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CONSTRUCTION OF MULTIVARIATE COPULAS IN N-BOXES

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In this paper we give an alternative proof of the construction of n-dimensional ordinal sums given in Mesiar and Sempi [17], we also provide a new methodology to construct n-copulas extending the patchwork methodology of Durante, Saminger-Platz and Sarkoci in [6] and [7]. Finally, we use the gluing method of Siburg and Stoimenov [20] and its generalization in Mesiar et al. [15] to give an alternative method of patchwork construction of n-copulas, which can be also used in composition with our patchwork method.

Keywords: n-copulas, modular functions, rectangular patchwork

Classification: 60A10, 60E05

1. INTRODUCTION

The idea of patching a 2-copula C, or simply copula, in a rectangular region R, by redefining C using another function D on R, is of great interest when modelling some bivariate data. It is well known that in many applications such as Mathematical Finances, Risk Theory, Ecology, etc., the researchers know from previous data what is the behavior of their observations in the tails, but if they try to fit a known model, many times, this model does not agree with these tail behaviors. In this case, it is important to take a base copula C and try to modify it in the regions of interest using some other copulas which have the behavior that we are looking for. This is now possible using the general approach of rectangular patchwork construction.

In this paper we will generalize these results for n copulas with dimensions $n \geq 3$.

Definition 1.1.
Let $n \geq 2$ and let $R = [u_1, v_1] \times [u_2, v_2] \times \cdots \times [u_n, v_n] =: \Pi_{i=1}^{n}[u_i, v_i] \subset \mathbf{R}^n$ be an n**-box**, that is, for every $i \in \{1, \dots, n\}$, $-\infty < u_i \leq v_i < \infty$. We will call R a **non trivial** n**-box** if for every $i \in \{1, \dots, n\}$, $-\infty < u_i < v_i < \infty$. For any $1 \leq k \leq n$ and for every $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ define

$$R_{i_1,\dots,i_k} = \{\langle x_1, \dots, x_n\rangle \in R \mid \text{for every } j \in \{1, \dots, k\}, \text{ either } x_{i_j} = u_{i_j} \text{ or } x_{i_j} = v_{i_j}\}. \tag{1}$$

Then we call $R_{i_1,\dots,i_k}$ an $(n-k)$**-dimensional face of** R.

If $k = 1$ an $(n-1)$-dimensional face is usually called simply a **face of** R, in this last case we make a distinction. We will denote by

$$R_i^l = \{\langle x_1, \ldots, x_n\rangle \in R \mid x_i = u_i\}$$

and call it the ith-lower face, and

$$R_i^u = \{\langle x_1, \ldots, x_n\rangle \in R \mid x_i = v_i\}$$

and call it the ith-upper face.

Let $n \geq 2$ and let $C : [0,1]^n \to [0,1]$ be a function which satisfies:

i) $C(u_1, \ldots, u_n) = 0$ if there exists at least one $i \in \{1, \ldots n\}$ such that $u_i = 0$.

ii) $C(1, \ldots, 1, u_i, 1, \ldots, 1) = u_i$ for every $i \in \{1, \ldots, n\}$ and for every $u_i \in [0,1]$.

iii) C is an n-increasing function, that is, for any n-box $R = \Pi_{i=1}^n [u_i, v_i]$ such that $\mathrm{Vert}(R) \subset [0,1]^n$ we have that

$$V_C(R) := \sum_{\{\underline{\mathbf{c}} \in \mathcal{D} \mid \underline{\mathbf{c}} \in \mathrm{Vert}(R)\}} \mathrm{sgn}(\underline{\mathbf{c}}) C(\underline{\mathbf{c}}) \geq 0, \tag{2}$$

where

$$\mathrm{sgn}(\underline{\mathbf{c}}) = \begin{cases} 1, & \text{if } c_i = u_i \text{ for an even number of } i's \\ -1, & \text{if } c_i = u_i \text{ for an odd number of } i's. \end{cases}$$

Then we will call C an n-**copula**.

We start with a generalization of De Baets and De Meyer [2], the proof of this result follows the same ideas and it can be found in [11].

Theorem 1.2. Let $C : [0,1]^n \to [0,1]$ be an n-copula, let $R = \Pi_{i=1}^n [u_i, v_i] \subset [0,1]^n$ be a non trivial n-box. Let $D : R \to [0,1]$ be a function. Define $Q : [0,1]^n \to [0,1]$ by

$$Q(x_1, \ldots, x_n) = \begin{cases} D(x_1, \ldots, x_n) & \text{if} \quad \langle x_1, \ldots, x_n\rangle \in R, \\ C(x_1, \ldots, x_n) & \text{if} \quad \langle x_1, \ldots, x_n\rangle \in [0,1]^n \setminus R. \end{cases} \tag{3}$$

Then, Q is an n-copula if and only if $D = C$ on $\delta(R)$ and D is n-increasing.

Using Aczel and Dhombres [1], we give a characterization of functions that assign volume zero to any n-box R called modular, and a useful Lemma.

Lemma 1.3. Let $F : \mathcal{D} \to \mathbf{R}$ be a function where $\mathcal{D} \subset \mathbf{R}^n$ for some $n \geq 2$. Then F is modular if and only if there exist n functions $G_i : \mathbf{R}^{n-1} \to \mathbf{R}$ such that for every $\mathbf{x} = \langle x_1, x_2, \ldots, x_n\rangle \in \mathcal{D}$

$$F(\mathbf{x}) = G_1(x_2, x_3, \ldots, x_n) + \cdots + G_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) + \cdots + G_n(x_1, \ldots, x_{n-1}). \tag{4}$$

P r o o f. Let $R = \Pi_{i=1}^{n}[u_i, v_i]$ be an n-box and let $\underline{c} = \langle c_1, c_2 \ldots, c_n\rangle$ be a vertex of R, define $c_1^* = v_1$ if $c_1 = u_1$ or $c_1^* = u_1$ if $c_1 = v_1$. Then $\underline{c^*} = \langle c_1^*, c_2, \ldots, c_n\rangle$ is another vertex of R, and using equation (2), $\operatorname{sgn}(\underline{c})G_1(\underline{c}) + \operatorname{sgn}(\underline{c^*})G_1(\underline{c^*}) = 0$. Repeating the argument for the ith coordinate of $\underline{c}$ respectively, in the remaining $n-1$ functions we have the result. □

The importance of Lemma 1.3 is that the functions $G_1, \ldots, G_n$ in equation (4) are completely arbitrary. For example if $n = 3$ and we define $F(x, y, z) = G_1(x, y) + G_2(x, z) + G_3(y, z) + H_1(x) + H_2(y) + H_3(z) + K$ where G_1, G_2, G_3, H_1, H_2 and H_3 are arbitrary functions and K is a constant, then F is modular.

Lemma 1.4. Let $n \geq 2$, let $R = \Pi_{i=1}^{n}[u_i, v_i] \subset [0,1]^n$ be an n-box, let $D : R \to \mathbf{R}$ be an n-increasing function and let $E : R \to \mathbf{R}$ be a modular function. If we define $F : R \to \mathbf{R}$ by

$$F(\underline{\mathbf{x}}) = D(\underline{\mathbf{x}}) + E(\underline{\mathbf{x}}). \tag{5}$$

Then F is an n-increasing function.

The following Theorem is the main result in the patchwork construction of 2-copulas given in Durante et al. [7], and an alternative shorter proof using Lemma 1.3 and Lemma 1.4 is given in [11].

Theorem A. Let C be a copula, let $\{C_j\}_{j\in\mathcal{J}}$ be a family of copulas and let $\{R_j = [u_1^j, v_1^j] \times [u_2^j, v_2^j]\}_{j\in\mathcal{J}}$ be a family of 2-boxes, in this case rectangles in $[0,1]^2$, such that

$$R_j \cap R_k \subset \delta(R_j) \cap \delta(R_k) \text{ for every } j, k \in \mathcal{J} \text{ with } j \neq k.$$

Define for every $j \in \mathcal{J}$, $\lambda_j = V_C(R_j)$, and for every $x \in [u_1^j, v_1^j]$ and for every $y \in [u_2^j, v_2^j]$, $R_{j,x} = [u_1^j, x] \times [u_2^j, v_2^j]$ and $R_{j,y} = [u_1^j, v_1^j] \times [u_2^j, y]$. Let $\tilde{C} : [0,1]^2 \to [0,1]$ defined by

$$\tilde{C}(x, y) = \begin{cases} \lambda_j C_j\left(\frac{V_C(R_{j,x})}{\lambda_i}, \frac{V_C(R_{j,y})}{\lambda_j}\right) + \varphi_j^C(x, y) & \text{if} \quad (x, y) \in R_j \text{ and } \lambda_j > 0, \\ C(x, y), & \text{otherwise,} \end{cases} \tag{6}$$

where $\varphi_j^C(x, y) = h_{u_2^j}^C(x) + v_{u_1^j}^C(y) - h_{u_2^j}^C(u_1^j)$. Then $\tilde{C}$ is a copula.

An important result about n-increasing functions proved in [11] that will be used in this paper is the following

Lemma 1.5. Let $n \geq 2$ and let $R = \Pi_{i=1}^{n}[u_i, v_i] \subset [0,1]^n$ be a non trivial n-box, let $C : [0,1]^n \to [0,1]$ and $D : [0,1]^n \to [0,1]$ be two n-copulas. Let $\lambda = V_C(R)$ and assume that $\lambda > 0$. Define $E : R \to [0, \lambda]$ by

$$E(\underline{\mathbf{x}}) = \lambda D\left(\frac{V_C(R_{x_1})}{\lambda}, \ldots, \frac{V_C(R_{x_n})}{\lambda}\right) \quad \text{for every} \quad \underline{\mathbf{x}} \in R, \tag{7}$$

where for every $i \in \{1, \ldots, n\}$, $u_i \leq x_i \leq v_i$ and

$$R_{x_i} = [u_1, v_1] \times \cdots \times [u_{i-1}, v_{i-1}] \times [u_i, x_i] \times [u_{i+1}, v_{i+1}] \times \cdots \times [u_n, v_n]. \tag{8}$$

Then E is an n-increasing function on R.

Besides, if $\underline{\mathbf{x}} = \langle v_1, \ldots, v_{j-1}, x_j, v_{j+1}, \ldots, v_n \rangle$, for some $j \in \{1, \ldots, n\}$ and $u_j \leq x_j < v_j$, then $E(\underline{\mathbf{x}}) = V_C(R_{x_j})$, in particular $E(v_1, v_2, \ldots, v_n) = \lambda$.

In Section 2 we use our methodology to give an alternative proof of the ordinal sum construction of n-copulas for $n \geq 3$ first proved in Mesiar and Sempi [17].

In Section 3 we proposed a patchwork construction method for n-copulas with $n \geq 3$. We analyze in detail the difference between the cases $n = 2$ and $n \geq 3$ using the multivariate ordinal sum construction. We also provide some examples that show the importance of this construction in the multivariate case.

In section 4 we use the gluing method of Siburg and Stoimenov [20] and its generalization given in Mesiar et al. [15] to provide an alternative method of construction of copulas in n-boxes, with $n \geq 3$, and we show that this methodology can be composed with the patchwork construction given in Section 3 to provide new n-copulas.

