KYBERNETIKA — VOLUME *49* (2013), NUMBER 1, PAGES 181–187

LINEAR FRACTIONAL PROGRAM UNDER INTERVAL AND ELLIPSOIDAL UNCERTAINTY

Maziar Salahi and Saeed Fallahi

In this paper, the robust counterpart of the linear fractional programming problem under linear inequality constraints with the interval and ellipsoidal uncertainty sets is studied. It is shown that the robust counterpart under interval uncertainty is equivalent to a larger linear fractional program, however under ellipsoidal uncertainty it is equivalent to a linear fractional program with both linear and second order cone constraints. In addition, for each case we have studied the dual problems associated with the robust counterparts. It is shown that in both cases, either interval or ellipsoidal uncertainty, the dual of robust counterpart is equal to the optimistic counterpart of dual problem.

Keywords: linear fractional program, robust optimization, uncertainty, second order cone

Classification: 90C05, 90C25, 90C32

1. INTRODUCTION

Optimization problems are widespread in real life decision making situations. However, the data uncertainty is almost invariably present and cannot be avoided. Uncertainty in the data means that the exact values of at least some parts of the data are not known at the time when the solution has to be determined. Research shows that quite small perturbations of the data can heavily affect the feasibility and/or optimality properties of the nominal optimal solution and thus make this solution meaningless, so it is more reliable to model and deal with a practical optimization problem as a problem that includes some uncertain data [2]. In order to handle optimization problems under uncertainty and perturbed optimization problems, several techniques have been proposed. The most common approach is the robust optimization which has received much attention in recent years [3, 4, 7].

Among optimization problems, linear fractional programming problem (LFP) have attracted considerable interest due to the their importance in various disciplines such as production planning, financial and corporate planning, health care and hospital planning [5, 8, 9, 12]. In this paper, we consider LFP under linear inequality constraints. Equivalent formulations of its robust counterpart when constraints are uncertain for, both interval and ellipsoidal uncertainty sets are discussed. Moreover, it is shown that in both cases, either interval or ellipsoidal uncertainty, the dual of robust counterpart is equal to the optimistic counterpart of dual problem. It should be noted that the interval uncertainty case has been studied in [13] (see page 299–302), but our proofs are using duality theorems and are simpler and shorter.

2. UNCERTAIN LFP AND ITS ROBUST COUNTERPARTS

Throughout this paper, we consider the following LFP:

$$\begin{aligned} \min \quad & \frac{a^T x + \alpha}{b^T x + \beta} \\ & \upsilon_i^T x \le \delta_i, \quad i = 1, 2, \ldots, m, \quad \text{(LFP)} \end{aligned}$$

where a, b, υ_i's are in $\mathbb{R}^n$and α, β, δ_i's are in $\mathbb{R}$. Moreover we assume $b^T x + \beta > 0$ in the feasible region. We associate (LFP) with its robust counterpart:

$$\begin{aligned} \min \quad & \frac{a^T x + \alpha}{b^T x + \beta} \\ & \upsilon_i^T x \le \delta_i, \quad \forall\, \upsilon_i \in \varepsilon_i, \quad i = 1, 2, \ldots, m, \quad \text{(RLFP)} \end{aligned}$$

where ε_i's are the so called given uncertainty sets.

Theorem 2.1. If υ_i's involve interval uncertainties, namely $\varepsilon_i = [\underline{\upsilon}_i, \overline{\upsilon}_i]$ for $i = 1, 2, \ldots, m$, where $\overline{\upsilon}_i, \underline{\upsilon}_i$'s are given vectors in $\mathbb{R}^n$, then (RLFP) is equivalent to

$$\begin{aligned} \min \quad & \frac{a^T x + \alpha}{b^T x + \beta} \\ & \overline{\upsilon}_i{}^T r_i - \underline{\upsilon}_i{}^T s_i \le \delta_i, \ i = 1, 2, \ldots, m, \\ & r_i - s_i = x, \ i = 1, 2, \ldots, m, \\ & r_i, s_i \ge 0, \ i = 1, 2, \ldots, m, \end{aligned} \tag{1}$$

which is a larger (LFP).

