{dt} \right|_{(10)}$ and $V(\varepsilon) = v_0^q V(\varepsilon^0)$.

Finally, from (A.11), we obtain

$$\left. \frac{dV(\varepsilon)}{dt} \right|_{(10)} = V(\varepsilon)^{-\gamma} \left. \frac{dV(\varepsilon^0)}{dt} \right|_{(10)} V(\varepsilon^0)^{-\gamma} \leq -w_4 \rho^{1-\sigma} V(\varepsilon)^{-\gamma}, \ \varepsilon \in \mathcal{R}^n \setminus \{0\}. \tag{A.12}$$

As for the proof of (iii), it follows the same procedure as the proof of (iv). The main difference compared with the proof (iv) is that in the proof of (iii), the compact set is constructed from the following four parts: $\mathcal{S}_1 \cap \overline{\mathcal{P}}_{\rho^{-\sigma}}$, $(\overline{\mathcal{P}}_{(1+\pi_2)\rho^{-\sigma}} \setminus \mathcal{P}_{(1-\pi_2)\rho^{-\sigma}}) \cap \overline{\mathcal{B}}_{4,\pi_2}$, $\mathcal{F}_{\rho^{-h^*\sigma}} \cap (\overline{\mathcal{B}}_{1,1} \setminus \mathcal{B}_{4,\pi_2})$ and $(\overline{\mathcal{P}}_{\rho^{-\sigma}} \setminus \mathcal{P}_{\rho^{-h^*\sigma}}) \cap (\overline{\mathcal{B}}_{4,\pi_2} \setminus \mathcal{B}_{4,\pi_2})$, where $\pi_2 > 0$, $h^* > 2$ are two positive numbers.

This completes the proof. $\qquad \square$

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Yunyan Li, (Corresponding author) Room 14-18, Engineering I Building, University of Pretoria, Pretoria 0002. South Africa.
    e-mail: u04425596@tuks.co.za

Yanjun Shen, College of Science, China Three Gorges University, College Road No.8, Yichang, Hubei 443002. China.
    e-mail: shenyj@ctgu.edu.cn

Xiaohua Xia, Room 14-7, Engineering I Building, University of Pretoria, Pretoria 0002. South Africa.
    e-mail: xxia@postino.up.ac.za