In Section 5 we give some final observations.

2. MULTIVARIATE ORDINAL SUMS OF COPULAS

In this section we will use our results to prove that the construction of ordinal sums can be extended to $n \geq 3$. This result was first proved in Mesiar and Sempi [17]. We start with a general Proposition.

Proposition 2.1. Let C_1 be an n-copula for some $n \geq 2$, and let $0 \leq a_1 < b_1 \leq 1$, define $R = \Pi_{i=1}^n [a_1, b_1] = [a_1, b_1]^n$ an n-box in $[0,1]^n$. Define $C : [0,1]^n \to [0,1]$ by

$$C(\underline{\mathbf{x}}) = \begin{cases} a_1 + (b_1 - a_1) C_1 \left(\frac{\min\{x_1, b_1\} - a_1}{b_1 - a_1}, \ldots, \frac{\min\{x_n, b_1\} - a_1}{b_1 - a_1} \right) & \\ & \text{if} \quad \min\{x_1, \ldots, x_n\} \in [a_1, b_1], \\ \min\{x_1, \ldots, x_n\} & \text{elsewhere.} \end{cases} \tag{9}$$

Then C is an n-copula.

P r o o f. We will proceed by induction. Let $n = 2$, and let $0 \leq a_1 < b_1 \leq 1$. Define C as in equation (9). Let $A_1 = \{\langle x_1, x_2 \rangle \in [0,1]^2 \mid \min\{x_1, x_2\} \in [a_1, b_1]\}$, then it is clear that

$$\begin{aligned} A_1^c = \{\langle x_1, x_2 \rangle \in [0,1]^2 \mid x_1 < a_1 \text{ or } x_2 < a_1\} \cup \{\langle x_1, x_2 \rangle \in [0,1]^2 \mid x_1 > b_1 \text{ and } x_2 > b_1\} \\ = A_2 \cup A_3. \end{aligned}$$

Observe that A_1 is the union of the $3 = 2^n - 1$ rectangles $R_1 = [a_1, b_1]^2$, $R_2 = [a_1, b_1] \times [b_1, 1]$ and $R_3 = [b_1, 1] \times [a_1, b_1]$, see Figure 1.

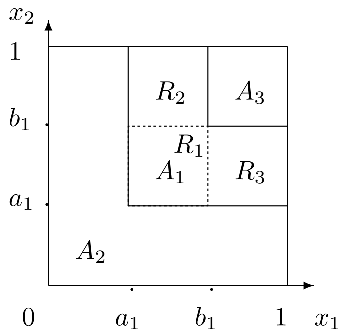

Fig. 1. Regions A_1, A_2 and A_3.

Define $D : A_1 \to \mathbf{R}$ by

$$D(x_1, x_2) = a_1 + (b_1 - a_1)C_1\left(\frac{\min\{x_1, b_1\} - a_1}{b_1 - a_1}, \frac{\min\{x_2, b_1\} - a_1}{b_1 - a_1}\right). \tag{10}$$

If $\langle x_1, x_2\rangle \in R_2$ then $D(x_1, x_2) = a_1 + (b_1 - a_1)C_1((x_1 - a_1)/(b_1 - a_1), 1) = a_1 + (x_1 - a_1) = x_1 = \min\{x_1, x_2\}$. Similarly, if $\langle x_1, x_2\rangle \in R_3$ then $D(x_1, x_2) = x_2 = \min\{x_1, x_2\}$. Using Theorem 1.2 we have to see that D is 2-increasing on R_1 and that D and $M(x_1, x_2) = \min\{x_1, x_2\}$ coincide on $\delta(R_1)$. We know that C_1 is a 2-copula, so, C_1 is 2-increasing. Define a function $h : R_1 \to [0, 1]^2$ by:

$$h(x_1, x_2) = \left(\frac{x_1 - a_1}{b_1 - a_1}, \frac{x_2 - a_1}{b_1 - a_1}\right). \tag{11}$$

Then it is clear that h is a bijection which takes R_1 onto $[0, 1]^2$, and also h is increasing in each coordinate. Let $S = [x_{1,1}, x_{1,2}] \times [x_{2,1}, x_{2,2}]$ where $a_1 \le x_{i,1} \le x_{i,2} \le b_1$ for $i = 1, 2$. Then S is a rectangle included in R_1, and the function h in equation (11), takes S onto

$$h[S] = \left[\frac{x_{1,1} - a_1}{b_1 - a_1}, \frac{x_{1,2} - a_1}{b_1 - a_1}\right] \times \left[\frac{x_{2,1} - a_1}{b_1 - a_1}, \frac{x_{2,2} - a_1}{b_1 - a_1}\right],$$

and using Lemma 1.4 with $E(x_1, x_2) = a_1$, we have from equation (10) that

$$V_D(S) = (b_1 - a_1)V_{C_1}(h[S]) \ge 0. \tag{12}$$

Therefore, from equation (12), D is 2-increasing on R_1. Now, we prove that D coincides with M on $\delta(R_1)$. Let $\underline{\mathbf{x}} = (x_1, a_1)$ or $\underline{\mathbf{x}} = (a_1, x_2)$, where $a_1 \le x_1, x_2 \le b_1$. Then, since C_1 is a 2-copula

$$D(x_1, a_1) = a_1 + (b_1 - a_1)C_1\left(\frac{x_1 - a_1}{b_1 - a_1}, \frac{a_1 - a_1}{b_1 - a_1}\right) = a_1 = M(x_1, a_1) \tag{13}$$

and

$$D(a_1, x_2) = a_1 + (b_1 - a_1)C_1\left(\frac{a_1 - a_1}{b_1 - a_1}, \frac{x_2 - a_1}{b_1 - a_1}\right) = a_1 = M(a_1, x_2). \tag{14}$$

Finally, if $\underline{\mathbf{x}} = (x_1, b_1)$ or $\underline{\mathbf{x}} = (b_1, x_2)$, where $a_1 \leq x_1, x_2 \leq b_1$. Then, since C_1 is a 2-copula

$$D(x_1, b_1) = a_1 + (b_1 - a_1)C_1\left(\frac{x_1 - a_1}{b_1 - a_1}, \frac{b_1 - a_1}{b_1 - a_1}\right) = a_1 + (b_1 - a_1)\frac{x_1 - a_1}{b_1 - a_1} = x_1 = M(x_1, b_1) \tag{15}$$

and

$$D(b_1, x_2) = a_1 + (b_1 - a_1)C_1\left(\frac{b_1 - a_1}{b_1 - a_1}, \frac{x_2 - a_1}{b_1 - a_1}\right) = a_1 + (b_1 - a_1)\frac{x_2 - a_1}{b_1 - a_1} = x_2 = M(b_1, x_2). \tag{16}$$

From equations (13), (14), (15) and (16) D and M coincide on $\delta(R_1)$ and C in equation (9) is a 2-copula according to Theorem 1.2.

Let $n > 2$ and let C_1 be an n-copula, let $R_1 = [a_1, b_1]^n$ be an n-box in $[0, 1]^n$ and let C be defined as in equation (9). Define $D : A_1 \to \mathbf{R}$, where

$$A_1 = \{\langle x_1, \ldots, x_n\rangle \in [0, 1]^n \mid \min\{x_1, \ldots, x_n\} \in [a_1, b_1]\},$$

then it is clear that

$$\begin{aligned} A_1^c &= \{\langle x_1, \ldots, x_n\rangle \in [0, 1]^n \mid \text{there exists } i \in \{1, \ldots, n\} \text{ such that } x_i < a_1\} \\ &\quad \cup \{\langle x_1, \ldots, x_n\rangle \in [0, 1]^2 \mid x_i > b_1 \text{ for every } i \in \{1, \ldots, n\}\} \\ &= A_2 \cup A_3. \end{aligned} \tag{17}$$

We will observe that in this case A_1 is the union of $2^n - 1$ n-boxes with disjoint interiors, which include R_1. Let $I_{1,i} = [a_1, b_1]$ and $I_{2,i} = [b_1, 1]$ for $i \in \{1, \ldots, n\}$ then

$$[a_1, 1]^n = ([a_1, b_1] \cup [b_1, 1])^n = \cup_{\langle j_1, \ldots, j_n\rangle \in \{1,2\}^n} \Pi_{i=1}^n I_{j_i, i}.$$

Therefore,

$$\begin{aligned} A_1 &= [a_1, 1]^n \backslash [b_1, 1]^n \\ &= \cup_{\langle j_1, \ldots, j_n\rangle \in \{1,2\}^n} \Pi_{i=1}^n I_{j_i, i} \backslash \Pi_{i=1}^n I_{2,i}, \end{aligned}$$

which is a union of $2^n - 1$ n-boxes with disjoint interiors. Define for every $\langle x_1, \ldots, x_n\rangle \in A_1$

$$D(x_1, \ldots, x_n) = a_1 + (b_1 - a_1)C_1\left(\frac{\min\{x_1, b_1\} - a_1}{b_1 - a_1}, \ldots, \frac{\min\{x_n, b_1\} - a_1}{b_1 - a_1}\right). \tag{18}$$

Using the same ideas as above, and the fact that C_1 is n-increasing, it is not difficult to see that D is n-increasing on A_1. So, by a natural generalization of Theorem 1.2, we only have to see that $M(x_1, \ldots, x_n) = \min\{x_1, \ldots, x_n\}$ and D coincide on $\delta(A_1) \cap \delta(A_2 \cup A_3)$.

Using equation (17), if $\langle x_1, \ldots, x_n\rangle \in \delta(A_1) \cap \delta(A_2 \cup A_3)$, then there exists $i \in \{1, \ldots, n\}$ such that $x_i = a_1$ and for every $j \in \{1, \ldots, n\} \backslash \{i\}$, $x_j \in [a_1, 1]$, or there exists $i \in \{1, \ldots, n\}$ such that $x_i = b_1$ and for every $j \in \{1, \ldots, n\} \backslash \{i\}$, $x_j \in [b_1, 1]$. In the first case using equation (18) and the frontier conditions of C_1, we have that

$$D(x_1, \ldots, x_n) = a_1 + (b_1 - a_1) \cdot 0 = a_1 = \min\{x_1, \ldots, x_n\},$$

and in the second case

$$D(x_1, \ldots, x_n) = a_1 + (b_1 - a_1)C(1, \ldots, 1, \frac{b_1 - a_1}{b_1 - a_1}, 1 \ldots, 1) = b_1 = \min\{x_1, \ldots, x_n\}.$$

Therefore, M and D coincide on $\delta(A_1) \cap \delta(A_2 \cup A_3)$ and C defined in equation (9) is an n-copula. $\square$

Remark 2.2. In the previous Proposition we define $R = [a_1, b_1]^n$. Let us denote by $\lambda = V_M(R)$, where $M(x_1, \ldots, x_n) = \min\{x_1, \ldots, x_n\}$, we will see that $\lambda = b_1 - a_1$. We know that using formula (2), we have that

$$V_M(R) = \sum_{\underline{\mathbf{c}} \in \mathrm{Vert}(R)} \mathrm{sgn}(\underline{\mathbf{c}}) M(\underline{\mathbf{c}}). \tag{19}$$

We observe that if $\underline{\mathbf{c}} = \langle c_1, \ldots, c_n \rangle \in \mathrm{Vert}(R)$, then for every $i \in \{1, \ldots, n\}$ $c_i = a_1$ or $c_i = b_1$. So, if there exists $i \in \{1, \ldots, n\}$ such that $c_i = a_1$ then $M(\underline{\mathbf{c}}) = a_1$, and the only vertex of R such that $M(\underline{\mathbf{c}}) = b_1$ is $\underline{\mathbf{c}} = \langle b_1, b_1, \ldots, b_1 \rangle =: \underline{\mathbf{b_1}}$. Then using (18), we have that

$$\lambda = V_M(R) = b_1 + a_1 \sum_{\underline{\mathbf{c}} \in \mathrm{Vert}(R) \setminus \{\underline{\mathbf{b_1}}\}} \mathrm{sgn}(\underline{\mathbf{c}}). \tag{20}$$

Observe that $\sum_{\underline{\mathbf{c}} \in \mathrm{Vert}(R)} \mathrm{sgn}(\underline{\mathbf{c}}) = 0$, this follows using the binomial expansion of $0 = ((-1)+1)^n$. But, in this case

$$0 = ((-1)+1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k (1)^{n-k} = 1 + \sum_{k=1}^{n} \binom{n}{k} (-1)^k (1)^{n-k}.$$

Therefore, $\sum_{k=1}^{n} \binom{n}{k} (-1)^k (1)^{n-k} = -1$, and using (20) we get that $\lambda = V_M(R) = b_1 - a_1$.