Proof. For a given $x \in \mathbb{R}^n$, to have $\upsilon_i^T x \le \delta_i, \ i = 1, 2, \ldots, m$ for all $\upsilon_i \in \varepsilon_i$, it is sufficient to have

$$\max_{\upsilon_i \in \varepsilon_i} \ \upsilon_i{}^T x \le \delta_i, \ \forall\, i = 1, 2, \ldots, m. \tag{2}$$

In [13] the authors use extreme points of the feasible region in (2) that are exponential, in order to give its equivalent model. They have replaced these constraints by exponential number of constraints. This makes the proof longer but we use duality approach which is shorter as follow. The dual of (2) for each $i = 1, 2, \ldots, m$ is

$$\begin{aligned} \min \quad & \overline{\upsilon}_i{}^T r_i - \underline{\upsilon}_i{}^T s_i \\ & r_i - s_i = x, \\ & r_i, s_i \ge 0, \end{aligned}$$

where r_i, s_i are in $\mathbb{R}^n$. On the other hand, by the weak duality for any feasible primal and dual solution we have

$$\upsilon_i^T x \le \overline{\upsilon}_i{}^T r_i - \underline{\upsilon}_i{}^T s_i.$$

Thus if we have

$$\begin{aligned} &\overline{\upsilon}_i{}^T r_i - \underline{\upsilon}_i{}^T s_i \le \delta_i, \\ &r_i - s_i = x, \\ &r_i, s_i \ge 0, \end{aligned} \tag{3}$$

then δ_i will be an upper bound for problem (2). Now by substituting (3) in (LFP) we get (1). □

Now it would be interesting to see the relation between the dual of the robust counterpart and the optimistic counterpart of the uncertain dual (LFP). Thus let us first consider the dual of (LFP) [6, 10, 11]:

$$\begin{aligned} \max \quad & \zeta \\ & b\zeta - \sum_{i=1}^{m} \upsilon_i \lambda_i = a, \\ & \beta\zeta + \sum_{i=1}^{m} \delta_i \lambda_i \le \alpha, \qquad \text{(DLFP)} \\ & \lambda_i \ge 0. \end{aligned}$$

We associate (DLFP) with its optimistic robust counterpart as follows [1]:

$$\begin{aligned} \max \quad & \zeta \\ & b\zeta - \sum_{i=1}^{m} \upsilon_i \lambda_i = a, \quad \text{for some} \quad \upsilon_i \in \varepsilon_i, \\ & \beta\zeta + \sum_{i=1}^{m} \delta_i \lambda_i \le \alpha, \qquad \text{(ORDLFP)} \\ & \lambda_i \ge 0. \end{aligned}$$

Theorem 2.2. The dual of the robust counterpart (1) and (ORDLFP) under the interval uncertainties are both given by

$$\begin{aligned} \max \quad & \zeta \\ & b\zeta - \sum_{i=1}^{m} \overline{\upsilon}_i \lambda_i \le a, \\ & b\zeta - \sum_{i=1}^{m} \underline{\upsilon}_i \lambda_i \ge a, \\ & \beta\zeta + \sum_{i=1}^{m} \delta_i \lambda_i \le \alpha, \\ & \lambda_i \ge 0. \end{aligned} \tag{4}$$

Proof. It is easy to see that the dual of (1) is exactly (4). On the other hand, in (ORDLFP) with $\varepsilon_i = [\underline{\upsilon}_i, \overline{\upsilon}_i] \ \ \forall i = 1, 2, \ldots, m$,

$$b\zeta - \sum_{i=1}^{m} \upsilon_i \lambda_i = a, \quad \text{for some} \quad \upsilon_i \in \varepsilon_i$$

is equivalent to

$$\begin{aligned} b\zeta - \sum_{i=1}^{m} \overline{\upsilon_i} \lambda_i &\leq a, \\ b\zeta - \sum_{i=1}^{m} \underline{\upsilon}_i \lambda_i &\geq a. \end{aligned} \tag{5}$$

Now by substituting (5) in (ORDLFP) we get (4). □

As one can see, dual of robust (LFP), namely (4) has much less constraints and variables compared to the robust (LFP). Thus it is reasonable to solve (4) instead of (1).

