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EXPONENTIAL ENTROPY ON INTUITIONISTIC FUZZY SETS

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In the present paper, based on the concept of fuzzy entropy, an exponential intuitionistic fuzzy entropy measure is proposed in the setting of Atanassov's intuitionistic fuzzy set theory. This measure is a generalized version of exponential fuzzy entropy proposed by Pal and Pal. A connection between exponential fuzzy entropy and exponential intuitionistic fuzzy entropy is also established. Some interesting properties of this measure are analyzed. Finally, a numerical example is given to show that the proposed entropy measure for Atanassov's intuitionistic fuzzy set is consistent by comparing it with other existing entropies.

Keywords: fuzzy set, fuzzy entropy, Atanassov's intuitionistic fuzzy set, intuitionistic fuzzy entropy, exponential entropy

Classification: 94A17

1. INTRODUCTION

The theory of fuzzy sets proposed by Zadeh [16] in 1965 has gained wide applications in many areas of science and technology e. g. clustering, image processing, decision making etc. because of its capability to model non-statistical imprecision or vague concepts. Fuzziness brings in a feature of uncertainty. The first attempt to quantify the fuzziness was made in 1968 by Zadeh [17], who introduced a probabilistic framework and defined the entropy of a fuzzy event as weighted Shannon entropy [11] (but this measure was not found adequate for measuring the fuzziness of a fuzzy event). In 1972, De Luca and Termini [5] formulated axioms which the fuzzy entropy measure should comply, and they defined the entropy of a fuzzy set based on Shannon's function. It may be regarded as the first correct measure of fuzziness of a fuzzy set.

Atanassov [1] introduced the notion of 'Atanassov's intuitionistic fuzzy set', which is a generalization of the concept of fuzzy set. Burillo and Bustince [3] defined the entropy on Atanassov's intuitionistic fuzzy set and on interval-valued fuzzy set. Vlachos and Sergiagis [13] proposed a measure of intuitionistic fuzzy entropy and revealed an intuitive and mathematical connection between the notions of entropy for fuzzy set and Atanassov's intuitionistic fuzzy set. Zhang and Jiang [18] defined a measure of intuitionistic (vague) fuzzy entropy on Atanassov's intuitionistic fuzzy sets by generalizing of the De Luca Termini [5] logarithmic fuzzy entropy.

In this paper, we propose a new information measure for Atanassov's intuitionistic fuzzy sets. We call it exponential intuitionistic fuzzy entropy. It is based on the concept of exponential fuzzy entropy defined by Pal and Pal [9]. To define this entropy function fuzzy set theoretic approach has been used. Such an approach is found particularly useful in situations where data is available in terms of intuitionistic fuzzy set values but implementation requirements are only fuzzy. So far the practice has been to simply ignore the hesitation part. A better result has been obtained by not ignoring but by merging the hesitation part suitably. We suggest a mathematical method for it. This may help application of IFS data in industry, where the tools used are of fuzzy set theory.

The paper is organized as follows: In Section 2 some basic definitions related to probability, fuzzy set theory and Atanassov's intuitionistic fuzzy set theory are briefly discussed. In Section 3 a new information measure called, '*exponential intuitionistic fuzzy entropy*' is proposed, which satisfies the axiomatic requirements [12]. Some mathematical properties of the proposed measure are then studied in this section. In Section 4 a numerical example is given comparing our measure with other entropies proposed in [14] and [18].

2. PRELIMINARIES

In this section we present some basic concepts related to probability theory, fuzzy sets and Atanassov's intuitionistic fuzzy sets, which will be needed in the following analysis.

First, let us cover probabilistic part of the preliminaries.

Let $\Delta_n = \{P = (p_1, \dots, p_n) : p_i \geq 0, \sum_{i=1}^{n} p_i = 1\}$, $n \geq 2$ be a set of n-complete probability distributions.