Remark 2.3. It is very important to observe that in the case $n = 2$, the construction of ordinal sums is made by modifying the copula M only on squares that have opposite vertices on the main diagonal, namely, if $R = [a, b]^2 \subset [0, 1]^2$ and D is a 2-copula, then the function $C : [0, 1]^2 \to [0, 1]$ given by

$$C(x_1, x_2) = \begin{cases} a + (b - a)D\left(\frac{x_1 - a}{b - a}, \frac{x_2 - b}{b - a}\right) & \text{if} \quad \langle x_1, x_2 \rangle \in [a, b]^2 \\ M(x_1, x_2) = \min\{x_1, x_2\} & \quad \text{elsewhere} \end{cases}$$

is a copula, see for example Nelsen (2006). If we try to extend directly this idea to larger dimensions the result is false, that is, if we take any $n \geq 3$, D an n-copula and $R = [a, b]^n \subset [0, 1]^n$ an n-box with opposite vertices on the main diagonal and we define a function $C : [0, 1]^n \to [0, 1]$ by

$$C(x_1, \ldots, x_n) = \begin{cases} a + (b - a)D\left(\frac{x_1 - a}{b - a}, \cdots, \frac{x_n - b}{b - a}\right) & \text{if} \quad \langle x_1, \ldots, x_n \rangle \in [a, b]^n \\ M(x_1, \ldots, x_n) = \min\{x_1, \ldots, x_n\} & \quad \text{elsewhere.} \end{cases}$$

Then C is not necessarily an n-copula. To see this, we give an easy example with $n = 3$, $D = \Pi$ and $R = [0, 1/3]^3$. If we define C as in the above equation, we have that

$$C(x_1, x_2, x_3) = \begin{cases} \frac{1}{3}\Pi\left(3x_1, 3x_2, 3x_3\right) & \text{if } \langle x_1, x_2, x_3\rangle \in [0, 1/3]^3 \\ M(x_1, x_2, x_3) = \min\{x_1, x_2, x_3\} & \text{elsewhere.} \end{cases} \tag{21}$$

In this case, if we take $\langle x_1, x_2, x_3\rangle = \langle 1/3, 1/4, 1/4\rangle$, then $\langle x_1, x_2, x_3\rangle$ is a point on an upper face of the 3-box R, but

$$\frac{1}{3}\Pi\left(3\frac{1}{3}, 3\frac{1}{4}, 3\frac{1}{4}\right) = \frac{3}{16} \neq \frac{1}{4} = \min\left\{\frac{1}{3}, \frac{1}{4}, \frac{1}{4}\right\}.$$

Therefore, by Theorem 1.2, C in equation (21) is not a 3-copula.

By Proposition 2.1, we know that if we define

$$C_1(x_1, x_2, x_3) = \begin{cases} \frac{1}{3}\Pi\left(\frac{\min\{x_1, 1/3\}}{1/3}, \frac{\min\{x_2, 1/3\}}{1/3}, \frac{\min\{x_3, 1/3\}}{1/3}\right) & \\ & \text{if } \min\{x_1, x_2, x_3\} \in [0, 1/3], \\ \min\{x_1, x_2, x_3\} & \text{elsewhere.} \end{cases} \tag{22}$$

Then C_1 is a 3-copula. Observe that the big difference between equations (21) and (22) is that in the first line of (21) the region is $R = [0, 1/3]^3$, and in the first line of equation (22) the region is $[0, 1]^3 \backslash (1/3, 1]^3$, that is, the complement of the 3-box $(1/3, 1]^3$.

Of course we can extend Proposition 2.1 to obtain the multivariate version of ordinal sums, as in Mesiar and Sempi [17]. See also the application given in Durante and Fernández-Sánchez [8].

Theorem B. Let $\{C_j\}_{j\in\mathcal{J}}$ be a family of n-copulas, let $\{[a_j, b_j]\}_{j\in\mathcal{J}}$ where $\mathcal{J} = \{1, \ldots, n\}$ or $\mathcal{J} = \{1, 2, \ldots\}$. Assume that for every $j \in \mathcal{J}$, $0 \leq a_j < b_j \leq 1$, and even more for every $j, j+1 \in \mathcal{J}$, $b_j \leq a_{j+1}$. Define $C : [0, 1]^n \to [0, 1]$ by

$$C(\underline{\mathbf{x}}) = \begin{cases} a_j + (b_j - a_j)C_j\left(\frac{\min\{x_1, b_j\} - a_j}{b_j - a_j}, \ldots, \frac{\min\{x_n, b_j\} - a_j}{b_j - a_j}\right) & \\ & \text{if} \min\{x_1, \ldots, x_n\} \in [a_j, b_j] \text{ for } j \in \mathcal{J} \\ M(\underline{\mathbf{x}}) = \min\{x_1, \ldots, x_n\} & \text{elsewhere.} \end{cases} \tag{23}$$

Then C is an n-copula.

Proof. The proof of Theorem B is an easy induction that uses the same arguments that we used on the proof of Theorem A in [11]. $\square$

3. A MULTIVARIATE PATCHWORK CONSTRUCTION

In this section we provide a multivariate patchwork construction of n-copulas in n-boxes by using the regions determined in multivariate ordinal sums. We will start by taking a $3-$copula and a 3-box R with $\langle 1, 1, 1\rangle$ as one of its vertices.

Theorem 3.1. Let C and C_1 be two 3-copulas and let $R = [u_1, 1] \times [u_2, 1] \times [u_3, 1]$ where $0 < u_i < 1$ for $i \in \{1, 2, 3\}$ and define $\underline{\mathbf{0}} = \langle 0, 0, 0\rangle$. Assume that $\lambda = V_C(R) > 0$, and for every $x_1 \in [u_1, 1]$, for every $x_2 \in [u_2, 1]$ and for every $x_3 \in [u_3, 1]$, define

$$\begin{aligned} R_{x_1} &= [u_1, x_1] \times [u_2, 1] \times [u_3, 1], \\ R_{x_2} &= [u_1, 1] \times [u_2, x_2] \times [u_3, 1], \\ R_{x_3} &= [u_1, 1] \times [u_2, 1] \times [u_3, x_3]. \end{aligned}$$

Let $\tilde{C} : [0, 1]^3 \to [0, 1]$ be defined in $\underline{\mathbf{x}} = \langle x_1, x_2, x_3\rangle$ by

$$\tilde{C}(\underline{\mathbf{x}}) = \begin{cases} \lambda C_1 \left(\frac{V_C(R_{x_1})}{\lambda}, \frac{V_C(R_{x_2})}{\lambda}, \frac{V_C(R_{x_3})}{\lambda} \right) + V_C([\underline{\mathbf{0}}, \underline{\mathbf{x}}] \setminus [\underline{\mathbf{u}}, \underline{\mathbf{x}}]) & \text{if } \underline{\mathbf{x}} \in R \\ C(\underline{\mathbf{x}}), & \text{otherwise,} \end{cases} \tag{24}$$

where $[\underline{\mathbf{a}}, \underline{\mathbf{b}}] = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$ for $\underline{\mathbf{a}} = \langle a_1, a_2, a_3\rangle$, $\underline{\mathbf{b}} = \langle b_1, b_2, b_3\rangle$, and $\underline{\mathbf{u}} = \langle u_1, u_2, u_3\rangle$. Then $\tilde{C}$ is a 3-copula.

Remark 3.2. The 3-box R can be written as $R = [\underline{\mathbf{u}}, \underline{\mathbf{1}}]$ where $\underline{\mathbf{1}} = \langle 1, 1, 1\rangle$.

P r o o f. Let $D(\underline{\mathbf{x}}) = E(\underline{\mathbf{x}}) + F(\underline{\mathbf{x}})$ with $E(\underline{\mathbf{x}}) = \lambda C_1 \left(\frac{V_C(R_{x_1})}{\lambda}, \frac{V_C(R_{x_2})}{\lambda}, \frac{V_C(R_{x_3})}{\lambda} \right)$ and $F(\underline{\mathbf{x}}) = V_C([\underline{\mathbf{0}}, \underline{\mathbf{x}}] \setminus [\underline{\mathbf{u}}, \underline{\mathbf{x}}])$. By Lemma 1.5 E is a 3-increasing function.
To see that D is also 3-increasing, using Lemma 1.4, we just need to prove that F is a modular function. Let $\underline{\mathbf{x}} \in R$ then,

$$\begin{aligned} F(\underline{\mathbf{x}}) &= V_C([\underline{\mathbf{0}}, \underline{\mathbf{x}}]) - V_C([\underline{\mathbf{u}}, \underline{\mathbf{x}}]) \\ &= C(\underline{\mathbf{x}}) - \sum_{\underline{\mathbf{c}} \in \mathrm{Vert}([\underline{\mathbf{u}}, \underline{\mathbf{x}}])} \mathrm{sgn}(\underline{\mathbf{c}}) C(\underline{\mathbf{c}}) \\ &= C(x_1, x_2, x_3) - \{C(x_1, x_2, x_3) \\ &\quad -C(u_1, x_2, x_3) - C(x_1, u_2, x_3) - C(x_1, x_2, u_3) \\ &\quad +C(x_1, u_2, u_3) + C(u_1, x_2, u_3) - C(u_1, u_2, x_3) \\ &\quad -C(u_1, u_2, u_3)\} \\ &= C(u_1, x_2, x_3) + C(x_1, u_2, x_3) + C(x_1, x_2, u_3) \\ &\quad -C(x_1, u_2, u_3) - C(u_1, x_2, u_3) - C(u_1, u_2, x_3) \\ &\quad +C(u_1, u_2, u_3), \end{aligned} \tag{25}$$

but equation (25) is a modular function by the observation just below Lemma 1.3.