In what follows we discuss the case where problem data involve ellipsoidal uncertainties, namely

$$\upsilon_i \in \varepsilon_i = \{\upsilon_i^0 + P_i u_i, \|u_i\| \leq 1\}, \quad \forall \ i = 1, 2, \ldots, m,$$

where $||.||$ denotes the Euclidean norm and P_i's are matrices in $\mathbb{R}^{n \times n}$.

Theorem 2.3. The robust counterpart of (LFP) with ellipsoidal uncertainties is equivalent to the following conic linear optimization problem:

$$\begin{aligned} \min \quad & a^T y + \alpha z \\ & b^T y + \beta z = 1, \\ & \left\|P_i^T y\right\| \leq \delta_i z - {\upsilon_i^0}^T y, \ i = 1, 2, \ldots, m, \\ & z \geq 0. \end{aligned} \tag{6}$$

Proof. For a given $x \in \mathbb{R}^n$, to have ${\upsilon_i}^T x \leq \delta$ for all $\upsilon_i \in \varepsilon_i$, $\forall i = 1, 2, \ldots, m$, it is sufficient to have

$$\max_{\upsilon_i \in \varepsilon_i} \ \upsilon_i^T x \leq \delta_i, \ \forall i = 1, 2, \ldots, m. \tag{7}$$

Since $\upsilon_i^T x = {\upsilon_i^0}^T x + (P_i u_i)^T x$, thus

$$\max_{\upsilon_i \in \varepsilon_i} \upsilon_i^T x = \quad {\upsilon_i^0}^T x + \left\|P_i^T x\right\| \quad .$$

Therefore, (7) holds if

$$\left\|P_i^T x\right\| \leq \delta_i - {\upsilon_i^0}^T x, \ \forall i = 1, 2, \ldots, m. \tag{8}$$

Now by substituting (8) in (LFP), letting

$$y = \frac{x}{b^T x + \beta}, \quad z = \frac{1}{b^T x + \beta}$$

and using the Charnes–Cooper transformation we have (6). □

In the next theorem we discuss the relation between the dual of robust (LFP) and the optimistic counterpart of (DLFP) under the ellipsoidal uncertainties.

Theorem 2.4. The dual of the robust counterpart (6) and (ORDLFP) under the ellipsoidal uncertainties are both given by

$$\begin{aligned} \max \quad & \zeta \\ & b\zeta + \sum_{i=1}^{m} \left[P_i \eta_i - \sigma_i \upsilon_i^0\right] = a, \\ & \beta\zeta + \sum_{i=1}^{m} \sigma_i \delta_i \le b, \\ & \|\eta_i\| \le \sigma_i, \ i = 1, 2, \ldots, m. \end{aligned} \tag{9}$$

Proof. The dual of (6) is exactly (9). On the other hand, in (ORDLFP) for each $i = 1, 2, \ldots, m$,

$$b\zeta - \textstyle\sum_{i=1}^{m} \upsilon_i \lambda_i = a, \quad \text{for some } \upsilon_i \in \varepsilon_i = \{\upsilon_i{}^0 + P_i u_i, \|u_i\| \le 1\}$$

is equivalent to

$$\begin{aligned} & b\zeta - a + \sum_{i=1}^{m} \left(\min\left(P_i u_i \lambda_i\right) - \upsilon_i^0 \lambda_i\right) \ge 0, \\ & b\zeta - a + \sum_{i=1}^{m} \left(\max\left(P_i u_i \lambda_i\right) - \upsilon_i^0 \lambda_i\right) \le 0 \end{aligned}$$