For any probability distribution $P = (p_1, \dots, p_n) \in \Delta_n$, Shannon's entropy [11], is defined as

$$H(P) = -\sum_{i=1}^{n} p_i \log p_i. \tag{1}$$

It is to be noted from the logarithmic entropic measure (1) that as $p_i \to 0$, it's corresponding self information of this event, $I(p_i) = -log(p_i) \to \infty$ but $I(p_i = 1) = -log(1) = 0$. Thus we see that self information of an event has conceptual problem, as in practice, the self information of an event, whether highly probable or highly unlikely, is expected to lie between two finite limits.

Some advantages for considering exponential entropy: In Shannon's theory, which is widely acclaimed, we find that the measure of self information of an event with probability p_i is taken as $log(1/p_i)$, a decreasing function of p_i. The same decreasing character alternatively may be maintained by considering it as a function of $(1 - p_i)$ rather than of $(1/p_i)$.

The additive property, which is considered crucial in Shannon's approach, of the self information function for independent events may not have a strong relevance (impact) in practice in some situations. Alternatively, as in the case of probability law, the joint self information may be product rather than sum of the self informations in two independent cases.

The above considerations suggest the self information as an exponential function of $(1 - p_i)$.

Based on the these considerations, Pal and Pal [9] proposed another measure called exponential entropy given by

$$_{e}H(P) = \sum_{i=1}^{n} p_i e^{(1-p_i)} - 1. \tag{2}$$

These authors point out that the exponential entropy has an advantage over Shannon's entropy. For example, for the uniform probability distribution $P = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})$, exponential entropy has a fixed upper bound

$$\lim_{n\to\infty} H\Big(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\Big) = e - 1 \tag{3}$$

which is not the case for Shannon's entropy.

Definition 2.1. *Fuzzy Set*: A fuzzy set $\tilde{A}$ defined in a finite universe of discourse $X = (x_1, \dots, x_n)$ is given by (Zadeh [16]):

$$\tilde{A} = \{\langle x, \mu_{\tilde{A}}(x)\rangle \,|x \in X\}, \tag{4}$$

where $\mu_{\tilde{A}} : X \to [0, 1]$ is the membership function of $\tilde{A}$. The number $\mu_{\tilde{A}}(x)$ describes the degree of membership of $x \in X$ to $\tilde{A}$.

De Luca and Termini [5] defined fuzzy entropy for a fuzzy set $\tilde{A}$ corresponding (1) as

$$H(\tilde{A}) = -\frac{1}{n}\sum_{i=1}^{n}\Big[\mu_{\tilde{A}}(x_i)\log\mu_{\tilde{A}}(x_i) + \big(1-\mu_{\tilde{A}}(x_i)\big)\log\big(1-\mu_{\tilde{A}}(x_i)\big)\Big]. \tag{5}$$

Fuzzy exponential entropy for fuzzy set $\tilde{A}$ corresponding to (2) has also been introduced by Pal and Pal [9] as

$$_{e}H(\tilde{A}) = \frac{1}{n(\sqrt{e}-1)}\sum_{i=1}^{n}\Big[\mu_{\tilde{A}}(x_i)e^{1-\mu_{\tilde{A}}(x_i)} + (1-\mu_{\tilde{A}}(x_i))e^{\mu_{\tilde{A}}(x_i)} - 1\Big]. \tag{6}$$

Further, Atanassov [1] generalized the idea of fuzzy sets, by what is called Atanassov's intuitionistic fuzzy sets, defined as follows:

Definition 2.2. *Atanassov's Intuitionistic Fuzzy Set*: An Atanassov's intuitionistic fuzzy set A in a finite universe of discourse $X = (x_1, \dots, x_n)$ is given by:

$$A = \big\{\langle x, \mu_A(x), \nu_A(x)\rangle \,|x \in X\big\}, \tag{7}$$

where

$$\mu_A : X \to [0, 1] \text{ and } \nu_A : X \to [0, 1] \tag{8}$$

with the condition

$$0 \le \mu_A(x) + \nu_A(x) \le 1, \qquad \forall\, x \in X. \tag{9}$$

The numbers $\mu_A(x)$ and $\nu_A(x)$ denote the *degree of membership* and *degree of non-membership* of $x \in X$ to A, respectively.