Now we will prove that $\tilde{C}$ is 3-increasing. Let $\underline{\mathbf{x}}$ be a point in one of the lower faces of R. Without loss of generality let $\underline{\mathbf{x}} = \langle x_1, x_2, u_3\rangle \in \delta(R)$ with $u_1 \le x_1 \le 1, u_2 \le x_2 \le 1$. Then $V_C(R_{u_3}) = 0$ and $D(\underline{\mathbf{x}}) = \lambda C_1 \left(\frac{V_C(R_{x_1})}{\lambda}, \frac{V_C(R_{x_2})}{\lambda}, \frac{0}{\lambda} \right) + V_C([0, x_1] \times [0, x_2] \times [0, u_3]) - 0 = C(\underline{\mathbf{x}})$. So, $D = C$ in the lower faces of R.

Using the proof of Theorem 1.2, see [11], we can see that $V_{\tilde{C}}(S) = V_C(S \cap \{[0, 1]^3 \setminus R\}) + V_D(S \cap R) \ge 0$ for any 3-box $S \subset [0, 1]^3$ and so $\tilde{C}$ is 3-increasing.

Finally, we prove that $\tilde{C}$ satisfies the boundary conditions of a copula.

First $\tilde{C}(x_1,x_2,x_3) = C(x_1,x_2,x_3) = 0$ if any of the $x_i = 0$ for $i \in \{1,2,3\}$, and $\tilde{C}(1,1,1) = \lambda C(1,1,1) + V_C([0,1]^3) - V_C([\underline{\mathbf{u}},\underline{\mathbf{1}}]) = \lambda + 1 - V_C(R) = 1$ by the definition of λ. Then, since $\tilde{C} = C$ in $[0,1]^3 \backslash R$ we just need to see that $\tilde{C}(\underline{\mathbf{x}}) = D(\underline{\mathbf{x}}) = x_i$, for $\underline{\mathbf{x}} = \langle x_1,1,1\rangle, \langle 1,x_2,1\rangle, \langle 1,1,x_3\rangle$ with $x_i \in [u_i,1], i = 1,2,3$. By the second part of Lemma 1.5 we have $E(x_1,1,1) = V_C(R_{x_1})$, $E(1,x_2,1) = V_C(R_{x_2}), E(1,1,x_3) = V_C(R_{x_3})$ for $x_i \in [u_i,1], i = 1,2,3$. Without losing generality let us assume $\underline{\mathbf{x}} = \langle x_1,1,1\rangle$, where $u_1 \le x_1 \le 1$, then

$$\begin{aligned}
D(\underline{\mathbf{x}}) &= E(x_1,1,1) + F(x_1,1,1) \\
&= V_C(R_{x_1}) + V_C([0,x_1]\times[0,1]\times[0,1]) - V_C([u_1,x_1]\times[u_2,1]\times[u_3,1]) \\
&= V_C(R_{x_1}) + C(x_1,1,1) - V_C(R_{x_1}) \\
&= x_1. \qquad (26)
\end{aligned}$$

Similar results as (26) hold if $\underline{\mathbf{x}} = \langle 1,x_2,1\rangle$ or if $\underline{\mathbf{x}} = \langle 1,1,x_3\rangle$. On the other hand if $\underline{\mathbf{x}} = \langle x_1,1,1\rangle$ where $0 \le x_1 \le u_1$, then $\tilde{C}(\underline{\mathbf{x}}) = C(\underline{\mathbf{x}}) = x_1$, since C is a 3-copula. Similar results are obtained for $\underline{\mathbf{x}} = \langle 1,x_2,1\rangle$ and $\underline{\mathbf{x}} = \langle 1,1,x_3\rangle$ when $0 \le x_2 \le u_2$ and $0 \le x_3 \le u_3$. Therefore, $\tilde{C}$ in equation (24) is a 3-copula. □

Remark 3.3. If we let $u_i = 0$ for some $i \in \{1,2,3\}$ in the previous Theorem the result still holds. For example if $u_1 = 0$, then $R = [0,1]\times[u_2,1]\times[u_3,1]$, and if we take $\underline{\mathbf{x}} = \langle 0,x_2,x_3\rangle$ where $u_2 \le x_2 \le 1$ and $u_3 \le x_3 \le 1$, then by definition (24) we have that

$$\begin{aligned}
\tilde{C}(\underline{\mathbf{x}}) &= \lambda C_1\left(\frac{V_C(R_{x_1=0})}{\lambda}, \frac{V_C(R_{x_2})}{\lambda}, \frac{V_C(R_{x_3})}{\lambda}\right) + V_C([\underline{\mathbf{0}},\underline{\mathbf{x}}]) - V_C([\underline{\mathbf{u}},\underline{\mathbf{x}}]) \\
&= 0 + V_C([0,0]\times[0,x_2]\times[0,x_3]) - V_C([0,0]\times[u_2,x_2]\times[u_3,x_3]) \\
&= 0.
\end{aligned}$$

Clearly, Theorem 3.1 can be generalized easily to larger dimensions.

Theorem 3.4. For every $n \ge 3$ let C and C_1 be two n-copulas and let $R = [u_1,1]\times[u_2,1]\times\cdots\times[u_n,1]$ where $0 \le u_i < 1$ for $i \in \{1,\ldots,n\}$. Assume that $\lambda = V_C(R) > 0$, and for every $i \in \{1,\ldots,n\}$ and for every $x_i \in [u_i,1]$ define $R_{x_i} = [u_1,1]\times\cdots\times[u_{i-1},1]\times[u_i,x_i]\times[u_{i+1},1]\times\cdots\times[u_n,1]$. Let $(C \biguplus_{\underline{\mathbf{u}}} C_1) : [0,1]^n \to [0,1]$ be defined in $\underline{\mathbf{x}} = \langle x_1,\ldots,x_n\rangle$ by

$$(C \biguplus_{\underline{\mathbf{u}}} C_1)(\underline{\mathbf{x}}) = \begin{cases} \lambda C_1\left(\frac{V_C(R_{x_1})}{\lambda},\ldots,\frac{V_C(R_{x_n})}{\lambda}\right) + V_C([\underline{\mathbf{0}},\underline{\mathbf{x}}] \setminus [\underline{\mathbf{u}},\underline{\mathbf{x}}]) & \text{if } \underline{\mathbf{x}} \in R, \\ C(\underline{\mathbf{x}}), & \text{otherwise}, \end{cases} \qquad (27)$$

where $[\underline{\mathbf{a}},\underline{\mathbf{b}}] = [a_1,b_1]\times\cdots\times[a_n,b_n]$ for $\underline{\mathbf{a}} = \langle a_1,\ldots,a_n\rangle, \underline{\mathbf{b}} = \langle b_1,\ldots,b_n\rangle$, and $\underline{\mathbf{u}} = \langle u_1,\ldots,u_n\rangle$. Then $(C \biguplus_{\underline{\mathbf{u}}} C_1)$ is an n-copula.

P r o o f. It follows the same steps as the proof of Theorem 3.1. □

Remark 3.5. For every $n \ge 3$ and for every n-copula C we can obtain from equation (27) every n-copula C_1. Let C and C_1 arbitrary n-copulas and let $R = [0,1]^n$, then

$V_C(R) = 1 = \lambda$, and for every $i \in \{1, \dots, n\}$ and for every $x_i \in [0,1]$, $R_{x_i} = [0,1] \times \cdots \times [0,1] \times [0, x_i] \times [0,1] \cdots \times [0,1]$. So, $V_C(R_{x_i}) = x_i$ for every $i \in \{1, \dots, n\}$ and $\underline{\mathbf{u}} = \underline{\mathbf{0}}$. Therefore, from equation (27) we have that $(C \biguplus_{\underline{\mathbf{u}}} C_1)(\underline{\mathbf{x}}) = C_1(\underline{\mathbf{x}})$ for every $\underline{\mathbf{x}} \in [0,1]^n$.

Using Theorem 3.4 we can construct many different new n-copulas. Observe that in the construction of the copula $(C \biguplus_{\underline{\mathbf{u}}} C_1)$ on $R = [\underline{\mathbf{u}}, \underline{\mathbf{1}}]$, given in equation (27), the copula C remains fixed on $[0,1]^n \backslash R$, and on R we have a rescaled version of the copula C_1. Using Theorem 3.4 we have the following

Definition 3.6. Let $\mathcal{C}^n$ be the family of all n-copulas for some $n \geq 3$, for every fixed $\underline{\mathbf{u}} \in [0,1)^n$, the function $\biguplus_{\underline{\mathbf{u}}} : \mathcal{C}^n \times \mathcal{C}^n \to \mathcal{C}^n$ defined by

$$\biguplus_{\underline{\mathbf{u}}}(C, C_1) = (C \biguplus_{\underline{\mathbf{u}}} C_1) \tag{28}$$

is an operator.

Lemma 3.7. Let $n = 3$, $\underline{\mathbf{u}} \in (0,1)^3$ and Π the product 3-copula and let C_1 and C_2 be 3-copulas. Then

$$(C_1 \biguplus_{\underline{\mathbf{u}}} C_2) = \Pi \tag{29}$$

if and only if $C_1 = C_2 = \Pi$.

Proof. Let $n = 3$, $\underline{\mathbf{u}} \in (0,1)^3$ and $R = [\underline{\mathbf{u}}, \underline{\mathbf{1}}]$. First, assume that $C_1 = C_2 = \Pi$, then $\lambda = V_\Pi(R) = (1-u_1)(1-u_2)(1-u_3)$. Now, we will see that $V_\Pi(R_{x_i})/\lambda = (x_i - u_i)/(1-u_i)$ for every $i \in \{1,2,3\}$. Without losing generality we will assume that $i = 1$. Since $R_{x_1} = [u_1, x_1] \times [u_2, 1] \times [u_3, 1]$ then

$$\frac{V_\Pi(R_{x_1})}{\lambda} = \frac{(x_1 - u_1)(1-u_2)(1-u_3)}{(1-u_1)(1-u_2)(1-u_3)} = \frac{(x_1 - u_1)}{(1-u_1)}.$$

So, using equations (24), (25) and Definition 3.6, we have that for $\underline{\mathbf{x}} \in R$

$$\begin{aligned}
(C_1 \biguplus_{\underline{\mathbf{u}}} C_2)(\underline{\mathbf{x}}) &= \lambda \Pi \left(\frac{V_\Pi(R_{x_1})}{\lambda}, \frac{V_\Pi(R_{x_2})}{\lambda}, \frac{V_\Pi(R_{x_3})}{\lambda} \right) + V_\Pi([\underline{\mathbf{0}}, \underline{\mathbf{x}}] \setminus [\underline{\mathbf{u}}, \underline{\mathbf{x}}]) \\
&= \lambda \Pi \left(\frac{(x_1 - u_1)}{(1-u_1)}, \frac{(x_2 - u_2)}{(1-u_2)}, \frac{(x_3 - u_3)}{(1-u_3)} \right) + V_\Pi([\underline{\mathbf{0}}, \underline{\mathbf{x}}] \setminus [\underline{\mathbf{u}}, \underline{\mathbf{x}}]) \\
&= (x_1 - u_1)(x_2 - u_2)(x_3 - u_3) + x_1 x_2 x_3 - (x_1 - u_1)(x_2 - u_2)(x_3 - u_3) \\
&= x_1 x_2 x_3 \\
&= \Pi(\underline{\mathbf{x}}).
\end{aligned}$$

Conversely, assume that equation (29) holds for some C_1 and C_2 3-copulas, then from equation (24) it is clear that C_1 must be Π in $[0,1]^3 \backslash R$. Then $\lambda = (1-u_1)(1-u_2)(1-u_3)$ and $V_{C_1}([\underline{\mathbf{0}}, \underline{\mathbf{x}}] \setminus [\underline{\mathbf{u}}, \underline{\mathbf{x}}]) = x_1 x_2 x_3 - (x_1 - u_1)(x_2 - u_2)(x_3 - u_3)$. But, from equations (24) and (29) this implies that $C_2 = \Pi$. $\square$

Of course, Lemma 3.7 also holds for $n > 3$.