or

$$b\zeta + \sum_{i=1}^{m} \left[P_i \eta_i - \lambda_i \upsilon_i^0\right] = a, \tag{10}$$

where $\eta_i = u\lambda_i$. Moreover

$$\|\eta_i\| \le \|u_i\| \, \lambda_i \le \lambda_i. \tag{11}$$

Thus by substituting (10) and (11) in (ORDLFP) we get (9). □

The case in which the uncertainty is considered in objective function coefficients, can be expressed in an equivalent way as uncertain parameters appear in the inequality constraints and thus the previous techniques can be easily applied.

3. CONCLUSIONS

In this paper, the robust counterpart of linear fractional programs for both interval and ellipsoidal uncertainty sets are discussed. Depending on the uncertainty sets, it leads to an equivalent linear fractional program with large number of linear constraints or second order cone constraints. Moreover, it is shown that the dual of robust problem is the same as the optimistic counterpart of dual problem. These results allow to solve the robust dual problem instead of original robust version since it has smaller number of constraints.

4. ACKNOWLEDGEMENT

The authors would like to thanks reviewers for their constructive comments on the early version of this paper.

(Received January 31, 2012)

REFERENCES

[1] A. Beck and A. Ben-Tal: Duality in robust optimization: primal worst equals dual best. Oper. Res. Lett. *37* (2009), 1–6.

[2] A. Ben-Tal and A. Nemirovski: Robust solutions of linear programming problems contaminated with uncertain data. Math. Programming *88* (2000), 411–424.

[3] A. Ben-Tal and A. Nemirovski: Robust solutions of uncertain linear programs. Oper. Res. Lett. *25* (1999), 1–13.

[4] A. Ben-Tal and A. Nemirovski: Robust convex optimization. Math. Oper. Res. *23* (1998), 769–805.

[5] A. Charnes and W. W. Cooper: Programming with linear fractional functional. Naval Res. Logist. Quart. *9* (1962), 181–186.

[6] A. Chinchuluun, D. Yuan, and P. M Pardalos: Optimality conditions and duality for nondifferentiable multiobjective fractional programming with generalized convexity. Ann. Oper. Res. *154* (2007), 133–147.

[7] D. Bertsimas, D. Pachamanova, and M. Sim: Robust linear optimization under general norms. Oper. Res. Lett. *32* (2004), 510–516.

[8] G. R. Bitran and A. J. Novaes: Linear programming with a fractional objective function. Oper. Res. *21* (1973), 22–29.

[9] P. M. Pardalos and A. Phillips: Global optimization of fractional programs. J. Global Optim. *1* (1991), 173–182.

[10] S. Schaible: Fractional programming a recent survey, Generalized convexity, generalized monotonicity, optimality conditions and duality in scalar and vector optimization. J. Statist. Management Syst. *5* (2002), 63–86.

[11] S. Schaible: Parameter-free convex equivalent and dual programs of fractional programming problems. Oper. Res. *18* (1974), 187–196.

[12] T. Gómez, M. Hernández, M. A. León, and R. Caballero: A forest planning problem solved via a linear fractional goal programming model. Forest Ecol. Management *227* (2006), 79–88.

[13] V. Jeyakumar and G. Y. Li: Robust duality for fractional programming problems with constraint-wise data uncertainty. J. Optim. Theory Appl. *151* (2011), 292–303.

Maziar Salahi, Department of Applied Mathematics, Faculty of Mathematical Sciences, University of Guilan, Rasht. Iran.
e-mail: salahim@guilan.ac.ir

Saeed Fallahi, Department of Applied Mathematics, Faculty of Mathematical Sciences, University of Guilan, Rasht. Iran.
e-mail: saeedf808@gmail.com