Definition 2.3. *Hesitation Margin*: For each Atanassov's intuitionistic fuzzy set A in X, if

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x), \tag{10}$$

then $\pi_A(x)$ is called the Atanassov's intuitionistic index (or a hesitation degree) of the element $x \in X$ to A.

For studying sets, there is need to consider set relations and operations, which in the study of Atanassov's intuitionistic fuzzy sets are defined as follows.

Definition 2.4. *Set Operations on Atanassov's Intuitionistic Fuzzy Set*[2]: Let $AIFS(X)$ denote the family of all Atanassov's intuitionistic fuzzy sets in the universe X, and let $A, B \in AIFS(X)$ given by

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \,|x \in X\},$$
$$B = \{\langle x, \mu_B(x), \nu_B(x)\rangle \,|x \in X\},$$

then some set operations can be defined as follows:

(i) $A \subseteq B$ iff $\mu_A(x) \le \mu_B(x)$ and $\nu_A(x) \ge \nu_B(x) \quad \forall\, x \in X$;

(ii) $A = B$ iff $A \subseteq B$ and $B \subseteq A$;

(iii) $A^C = \{\langle x, \nu_A(x), \mu_A(x)\rangle \,|x \in X\}$;

(iv) $A \cup B = \{\langle x, (\mu_A(x) \vee \mu_B(x)), (\nu_A(x) \wedge \nu_B(x))\rangle \,|x \in X\}$;

(v) $A \cap B = \{\langle x, (\mu_A(x) \wedge \mu_B(x)), (\nu_A(x) \vee \nu_B(x))\rangle \,|x \in X\}$;

(vi) $\Box A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle \,|x \in X\}$;

(vii) $\Diamond A = \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle \,|x \in X\}$;

(viii) $A@B = \left\{ \left\langle x, \frac{\mu_A + \mu_B}{2}, \frac{\nu_A + \nu_B}{2} \right\rangle |x \in X \right\}$.

Method for Transforming AIFSs into FSs: Li, Lu and Cai [8], as briefly outlined below, proposed a method for transforming 'Atanassov's intuitionistic fuzzy sets' (vague sets) into 'fuzzy sets' by distributing hesitation degree equally with membership and non-membership.

Definition 2.5. Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \,|x \in X\}$ be an Atanassov's intuitionistic fuzzy set defined in a finite universe of discourse X. Then the fuzzy membership function $\mu_{\tilde{A}^*}(x)$ to $\tilde{A}^*$ ($\tilde{A}^*$ be the fuzzy set corresponding to Atanassov's intuitionistic fuzzy set A) is defined as:

$$\mu_{\tilde{A}^*}(x) = \mu_A(x) + \frac{\pi_A(x)}{2} = \frac{\mu_A(x) + 1 - \nu_A(x)}{2}. \tag{11}$$

This area of study has attracted quite some attention for applications in decision-making. Finally we may as well mention some other related measures with which we compare our study.

Zhang and Jiang [18] presented a measure of intuitionistic (vague) fuzzy entropy based on a generalization of measure (5) as

$$E_{ZJ}(A) = -\frac{1}{n}\sum_{i=1}^{n}\left[\left(\frac{\mu_A(x_i)+1-\nu_A(x_i)}{2}\right)\log\left(\frac{\mu_A(x_i)+1-\nu_A(x_i)}{2}\right) + \left(\frac{\nu_A(x_i)+1-\mu_A(x_i)}{2}\right)\log\left(\frac{\nu_A(x_i)+1-\mu_A(x_i)}{2}\right)\right]. \quad (12)$$