Example 3.8. Let $n = 3$, let

$$C_1(\underline{\mathbf{x}}) = \exp\left(-\left[(-\ln(x_1))^\theta + (-\ln(x_2))^\theta + (-\ln(x_3))^\theta\right]^{1/\theta}\right)$$

for some $\theta \geq 1$, which is a member of the Gumbel–Hougaard Archimedean family, see Example 4.23 in Nelsen [18], and let $C = \Pi$. Let $R = [1/2, 1] \times [1/2, 1] \times [3/4, 1]$, then $\lambda = V_\Pi(R) = 1/16$, and $V_\Pi(R_{x_i}) = (1/8)(x_i - 1/2)$ for $i = 1, 2$ with $x_1, x_2 \in [1/2, 1]$, and $V_\Pi(R_{x_3}) = (1/4)(x_3 - 3/4)$ with $x_3 \in [3/4, 1]$. Besides, $V_\Pi([\underline{\mathbf{0}}, \underline{\mathbf{x}}]) = x_1x_2x_3$ and since $\underline{\mathbf{u}} = \langle 1/2, 1/2, 3/4\rangle$ then $V_\Pi([\underline{\mathbf{u}}, \underline{\mathbf{x}}]) = (x_1 - 1/2)(x_2 - 1/2)(x_3 - 3/4)$. Therefore, using Theorem 3.1 if we define $\tilde{C} = (C \uplus_{\underline{\mathbf{u}}} C_1)$ then

$$\tilde{C}(\underline{\mathbf{x}}) = \begin{cases} \frac{1}{16}C_1(2x_1 - 1, 2x_2 - 1, 4x_3 - 3) + x_1x_2x_3 - (x_1 - 1/2)(x_2 - 1/2)(x_3 - 3/4) \\ \hfill \text{if } \underline{\mathbf{x}} \in R \\ x_1x_2x_3 \hfill \text{otherwise.} \end{cases}$$

Then $\tilde{C}$ is a 3-copula, which behaves like a rescaled version of the Gumbel–Hougaard family on the upper 3-box R and on the rest is the product copula. It is also clear that $\tilde{C}$ is not an ordinal sum.

Example 3.9. We will see that even for $n = 3$ and a simple 3-box of the form $R = [0, v_1] \times [0, v_2] \times [0, v_3] \subset I^3$, if we take two 3-copulas C and D such that $\lambda = V_C(R) > 0$ and we define R_{x_i} for $x_i \in [0, v_i]$ as in equation (24) of Theorem 3.1, for every $i \in \{1, 2, 3\}$, and taking

$$E(\underline{\mathbf{x}}) = \lambda D\left(\frac{V_C(R_{x_1})}{\lambda}, \frac{V_C(R_{x_2})}{\lambda}, \frac{V_C(R_{x_3})}{\lambda}\right),$$

which is 3-increasing by Lemma 1.5. We can find functions $F_{1,2}(x_1, x_2)$, $F_{1,3}(x_1, x_3)$, $F_{2,3}(x_2, x_3)$ from $[0, 1]^2$ into $\mathbf{R}$, and functions $H_1(x_1)$, $H_2(x_2)$ and $H_2(x_2)$ from $[0, 1]$ into $\mathbf{R}$, such that if we define for every $\underline{\mathbf{x}} \in R$

$$\varphi(\underline{\mathbf{x}}) = F_{1,2}(x_1, x_2) + F_{1,3}(x_1, x_3) + F_{2,3}(x_2, x_3) + H_1(x_1) + H_2(x_2) + H_3(x_3),$$

then the function $Q : [0, 1]^3 \to [0, 1]$ defined by

$$Q(\underline{\mathbf{x}}) = \begin{cases} E(\underline{\mathbf{x}}) + \varphi(\underline{\mathbf{x}}) & \text{if} \quad \underline{\mathbf{x}} \in R \\ C(\underline{\mathbf{x}}) & \text{if} \quad \underline{\mathbf{x}} \in [0, 1]^3 \backslash R \end{cases}$$

satisfies that Q is continuous. However Q is not in general a 3-copula.

We know that the function φ defined above is a modular function by Lemma 1.3, and by Lemma 1.4 the first row in the definition of Q is also 3-increasing. Besides, since C and D are 3-copulas we also know that $E(\underline{\mathbf{x}})$ is continuous. The idea now is try to find an appropriate continuous function φ, such that it makes Q continuous.

First we observe that $\delta(R)$ is given by the union of $\{\underline{\mathbf{x}} \in R \,|\, \text{there exists } i \in \{1, 2, 3\}$ such that $x_i = 0\}$ with $\{\underline{\mathbf{x}} \in R \,|\, \text{ there exists } i \in \{1, 2, 3\} \text{ such that } x_i = v_i\}$. Since C is continuous we only have to find φ such it makes coincide the first row with the second row of Q on the upper faces of R, that is, on

$$\{\underline{\mathbf{x}} \in R \,|\, \text{there exists } i \in \{1, 2, 3\} \text{ such that } x_i = v_i\}.$$

In this case we want to find adequate functions in the definition of φ such that the following three conditions hold

$$\begin{aligned} Q(v_1,x_2,x_3) &= C(v_1,x_2,x_3) \quad \text{for every} \quad \langle x_2,x_3\rangle \in [0,v_2]\times[0,v_3], \\ Q(x_1,v_2,x_3) &= C(x_1,v_2,x_3) \quad \text{for every} \quad \langle x_1,x_3\rangle \in [0,v_1]\times[0,v_3], \end{aligned}$$

and

$$Q(x_1,x_2,v_3) = C(x_1,x_2,v_3) \quad \text{for every} \quad \langle x_1,x_2\rangle \in [0,v_1]\times[0,v_2].$$

We define for $\langle x_1,x_2\rangle \in [0,v_1]\times[0,v_2]$

$$F_{1,2}(x_1,x_2) = -\lambda D\left(\frac{V_C(R_{x_1})}{\lambda}, \frac{V_C(R_{x_2})}{\lambda}, 1\right) + C(x_1,x_2,v_3),$$

for $\langle x_1,x_3\rangle \in [0,v_1]\times[0,v_3]$

$$F_{1,3}(x_1,x_3) = -\lambda D\left(\frac{V_C(R_{x_1})}{\lambda}, 1, \frac{V_C(R_{x_3})}{\lambda}\right) + C(x_1,v_2,x_3),$$

and for $\langle x_2,x_3\rangle \in [0,v_2]\times[0,v_3]$

$$F_{2,3}(x_2,x_3) = -\lambda D\left(1, \frac{V_C(R_{x_2})}{\lambda}, \frac{V_C(R_{x_3})}{\lambda}\right) + C(v_1,x_2,x_3).$$

We also define for $x_1 \in [0,v_1], x_2 \in [0,v_2]$ and $x_3 \in [0,v_3]$

$$H_1(x_1) = V_C(R_{x_1}) - C(x_1,v_2,v_3),\ H_2(x_2) = V_C(R_{x_2}) - C(v_1,x_2,v_3)$$

and

$$H_3(x_3) = V_C(R_{x_3}) - C(v_1,v_2,x_3).$$

Then if we take $\langle v_1,x_2,x_3\rangle \in R$, we observe that $R_{v_1} = R$, $\lambda = V_C(R) = C(v_1,v_2,v_3)$, and using the frontier properties of D, we have that

$$\begin{aligned} Q(v_1,x_2,x_3) &= \lambda D\left(1, \frac{V_C(R_{x_2})}{\lambda}, \frac{V_C(R_{x_3})}{\lambda}\right) \\ &\quad -\lambda D\left(1, \frac{V_C(R_{x_2})}{\lambda}, 1\right) + C(v_1,x_2,v_3) \\ &\quad -\lambda D\left(1, 1, \frac{V_C(R_{x_3})}{\lambda}\right) + C(v_1,v_2,x_3) \\ &\quad -\lambda D\left(1, \frac{V_C(R_{x_2})}{\lambda}, \frac{V_C(R_{x_3})}{\lambda}\right) + C(v_1,x_2,x_3) \\ &\quad +V_C(R_{v_1}) - C(v_1,v_2,v_3) + V_C(R_{x_2}) - C(v_1,x_2,v_3) + V_C(R_{x_3}) \\ &\quad -C(v_1,v_2,x_3) \\ &= -V_C(R_{x_2}) - V_C(R_{x_3}) + C(v_1,x_2,x_3) + \lambda - \lambda + V_C(R_{x_2}) + V_C(R_{x_3}) \\ &= C(v_1,x_2,x_3). \end{aligned}$$

Therefore, the first condition above holds. Analogously it is easy to see that the other two conditions also hold with the same definition of φ which is clearly continuous. Hence, Q is continuous.

Now, if $\underline{\mathbf{x}} \in R$ is such that $x_i = 0$ for some $i \in \{1,2,3\}$, then it is clear that $V_C(R_{x_i}) = 0$ and since D and C are 3-copulas then $E(\underline{\mathbf{x}}) = 0 = C(\underline{\mathbf{x}})$. However, if we take $\underline{\mathbf{x}} = \langle 0, x_2, x_3\rangle$ with $x_2 \in (0, v_2]$ and $x_3 \in (0, v_3]$, then $E(\underline{\mathbf{x}}) = 0$, $F_{1,2}(0, x_2) = 0$, $F_{1,3}(0, x_3) = 0$, $F_{2,3}(x_2, x_3) = -\lambda D(1, V_C(R_{x_2})/\lambda, V_C(R_{x_3})/\lambda) + C(v_1, x_2, x_3)$, $H_1(0) = 0$, $H_2(x_2) = V_C(R_{x_2}) - C(v_1, x_2, v_3) = C(v_1, x_2, v_3) - C(v_1, x_2, v_3) = 0$ and $H_3(x_3) = V_C(R_{x_3}) - C(v_1, v_2, x_3) = C(v_1, v_2, x_3) - C(v_1, v_2, x_3) = 0$. So, $\varphi(\underline{\mathbf{x}}) = -\lambda D(1, C(v_1, x_2, v_3)/\lambda, C(v_1, v_2, x_3)/\lambda) + C(v_1, x_2, x_3)$, which in general is not zero. Therefore, Q is not a 3-copula even if it is continuous.

This last example shows that it is not easy to find a modular function φ, such that the function Q satisfies being a 3-copula. The problem is to find a modular function φ such that the first and second rows in the definition of Q coincide on the lower and upper faces of R. It seems that we can make them coincide if we take only the lower faces or only the upper faces, but not both at the same time, in order to get a 3-copula. Maybe if we define a different function φ we could obtain a 3-copula, but this still remains an open problem.