Ye [15] introduced two effective measures of intuitionistic fuzzy entropy based on a generalization of the fuzzy entropy defined by Prakash et al. [10] given by

$$E_{JY1}(A) = \frac{1}{n}\sum_{i=1}^{n}\left[\left\{\sin\pi\left(\frac{\mu_A(x_i)+1-\nu_A(x_i)}{4}\right) + \sin\pi\left(\frac{\nu_A(x_i)+1-\mu_A(x_i)}{4}\right) - 1\right\} \times \frac{1}{\sqrt{2}-1}\right], \quad (13)$$

$$E_{JY2}(A) = \frac{1}{n}\sum_{i=1}^{n}\left[\left\{\cos\pi\left(\frac{\mu_A(x_i)+1-\nu_A(x_i)}{4}\right) + \cos\pi\left(\frac{\nu_A(x_i)+1-\mu_A(x_i)}{4}\right) - 1\right\} \times \frac{1}{\sqrt{2}-1}\right]. \quad (14)$$

Later, Wei et al. [14] have shown that the two entropy functions (13) and (14) proposed by Ye [15] are mathematically the same and gave a simplified version as

$$E_{WGG}(A) = \frac{1}{n}\sum_{i=1}^{n}\left[\left\{\sqrt{2}\cos\pi\left(\frac{\mu_A(x_i)-\nu_A(x_i)}{4}\right) - 1\right\} \times \frac{1}{\sqrt{2}-1}\right]. \quad (15)$$

Throughout this paper, we denote the set of all Atanassov's intuitionistic fuzzy sets in X by $AIFS(X)$. Similarly, $FS(X)$ is the set of all fuzzy sets defined in X.

In the next section we introduce an entropy measure on Atanassov's intuitionistic fuzzy sets called "*exponential intuitionistic fuzzy entropy*" corresponding to (6) and verify axiomatic basis of the same.

3. EXPONENTIAL INTUITIONISTIC FUZZY ENTROPY

Let A be an Atanassov's intuitionistic fuzzy set defined in the finite universe of discourse, $X = (x_1, \ldots, x_n)$. Then, according to the Definition 2.5, an Atanassov's intuitionistic fuzzy set can be transformed into a fuzzy set to structure an entropy measure of the intuitionistic fuzzy set by means of

$$\mu_{\tilde{A}^*}(x_i) = \mu_A(x_i) + \frac{\pi_A(x_i)}{2} = \frac{\mu_A(x_i)+1-\nu_A(x_i)}{2}.$$

Then, in analogy with the definition of exponential fuzzy entropy given in (6), we propose the exponential intuitionistic fuzzy entropy measure for Atanassov's intuitionistic fuzzy set A as follows:

$$ {}_eE(A) = \frac{1}{n(\sqrt{e}-1)} \sum_{i=1}^{n} \Bigg[\left(\frac{\mu_A(x_i) + 1 - \nu_A(x_i)}{2} \right) e^{1-\left(\frac{\mu_A(x_i)+1-\nu_A(x_i)}{2}\right)} + \left(1 - \frac{\mu_A(x_i) + 1 - \nu_A(x_i)}{2} \right) e^{\left(\frac{\mu_A(x_i)+1-\nu_A(x_i)}{2}\right)} - 1 \Bigg], \tag{16} $$

which can also be written as

$$ {}_eE(A) = \frac{1}{n(\sqrt{e}-1)} \sum_{i=1}^{n} \Bigg[\left(\frac{\mu_A(x_i) + 1 - \nu_A(x_i)}{2} \right) e^{\left(\frac{\nu_A(x_i)+1-\mu_A(x_i)}{2}\right)} + \left(\frac{\nu_A(x_i) + 1 - \mu_A(x_i)}{2} \right) e^{\left(\frac{\mu_A(x_i)+1-\nu_A(x_i)}{2}\right)} - 1 \Bigg]. \tag{17} $$

In the next theorem, we establish properties that according to Szmidt and Kacprzyk [12], justify our proposed measure to be a bonafide/valid 'intuitionistic fuzzy entropy':

Theorem 3.1. The ${}_eE(A)$ measure in (17) of the exponential intuitionistic fuzzy entropy satisfies the following propositions:

P1. ${}_eE(A) = 0$ iff A is a crisp set, i. e., $\mu_A(x_i) = 0, \nu_A(x_i) = 1$ or $\mu_A(x_i) = 1, \nu_A(x_i) = 0$ for all $x_i \in X$.

P2. ${}_eE(A) = 1$ iff $\mu_A(x_i) = \nu_A(x_i)$ for all $x_i \in X$.

P3. ${}_eE(A) = {}_eE(A)$ iff $A \le B$, i. e., $\mu_A(x_i) \le \mu_B(x_i)$ and $\nu_A(x_i) \ge \nu_B(x_i)$, for $\mu_B(x_i) \le \nu_B(x_i)$ or $\mu_A(x_i) \ge \mu_B(x_i)$ and $\nu_A(x_i) \le \nu_B(x_i)$, for $\mu_B(x_i) \ge \nu_B(x_i)$ for any $x_i \in X$.

P4. ${}_eE(A) = {}_eE(A^C)$.

Proof. ***P1.*** Let A be a crisp set with membership values being either 0 or 1 for all $x_i \in X$. Then from (17) we simply obtain that

$$ {}_eE(A) = 0. \tag{18} $$

Now, let

$$ \frac{\mu_A(x_i) + 1 - \nu_A(x_i)}{2} = z_A(x_i). \tag{19} $$

In view of (19), expression in (17) can be written as

$$ {}_eE(A) = \frac{1}{n(\sqrt{e}-1)} \sum_{i=1}^{n} \Big[z_A(x_i) e^{1-z_A(x_i)} + (1 - z_A(x_i)) e^{z_A(x_i)} - 1 \Big]. \tag{20} $$

From Pal and Pal [9], we know that (20) becomes zero if and only if $z_A(x_i) = 0$ or 1, $\forall\, x_i \in X$ i. e.,

$$\frac{(\mu_A(x_i) + 1 - \nu_A(x_i))}{2} = 0 \text{ i. e., } \nu_A(x_i) - \mu_A(x_i) = 1,\ \forall\, x_i \in X \tag{21}$$

or

$$\frac{(\mu_A(x_i) + 1 - \nu_A(x_i))}{2} = 1 \text{ i. e., } \mu_A(x_i) - \nu_A(x_i) = 1,\ \forall\, x_i \in X. \tag{22}$$

And from Definition 2.2,we have

$$\mu_A(x_i) + \nu_A(x_i) \le 1,\ \forall\, x_i \in X. \tag{23}$$

Now solving equation (21) with (23), we get

$$\mu_A(x_i) = 0, \nu_A(x_i) = 1,\ \forall\, x_i \in X.$$

Next solving equation (22) with (23), we get

$$\mu_A(x_i) = 1, \nu_A(x_i) = 0,\ \forall\, x_i \in X.$$

Therefore ${}_eE(A)$ reduces to zero only if either $\mu_A(x_i) = 0$, $\nu_A(x_i) = 1$ or $\mu_A(x_i) = 1$, $\nu_A(x_i) = 0$ for all $x_i \in X$, proving the result.

P2. Let $\mu_A(x_i) = \nu_A(x_i)$ for all $x_i \in X$. From (17) we obtain ${}_eE(A) = 1$.
From equation (20), we have

$${}_eE(A) = \frac{1}{n}\sum_{i=1}^{n} f(z_A(x_i)),$$

where

$$f(z_A(x_i)) = \Big[\frac{z_A(x_i)e^{1-z_A(x_i)} + (1 - z_A(x_i))e^{z_A(x_i)} - 1}{(\sqrt{e} - 1)}\Big] \quad \forall\, x_i \in X. \tag{24}$$

Now, let us suppose that ${}_eE(A) = 1$, i. e.

$$\frac{1}{n}\sum_{i=1}^{n} f(z_A(x_i)) = 1$$

or

$$f(z_A(x_i)) = 1 \quad \forall\, x_i \in X. \tag{25}$$

Differentiating (25) with respect to $z_A(x_i)$ and equating to zero, we get

$$\frac{\partial f}{\partial(z_A(x_i))} = e^{1-z_A(x_i)} - z_A(x_i)e^{1-z_A(x_i)} - e^{z_A(x_i)} + (1 - z_A(x_i))e^{z_A(x_i)} = 0$$

or

$$(1 - z_A(x_i))e^{1-z_A(x_i)} = z_A(x_i)e^{z_A(x_i)} \quad \forall\, x_i \in X. \tag{26}$$