3.1. Patchwork Construction Method for Dimensions Larger Than or Equal to Three

We first observe that the construction of new copulas in Theorem 3.4 is restricted to n-boxes of the form $R = [\underline{\mathbf{u}}, \underline{\mathbf{1}}] \subset [0,1]^n$ where $\underline{\mathbf{u}} = \langle u_1, \ldots, u_n\rangle$ and $\underline{\mathbf{1}} = \langle 1, \ldots, 1\rangle$. The question here is, how to do a construction of a new copula when we want to modify a base n-copula C on an arbitrary n-box $R = [\underline{\mathbf{u}}, \underline{\mathbf{v}}] \subset [0,1]^n$ by another rescaled n-copula D?

In order to answer this question we will propose a new methodology which is based on Theorem 3.4.

Let us assume that $R = [\underline{\mathbf{u}}, \underline{\mathbf{v}}] \subset [0,1]^n$ is a non trivial n-box such that for every $i \in \{1, \ldots, n\}$, $0 < u_i < v_i < 1$. Let $\mathrm{Vert}(R)$ be the set of vertices of R. We will first establish an order in this set of vertices. Let $\underline{\mathbf{c}} \in \mathrm{Vert}(R)$, define a bijection $f : \mathrm{Vert}(R) \to \{1,2\}^n$ given by $f(\underline{\mathbf{c}}) = \langle l_1, l_2, \ldots, l_n\rangle \in \{1,2\}^n$, where $l_i = 1$ if $c_i = u_i$ and $l_i = 2$ if $c_i = v_i$. Define $Q_{f(\underline{\mathbf{c}})} = \{k \in \{1, \ldots, n\} \mid f(\underline{\mathbf{c}})_k = 2\}$, where $f(\underline{\mathbf{c}})_k$ is the kth coordinate of $f(\underline{\mathbf{c}})$. Now we define the composition $\varphi : \mathrm{Vert}(R) \to \{1, 2, \ldots, 2^n\}$ given by

$$\varphi(f(\underline{\mathbf{c}})) = 1 + \sum_{k \in Q_{f(\underline{\mathbf{c}})}} 2^{n-k} \quad \text{for every} \quad \underline{\mathbf{c}} \in \mathrm{Vert}(R).$$

Of course, $Q_{f(\underline{\mathbf{c}})} = \emptyset$ if and only if $\underline{\mathbf{c}} = \langle u_1, u_2, \ldots, u_n\rangle$ and in this case $\varphi(f(u_1, u_2, \ldots, u_n)) = 1$. Also observe that if $\underline{\mathbf{c}} = \langle v_1, v_2, \ldots, v_n\rangle$, then $\varphi(f(v_1, v_2, \ldots, v_n)) = 1 + \sum_{k=1}^{n} 2^{n-k} = 1 + \sum_{j=0}^{n-1} 2^j = 2^n$. It is easy to see that φ is a bijection which establishes an order among the vertices of R, in fact, this order gives the number one to the "lowest" vertex and the number 2^n to the "highest" vertex.

Let C be an n-copula which we will call base copula, let D be an n-copula which we will call modifying copula, and let $R = [\underline{\mathbf{u}}, \underline{\mathbf{v}}] \subset [0,1]^n$ be a non trivial n-box. Then inductively define

- Let $C_1 = (C \uplus_{\underline{\mathbf{u}}} D)$ where $1 = \varphi(\underline{\mathbf{u}})$.
- Given C_k for $1 \leq k < 2^n$, let $\underline{\mathbf{c}} \in \mathrm{Vert}(R)$ such that $\varphi(\underline{\mathbf{c}}) = k + 1$ and define $C_{k+1} = (C_k \uplus_{\underline{\mathbf{c}}} C)$.
- The n-copula C_{2^n} is our target copula.

Observe that if we follow this construction, then in the first step in C_1 we introduce the rescaled version of the n-copula D on $[\underline{\mathbf{u}}, \underline{\mathbf{1}}]$, and in the remaining steps we keep unaltered C_1 at least in the semi open n-box $[\underline{\mathbf{u}}, \underline{\mathbf{v}}) = \Pi_{i=1}^n [u_i, v_i)$.

Example 3.10. Let us assume that $n = 3$ and that we want to construct a 3-copula C which has a desired behavior in each of the eight vertices of $I^3 := [0,1]^3$. We will use the 3-box given by $R = [0, 1/2]^3$ as an auxiliary tool. We first establish a bijection $h : \mathrm{Vert}(I^3) \to \mathrm{Vert}(R)$, among the vertices of I^3 and the vertices of R. If $\underline{\mathbf{c}} \in \mathrm{Vert}(I^3)$ then the coordinates of $h(\underline{\mathbf{c}})$ are given by

$$h(\underline{\mathbf{c}})_i = \begin{cases} 0 & \text{if} \quad \underline{\mathbf{c}}_i = 0 \\ 1/2 & \text{if} \quad \underline{\mathbf{c}}_i = 1, \end{cases}$$

for $i = 1, 2, 3$. Using the order established at the beginning of this subsection we have Table 1:

$\underline{\mathbf{c}}$	$f(\underline{\mathbf{c}})$	$Q_{f(\underline{\mathbf{c}})}$	$\varphi(f(\underline{\mathbf{c}}))$
$\langle 0,0,0\rangle$	$\langle 1,1,1\rangle$	$\emptyset$	1
$\langle 0,0,1\rangle$	$\langle 1,1,2\rangle$	$\{3\}$	$1 + 2^{3-3} = 2$
$\langle 0,1,0\rangle$	$\langle 1,2,1\rangle$	$\{2\}$	$1 + 2^{3-2} = 3$
$\langle 0,1,1\rangle$	$\langle 1,2,2\rangle$	$\{2,3\}$	$1 + 2^{3-2} + 2^{3-3} = 4$
$\langle 1,0,0\rangle$	$\langle 2,1,1\rangle$	$\{1\}$	$1 + 2^{3-1} = 5$
$\langle 1,0,1\rangle$	$\langle 2,1,2\rangle$	$\{1,3\}$	$1 + 2^{3-1} + 2^{3-3} = 6$
$\langle 1,1,0\rangle$	$\langle 2,2,1\rangle$	$\{1,2\}$	$1 + 2^{3-1} + 2^{3-2} = 7$
$\langle 1,1,1\rangle$	$\langle 2,2,2\rangle$	$\{1,2,3\}$	$1 + 2^{3-1} + 2^{3-2} + 2^{3-3} = 8$

Tab. 1. Order of the vertices of I^3.

Of course, the order of the vertices of R is the same, that is,

$$\varphi(f(h(\underline{\mathbf{c}}))) = \varphi(f(\underline{\mathbf{c}})) \quad \text{for every} \quad \underline{\mathbf{c}} \in \mathrm{Vert}(I^3).$$

Assume that we have selected eight 3-copulas $\{C_j\}_{j=1}^8$, such that if $\underline{\mathbf{c}} \in \mathrm{Vert}(I^3)$ and it satisfies that $\varphi(f(\underline{\mathbf{c}})) = j$, then C_j has the desired behavior near the vertex $\underline{\mathbf{c}}$, for every $j \in \{1, \ldots, 8\}$. Now we proceed with the construction of a 3-copula which satisfies the desired properties using the 3-box R:

- Let $j = 1$, $R^1 = [0,1]^3$ and define $D_1 = C_1$ on R^1, then D_1 is exactly C_1 near $\underline{\mathbf{c}} = \underline{\mathbf{0}} \in \mathrm{Vert}(R)$, $\underline{\mathbf{0}} \in \mathrm{Vert}(I^3)$ and $\varphi(f(\underline{\mathbf{0}})) = 1$.
- Let $j = 2$, $R^2 = [0,1]\times[0,1]\times[1/2,1]$, $\underline{\mathbf{u}}^2 = \langle 0,0,1/2\rangle \in \mathrm{Vert}(R)$, then $\varphi(f(h(\underline{\mathbf{u}}^2))) = 2 = j$. Define $D_2 = (D_1 \biguplus_{\underline{\mathbf{u}}^2} C_2)$, then from equation (27), D_2 behaves like C_1 near $\underline{\mathbf{0}}$ and like C_2 near $\langle 0,0,1\rangle = \underline{\mathbf{c}}$, where $\varphi(f(\underline{\mathbf{c}})) = 2 = j$.
- Inductively, given D_{j-1} for $3 \le j \le 8$, let $R^j = [c_1^j,1]\times[c_2^j,1]\times[c_3^j,1]$, where $\underline{\mathbf{u}}^j = \langle c_1^j, c_2^j, c_3^j\rangle \in \mathrm{Vert}(R)$ is such that $\varphi(f(h(\underline{\mathbf{u}}^j))) = j$. Define $D_j = (D_{j-1} \biguplus_{\underline{\mathbf{u}}^j} C_j)$, then from equation (27), D_j behaves like C_k near $\underline{\mathbf{c}}^k \in \mathrm{Vert}(I^3)$ for every $k = \{1,2,\ldots,j\}$.
- Then if we define $C := D_8$, C has the desired properties.

The last example has of course generalizations to larger dimensions, and it is of great importance because it allows to model tail dependence, see for example Joe [12] and Nelsen [18]. In fact, in Finance and Risk Theory one of the biggest problems is to model tail dependence for economic variables using copulas, see for example Embrechts et al. [9], Cherubini et al. [3], McNeil et al. [16], Malevergne et al. [14], Zhang [21], just to mention some references. The use of copulas for modeling dependence has been also used in other areas such as ecology, hidrology, medicine, etc., see for example Dorey and Joubert [4], Erdely and Díaz-Vieira [10] or Salvadori et al. [19]. The last example provides a method for modeling the tail dependence in dimensions larger than or equal to three.

Of course this methodology also allows us to study the multidimensional versions of the horizontal and vertical sections of 2-copulas and the construction of 2-copulas with given horizontal or vertical sections, see for example see Nelsen [18], Durante et al. [5] or Klement et al. [13]. For $n \ge 3$, we can analyze the structure of the $n-1$ dimensional faces of an n-copula, of the form

$$C_i(\underline{\mathbf{x}}) = \{C(x_1,\ldots,x_{i-i},a_i,x_{i+1},\ldots,x_n) \mid x_j \in [0,1] \text{ for every } j \in \{1,\ldots,n\}\backslash\{i\}\},$$

where $a_i \in [0,1]$ is fixed and $i \in \{1,\ldots,n\}$; which are clearly equivalent to the horizontal and vertical sections of a 2-copula.

Example 3.11. Let us assume that $n = 3$ and $R = [0,1/2]^3$ in Example 3.10, and let us select the first six copulas as $C_1 = \cdots = C_6 = \Pi$, $C_7 = M$ and C_8 is a Gumbel–Hougaard copula with parameter $\theta = 2$, that is,

$$C_8(\underline{\mathbf{x}}) = \exp\left\{-\left[(\ln x_1)^2 + (\ln x_2)^2 + (\ln x_3)^2\right]^{1/2}\right\}.$$

If we proceed with the order given in Table 1 and the steps of Example 3.10, we get that $D_1 = D_2 = \cdots = D_6 = \Pi$ by Lemma 3.7, so, we can start at the seventh step.