Using the fact that $f(x) = xe^x$ is a bijection function, we can write

$$(1 - z_A(x_i)) = z_A(x_i) \quad \forall\, x_i \in X \tag{27}$$

or

$$z_A(x_i) = 0.5 \quad \forall\, x_i \in X \tag{28}$$

and find

$$\left[\frac{\partial^2 f}{\partial(z_A(x_i))^2}\right]_{z_A(x_i)=0.5} < 0 \quad \forall\, x_i \in X. \tag{29}$$

Hence $f(z_A(x_i))$ is a concave function and has a global maximum at $z_A(x_i) = 0.5$. Since ${}_eE(A) = \frac{1}{n}\sum_{i=1}^{n} f(z_A(x_i))$, So ${}_eE(A)$ attains the maximum value when $z_A(x_i) = 0.5$ or $\mu_A(x_i) = \nu_A(x_i)$ for all $x_i \in X$.

P3. In order to show that (17) fulfills *P3*, it suffices to prove that the function

$$g(x,y) = \left[\left(\frac{x+1-y}{2}\right)e^{\left(\frac{y+1-x}{2}\right)} + \left(\frac{y+1-x}{2}\right)e^{\left(\frac{x+1-y}{2}\right)} - 1\right] \tag{30}$$

where $x, y \in [0,1]$, is increasing with respect to x and decreasing for y. Taking the partial derivatives of g with respect to x and y, respectively, yields

$$\frac{\partial g}{\partial x} = \frac{1}{2}\left[\left(\frac{y+1-x}{2}\right)e^{\left(\frac{y+1-x}{2}\right)} - \left(\frac{x+1-y}{2}\right)e^{\left(\frac{x+1-y}{2}\right)}\right] \tag{31}$$

$$\frac{\partial g}{\partial y} = \frac{1}{2}\left[\left(\frac{x+1-y}{2}\right)e^{\left(\frac{x+1-y}{2}\right)} - \left(\frac{y+1-x}{2}\right)e^{\left(\frac{y+1-x}{2}\right)}\right] \tag{32}$$

In order to find critical point of g, we set $\frac{\partial g}{\partial x} = 0$ and $\frac{\partial g}{\partial y} = 0$. This gives

$$x = y. \tag{33}$$

From (31) and (33), we have

$$\frac{\partial g}{\partial x} \geq 0, \qquad \text{when } x \leq y \tag{34}$$

and

$$\frac{\partial g}{\partial x} \leq 0, \qquad \text{when } x \geq y \tag{35}$$

for any $x, y \in [0,1]$, Thus $g(x,y)$ is increasing with respect to x for $x \leq y$ and decreasing when $x \geq y$.

Similarly, we obtain that

$$\frac{\partial g}{\partial y} \leq 0, \qquad \text{when } x \leq y \tag{36}$$

and

$$\frac{\partial g}{\partial y} \geq 0, \qquad \text{when } x \geq y. \tag{37}$$

Let us now consider two sets $A, B \in IFS(X)$ with $A \subseteq B$. Assume that the finite universe of discourse $X = (x_1, \ldots, x_n)$ is partitioned into two disjoint sets X_1 and X_1 with $X_1 \cup X_2$.
Let us further suppose that all $x_i \in X_1$ are dominated by the condition

$$\mu_A(x_i) \leq \mu_B(x_i) \leq \nu_B(x_i) \leq \nu_A(x_i),$$

while for all $x_i \in X_2$

$$\mu_A(x_i) \geq \mu_B(x_i) \geq \nu_B(x_i) \geq \nu_A(x_i).$$

Then from the monotonicity of $g(x, y)$ and (17), we obtain that ${}_eE(A) \leq {}_eE(B)$ when $A \subseteq B$.