Let $\underline{\mathbf{u}}^7 = \langle 1/2,1/2,0\rangle \in \mathrm{Vert}(R)$, $R^7 = [1/2,1]\times[1/2,1]\times[0,1]$ and define $D_7 = (D_6 \biguplus_{\underline{\mathbf{u}}^7} C_7)$, after some calculations we have that $\lambda_7 = V_\Pi(R^7) = 1/4$,

$$V_\Pi(R^7_{x_1}) = x_1/2 - 1/4, \quad V_\Pi(R^7_{x_2}) = x_2/2 - 1/4, \quad V_\Pi(R^7_{x_3}) = x_3/4,$$

$$V_\Pi([\underline{\mathbf{u}}^7, \underline{\mathbf{x}}]) = x_1x_2x_3 - (1/2)(x_2x_3 + x_1x_3) + x_3/4$$

and

$$D_7(\underline{\mathbf{x}}) = \begin{cases} \frac{1}{4}\min\{2x_1-1, 2x_2-1, x_3\} + \frac{1}{2}(x_2x_3+x_1x_3) - \frac{1}{4}x_3 & \text{if } \underline{\mathbf{x}} \in R^7 \\ x_1x_2x_3 & \text{otherwise.} \end{cases}$$

For the last step we have $\underline{\mathbf{u}}^8 = \langle 1/2, 1/2, 1/2\rangle \in \mathrm{Vert}(R)$, $R^8 = [1/2,1]\times[1/2,1]\times[1/2,1]$. Define $D_8 = (D_7 \uplus_{\underline{\mathbf{u}}^8} C_8)$, then we get $\lambda_8 = V_{D_7}(R^8) = 1/8$,

$$\begin{aligned}
&V_{D_7}(R^8_{x_1}) = x_1/2 - 1/4 - C_7(2x_1-1, 1, 1/2)/4,\\
&V_{D_7}(R^8_{x_2}) = x_2/2 - 1/4 - C_7(1, 2x_2-1, 1/2)/4,\\
&V_{D_7}(R^8_{x_3}) = x_3/4 - 1/8,\\
&V_{D_7}([\underline{\mathbf{u}}^8, \underline{\mathbf{x}}]) = D_7(x_1,x_2,x_3) - C_7(2x_1-1, 2x_2-1, 1/2)/4 + x_3/4 - (x_2x_3+x_1x_3)/2
\end{aligned}$$

and

$$D_8(\underline{\mathbf{x}}) = \begin{cases} \frac{1}{8}C_8\left(4x_1-2-2\min\{2x_1-1,\frac{1}{2}\}, 4x_2-2-2\min\{2x_2-1,\frac{1}{2}\}, 2x_3-1\right) & \\ \qquad +\frac{1}{4}\min\{2x_1-1, 2x_2-1, \frac{1}{2}\} - \frac{1}{4}x_3 + \frac{1}{2}(x_2x_3+x_1x_3) & \text{if } \underline{\mathbf{x}} \in R^8 \\ \frac{1}{4}\min\{2x_1-1, 2x_2-1, x_3\} & \\ \qquad +\frac{1}{2}(x_2x_3+x_1x_3) - \frac{1}{4}x_3 & \text{if } \underline{\mathbf{x}} \in [1/2,1]\times[1/2,1]\times[0,1/2] \\ x_1x_2x_3 & \text{otherwise.} \end{cases}$$

We can see that this last copula behaves similar to the copula M in the vertex $\langle 1,1,0\rangle$ and similar to a Gumbel–Hougaard copula near the vertex $\langle 1,1,1\rangle$.

4. AN ALTERNATIVE PATCHWORK USING GLUING COPULAS

In Siburg and Stoimenov [20] a new methodology of constructing n-copulas is proposed. The main idea is to glue two rescaled copulas on adjacent n-boxes of $[0,1]^n$ whose union is $[0,1]^n$, two n-boxes R and S are adjacent if their intersection is a common face. Their main result is the following:

Theorem C. Let $n \geq 2$ and let C_1, C_2 be two n-copulas. Let $0 \leq \theta \leq 1$ and define $R^l_{i,\theta} = [0,1]\times\cdots\times[0,1]\times[0,\theta]\times[0,1]\times\cdots\times[0,1]$, where the interval $[0,\theta]$ is located on the ith coordinate, for some $i \in \{1,2,\ldots,n\}$, similarly define $R^u_{i,\theta} = [0,1]\times\cdots\times[0,1]\times[\theta,1]\times[0,1]\times\cdots\times[0,1]$. Define for every $\underline{\mathbf{x}} = (x_1,\ldots,x_i,\ldots,x_n) \in [0,1]^n$

$$\left(C_1 \underset{x_i=\theta}{\circledast} C_2\right)(\underline{\mathbf{x}}) = \begin{cases} \theta C_1\left(x_1,\ldots,\frac{x_i}{\theta},\ldots,x_n\right) & \text{if } \underline{\mathbf{x}} \in R^l_{i,\theta} \\ (1-\theta)C_2\left(x_1,\ldots,\frac{x_i-\theta}{1-\theta},\ldots,x_n\right) + \theta C_1(x_1,\ldots,1,\ldots,x_n) & \\ & \text{if } \underline{\mathbf{x}} \in R^u_{i,\theta}. \end{cases} \tag{30}$$

Then $C_1 \circledast_{x_i=\theta} C_2$ is an n-copula.

Proof. Observe that the first row in equation (30) it is simply C_1 rescaled on $R^l_{i,\theta}$ and the second row is C_2 rescaled on $R^u_{i,\theta}$ plus the value of the rescaled C_1 in $\langle x_1,\ldots,1,\ldots x_n\rangle$.

It is clear that $C_1 \circledast_{x_i=\theta} C_2$ satisfies the frontier conditions of an n-copula. Using that $R^l_{i,\theta}$ and $R^u_{i,\theta}$ are adjacent n-boxes with common face $R_{i,\theta} = \{\underline{\mathbf{x}} \in [0,1]^n \mid x_i = \theta\}$, and observing that on $R_{i,\theta}$ both rows of equation (30) coincide, it is clear that $C_1 \circledast_{x_i=\theta} C_2$ is an n-increasing function. Therefore, $C_1 \circledast_{x_i=\theta} C_2$ is an n-copula. □

Of course, the binary operation $\circledast_{x_i=\theta}$, where $i \in \{1,\ldots,n\}$ and $\theta \in [0,1]$, is defined on the family $\mathcal{C}^n$ of all n-copulas. Of course, $\circledast_{x_i=\theta}$ is not a commutative operation. In Proposition 3.1 of Siburg and Stoimenov [20], it is shown that if Π is the n-product copula, $i \in \{1,\ldots,n\}$ and $\theta \in [0,1]$. Then

$$C_1 \underset{x_i=\theta}{\circledast} C_2 = \Pi \quad \text{if and only if} \quad C_1 = C_2 = \Pi. \tag{31}$$

This result follows directly from equation (30).

Remark 4.1. For dimension $n = 2$ the constructions proposed in Theorem A in Section 1 and the one given in Theorem C are quite different. Let $C = M$, $\mathcal{J} = \{1\}$, $R_1 = [0,1/2,1] \times [0,1]$ and $C_1 = \Pi$ in Theorem A. Then $\lambda_1 = V_M(R_1) = 1/2$, $R_{1,x} = [0,x] \times [0,1]$, $R_{1,y} = [0,1/2],[0,y]$ for every $x \in [0,1/2]$ and for every $y \in [0,1]$. So, $V_M(R_{1,x}) = x$ and $V_M(R_{1,y}) = \min\{1/2, y\}$ and $\varphi^M_1(x,y) = h^M_0(x) + v^M_0(y) - h^M_0(0) = 0$. Therefore, using equation (6)

$$\begin{aligned}
\tilde{C}(x,y) &= \begin{cases} \frac{1}{2}\Pi(2x, 2\min\{1/2,y\}) & \text{if } \langle x,y\rangle \in [0,1/2]\times[0,1] \\ \min\{x,y\} & \text{if } \langle x,y\rangle \in [1/2,1]\times[0,1]. \end{cases} \\
&= \begin{cases} 2x \cdot \min\{1/2,y\} & \text{if } \langle x,y\rangle \in [0,1/2]\times[0,1] \\ \min\{x,y\} & \text{if } \langle x,y\rangle \in [1/2,1]\times[0,1]. \end{cases}
\end{aligned}$$

On the other hand, if we let $C_1 = \Pi$, $C_2 = M$ and $\theta = 1/2$ in Theorem C, then using equation (30), we get that

$$\begin{aligned}
\left(\Pi \underset{x=1/2}{\circledast} M\right)(x,y) &= \begin{cases} \frac{1}{2}\Pi(2x, y) & \text{if } \langle x,y\rangle \in [0,1/2]\times[0,1] \\ \frac{1}{2}\min\{2x-1,y\} + \frac{1}{2}\min\{1,y\} & \text{if } \langle x,y\rangle \in [1/2,1]\times[0,1]. \end{cases} \\
&= \begin{cases} xy & \text{if } \langle x,y\rangle \in [0,1/2]\times[0,1] \\ \frac{1}{2}\min\{2x+y-1, 2y\} & \text{if } \langle x,y\rangle \in [1/2,1]\times[0,1]. \end{cases}
\end{aligned}$$

Of course, $\tilde{C} \neq \big(\Pi \circledast_{x=1/2} M\big)$.

Example 4.2. We will see that even for $n = 3$ the patchwork construction given in Theorem 3.1 is also different from the gluing proposal in Theorem C. In the case where Π is the base copula they may coincide, but in general they are different. Let us glue Π with some 3-copula C_1 in $x_1 = 1/2$, using equation (30) we get

$$\left(\Pi \underset{x_1=1/2}{\circledast} C_1\right)(\underline{\mathbf{x}}) = \begin{cases} 1/2\Pi\left(\frac{x_1}{1/2}, x_2, x_3\right) & \text{if } \underline{\mathbf{x}} \in [0,1/2]\times[0,1]\times[0,1] \\ \\ (1-1/2)C_1\left(\frac{x_1-1/2}{1-1/2}, x_2, x_3\right) & \\ \quad +1/2\Pi(1, x_2, x_3) & \text{if } \underline{\mathbf{x}} \in [1/2,1]\times[0,1]\times[0,1] \end{cases}$$

$$= \begin{cases} x_1x_2x_3 & \text{if } \underline{\mathbf{x}} \in [0,1/2]\times[0,1]\times[0,1] \\ \frac{1}{2}C_1\left(2x_1-1,x_2,x_3\right) \\ +\frac{1}{2}x_2x_3 & \text{if } \underline{\mathbf{x}} \in [1/2,1]\times[0,1]\times[0,1]. \end{cases} \tag{32}$$

Now, using Remark 3.3, let us patch the 3-copula C_1 in $R=[1/2,1]\times[0,1]\times[0,1]$ with the 3-copula Π. In this case, $\underline{\mathbf{u}}=\langle 1/2,0,0\rangle$, $\lambda=V_\Pi(R)=1/2$, $V_\Pi(R_{x_1})=x_1-\frac{1}{2}$, $V_\Pi(R_{x_2})=\frac{x_2}{2}$, $V_\Pi(R_{x_3})=\frac{x_3}{3}$, $V_\Pi([\underline{\mathbf{0}},\underline{\mathbf{x}}])=\Pi(x_1,x_2,x_3)$ and $V_\Pi([\underline{\mathbf{u}},\underline{\mathbf{x}}])=V_\Pi([1/2,x_1]\times[0,x_2]\times[0,x_3])=(x_1-1/2)x_2x_3$, for every $x_1\in[1/2,1]$ and for every $x_2,x_3\in[0,1]$. Then using equation (24) in Theorem 3.1 we have that

$$\left(\Pi \biguplus_{\underline{\mathbf{u}}} C_1\right)(\underline{\mathbf{x}}) = \begin{cases} \Pi\left(x_1,x_2,x_3\right) & \text{if } \underline{\mathbf{x}} \in [0,1/2]\times[0,1]\times[0,1] \\ \frac{1}{2}C_1\left(\frac{x_1-1/2}{1/2},\frac{x_2/2}{1/2},\frac{x_3/2}{1/2}\right)+\Pi(x_1,x_2,x_3) \\ \quad -\mathrm{V}_\Pi([1/2,x_1]\times[0,x_2]\times[0,x_3]) \\ & \text{if } \underline{\mathbf{x}} \in R=[1/2,1]\times[0,1]\times[0,1] \end{cases}$$

$$= \begin{cases} x_1x_2x_3 & \text{if } \underline{\mathbf{x}} \in [0,1/2]\times[0,1]\times[0,1] \\ \frac{1}{2}C_1\left(2x_1-1,x_2,x_3\right)+\frac{1}{2}x_2x_3 & \text{if } \underline{\mathbf{x}} \in R=[1/2,1]\times[0,1]\times[0,1]. \end{cases} \tag{33}$$

Then from equations (32) and (33), clearly, $(\Pi \circledast_{x_1=1/2} C_1) = (\Pi \biguplus_{\underline{\mathbf{u}}} C_1)$. But if we reverse the order and we glue some copula C_1, say $C_1=\frac{M+\Pi}{2}$, with Π in $x=1/2$ we get from equation (30) that

$$\left(C_1 \underset{x_1=1/2}{\circledast} \Pi\right)(\underline{\mathbf{x}}) = \begin{cases} \frac{1}{2}C_1\left(\frac{x_1}{1/2},x_2,x_3\right) & \text{if } \underline{\mathbf{x}} \in [0,1/2]\times[0,1]\times[0,1] \\ (1-1/2)\Pi\left(\frac{x_1-1/2}{1-1/2},x_2,x_3\right) & \text{if } \underline{\mathbf{x}} \in [1/2,1]\times[0,1]\times[0,1] \\ +1/2C_1(1,x_2,x_3) \end{cases}$$

$$= \begin{cases} \frac{1}{4}\min\{2x_1,x_2,x_3\}+\frac{1}{2}x_1x_2x_3 & \text{if } \underline{\mathbf{x}} \in [0,1/2]\times[0,1]\times[0,1] \\ x_1x_2x_3-\frac{x_2x_3}{4}+\frac{\min\{x_2,x_3\}}{4} & \text{if } \underline{\mathbf{x}} \in [1/2,1]\times[0,1]\times[0,1], \end{cases} \tag{34}$$

and using Theorem 3.1 and Remark 3.3 again, if we patch the same copula C_1 in $R=[1/2,1]\times[0,1]\times[0,1]$ with Π we have that $\underline{\mathbf{u}}=\langle 1/2,0,0\rangle$, $\lambda=V_{C_1}(R)=C_1(1,1,1)-C_1(1/2,1,1)=1/2$, $V_{C_1}(R_{x_1})=C_1(x_1,1,1)-C_1(1/2,1,1)=x_1-1/2$, $V_{C_1}(R_{x_2})=C_1(1,x_2,1)-C_1(1/2,x_2,1)=(3x_2-\min\{1,2x_2\})/4$, $V_{C_1}(R_{x_3})=C_1(1,1,x_3)-C_1(1/2,1,x_3)$ $=(3x_3-\min\{1,2x_3\})/4$, $V_{C_1}([\underline{\mathbf{0}},\underline{\mathbf{x}}]\setminus[\underline{\mathbf{u}},\underline{\mathbf{x}}])=C_1(x_1,x_2,x_3)-(C_1(x_1,x_2,x_3)-C_1(1/2,x_2,x_3))$ $=C_1(1/2,x_2,x_3)$, for every $x_1\in[1/2,1]$ and for every $x_2,x_3\in[0,1]$. So, by equation (24) in Theorem 3.1 we have that

$$\left(C_1 \biguplus_{\underline{\mathbf{u}}} \Pi\right)(\underline{\mathbf{x}}) = \begin{cases} C_1\left(x_1,x_2,x_3\right) & \text{if } \underline{\mathbf{x}} \in [0,1/2]\times[0,1]\times[0,1] \\ \frac{1}{2}\Pi\left(\frac{x_1-1/2}{1/2},\frac{(3x_2-\min\{1,2x_2\})/4}{1/2},\frac{(3x_3-\min\{1,2x_3\})/4}{1/2}\right) \\ \quad +C_1(1/2,x_2x_3) & \text{elsewhere} \end{cases}$$

$$= \begin{cases} \frac{1}{2}\min\{x_1,x_2,x_3\} + \frac{1}{2}x_1x_2x_3 & \text{if } \underline{\mathbf{x}} \in [0,1/2]\times[0,1]\times[0,1] \\ \frac{1}{4}\left(x_1-\frac{1}{2}\right)(3x_2-\min\{1,2x_2\})(3x_3-\min\{1,2x_3\}) & \\ \quad +\frac{1}{2}\min\{1/2,x_2,x_3\}+\frac{1}{4}x_2x_3 & \text{if } \underline{\mathbf{x}} \in [1/2,1]\times[0,1]\times[0,1] \end{cases} \tag{35}$$

If $\underline{\mathbf{x}} = \langle 1/4, 1/4, 1/4\rangle$ then from equations (34) and (35), $(C_1 \circledast_{x_1=1/2} \Pi)(\underline{\mathbf{x}}) = 9/128$, but $(C_1 \uplus_{\underline{\mathbf{u}}} \Pi)(\underline{\mathbf{x}}) = 9/64$. So, gluing and patching copulas do not always yield the same result. This example also shows that the operators gluing and patching are not commutative.

4.1. Proposed Methodology Using Gluing Copulas

Let $R = \Pi_{i=1}^n [u_i, v_i] \subset [0,1]^n$ be a non trivial n-box, we want to define an n-copula C, such that on R it behaves as a rescaled version of another n-copula C_1. We propose to take as a background copula $\Pi(x_1,\ldots,x_n) = x_1x_2\cdots x_n$ the n-product copula and to give a rescaled version of C_1 on R. In order to do so, we can use the gluing method as follows:

- Since R is non trivial then for every $i \in \{1,\ldots,n\}$, $0 \le u_i < v_i \le 1$, then there exists $a_i \in [0,1)$ such that $a_i v_i = u_i$.
- Define $T_1 = \Pi \circledast_{x_1=a_1} C_1$, then T_1 is an n-copula such that for $x_1 \in [0,a_1]$ behaves like Π, and for $x_1 \in [a_1,1]$ behaves like a rescaled version of C_1.
- Inductively, let $T_k = \Pi \circledast_{x_k=a_k} T_{k-1}$ for every $k \in \{2,\ldots,n\}$, then T_k is an n-copula such that for $x_k \in [0,a_k]$ behaves like Π.
- Define inductively $T_{n+j} = T_{n+j-1} \circledast_{x_j=v_j} \Pi$ for every $j \in \{1,2,\ldots,n\}$.
- Let $C = T_{2n}$, then C has the desired properties.

Example 4.3. We will see that this proposed methodology does not yield to the ordinal sum. In order to do so we will consider the copula Π. Let us define

$$T_1(x_1,x_2) = \left(M \underset{x_1=1/2}{\circledast} \Pi\right)(x_1,x_2) = \begin{cases} \min\left\{x_1, \frac{1}{2}x_2\right\} & \text{if } \langle x_1,x_2\rangle \in [0,1/2]\times[0,1] \\ x_1x_2 & \text{if } \langle x_1,x_2\rangle \in [1/2,1]\times[0,1]. \end{cases}$$

In the next step we glue M with the copula T_1 at $x_2 = 1/2$, that is, we define

$T_2(x_1,x_2) =$

$$= \left(M \underset{x_2=1/2}{\circledast} T_1\right)(x_1,x_2) = \begin{cases} \min\left\{\frac{1}{2}x_1, x_2\right\} & \text{if } \langle x_1,x_2\rangle \in [0,1]\times[0,1/2] \\ \min\left\{x_1, \frac{2x_1+2x_2-1}{4}\right\} & \text{if } \langle x_1,x_2\rangle \in [0,1/2]\times[1/2,1] \\ x_1x_2 & \text{if } \langle x_1,x_2\rangle \in [1/2,1]\times[1/2,1]. \end{cases}$$

The resulting copula T_2 behaves like Π on $[1/2,1]^2$, on $[0,1]\times[0,1/2]$ has support the line joining $\langle 0,0\rangle$ and $\langle 1,1/2\rangle$ with mass $1/2$ and on $[0,1/2,1]\times[1/2,1]$ has support the line joining $\langle 0,1/2\rangle$ and $\langle 1/2,1\rangle$ with mass $1/4$, see Figure 2 below.

On the other hand the ordinal sum of Π on $[1/2,1]^2$ is given by

$$C(x_1,x_2)=\begin{cases} 2x_1x_2-x_1-x_2+1 & \text{if } \langle x_1,x_2\rangle\in[1,2,1]^2\\ \min\{x_1,x_2\} & \text{otherwise.}\end{cases}$$

Of course, the gluing copula T_2 is different from C the ordinal sum.

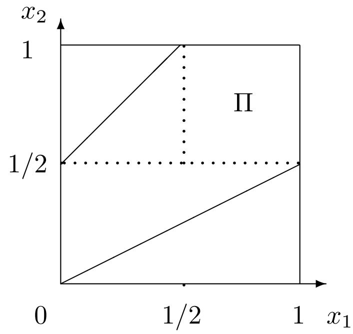

Fig. 2. Support of the gluing copula T_2.

5. FINAL REMARKS

In this paper we give a new multivariate patchwork construction of n-copulas using a base n-copula C and an n-box $R\subset[0,1]^n$ and a function $D:R\to[0,1]$ which is easily proved to be n-increasing on R, then we try to construct a modular function E on R such that $C=D+E$ on the frontier of R, $\delta(R)$. Of course, this procedure can be carried out in many different ways, and in this paper we explore just a few possibilities. However, other proposals may lead to different solutions that the one given in this paper.

The construction described in Section three for arbitrary n-boxes is sequential and it is more complicated than the one given in Durante et al. in [7]. The reason for this complication is related to the dimension n, and this problem is very common with multidimensional generalizations.

Of course we can construct new n-copulas composing our methodology with the gluing copulas of Siburg and Stoimenov [20] or its generalization given in Mesiar et al. [15], because as we noticed in our examples both constructions are not necessarily equivalent.

The study of tail behavior and the multivariate generalization of horizontal and vertical sections of dimension two can also be studied using our methodology, but it goes beyond the goal of this article.

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