P4. It is clear that $A^C = \{\langle x, \nu_A(x_i), \mu_A(x_i)\rangle \,|x \in X\}$ for all $x_i \in X$, i. e.,

$$\mu_{A^C}(x_i) = \nu_A(x_i) \; and \; \nu_{A^C}(x_i) = \mu_A(x_i)$$

then, from (17) we have

$${}_eE(A) = {}_eE(A^C).$$

Hence ${}_eE(A)$ is a valid measure of Atanassov's intuitionistic fuzzy entropy.

This proves the theorem. □

Particular case: It is interesting to notice that if an Atanassov's intuitionistic fuzzy set is an ordinary fuzzy set, i. e., for all $x_i \in X$, $\nu_A(x_i) = 1 - \mu_A(x_i)$, then the exponential intuitionistic fuzzy entropy reduces to exponential fuzzy entropy as proposed in [9].

We now turn to study of properties of ${}_eE(A)$. The proposed exponential intuitionistic fuzzy entropy ${}_eE(A)$, just like fuzzy entropy measure, satisfies the following interesting properties.

Theorem 3.2. Let A and B two Atanassov's intuitionistic fuzzy sets in a finite universe of discourse $X = (x_1, \ldots, x_n)$, where $A(x_i) = \langle \mu_A(x_i), \nu_A(x_i)\rangle$, $B(x_i) = \langle \mu_B(x_i), \nu_B(x_i)\rangle$ such that they satisfy for any $x_i \in X$ either $A \subseteq B$ or $A \supseteq B$, then we have

$${}_eE(A \cup B) + {}_eE(A \cap B) = {}_eE(A) + {}_eE(B).$$

P r o o f. Let us separate X into two parts X_1 and X_2, where

$$X_1 = \{x_i \in X : A(x_i) \subseteq B(x_i)\} \text{ and } X_2 = \{x_i \in X : A(x_i) \supseteq B(x_i)\}.$$

That is, for all $x_i \in X_1$

$$\mu_A(x_i) \leq \mu_B(x_i) \text{ and } \nu_A(x_i) \geq \nu_B(x_i) \tag{38}$$

and for all $x_i \in X_2$

$$\mu_A(x_i) \geq \mu_B(x_i) \text{ and } \nu_A(x_i) \leq \nu_B(x_i). \tag{39}$$

From definition in (17), we have

$$
\begin{aligned}
{}_eE(A\cup B) = \frac{1}{n(\sqrt{e}-1)}\sum_{i=1}^{n}\Bigg[\left(\frac{\mu_{A\cup B}(x_i)+1-\nu_{A\cap B}(x_i)}{2}\right)e^{\left(\frac{\nu_{A\cap B}(x_i)+1-\mu_{A\cup B}(x_i)}{2}\right)} \\
+\left(\frac{\nu_{A\cap B}(x_i)+1-\mu_{A\cup B}(x_i)}{2}\right)e^{\left(\frac{\mu_{A\cup B}(x_i)+1-\nu_{A\cap B}(x_i)}{2}\right)}-1\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
= \frac{1}{n(\sqrt{e}-1)}\Bigg[\Bigg\{\sum_{x_i\in X_1}\left(\frac{\mu_B(x_i)+1-\nu_B(x_i)}{2}\right)e^{\left(\frac{\nu_B(x_i)+1-\mu_B(x_i)}{2}\right)} \\
+\left(\frac{\nu_B(x_i)+1-\mu_B(x_i)}{2}\right)e^{\left(\frac{\mu_B(x_i)+1-\nu_B(x_i)}{2}\right)}-1\Bigg\} \\
+\Bigg\{\sum_{x_i\in X_2}\left(\frac{\mu_A(x_i)+1-\nu_A(x_i)}{2}\right)e^{\left(\frac{\nu_A(x_i)+1-\mu_A(x_i)}{2}\right)} \\
+\left(\frac{\nu_A(x_i)+1-\mu_A(x_i)}{2}\right)e^{\left(\frac{\mu_A(x_i)+1-\nu_A(x_i)}{2}\right)}-1\Bigg\}\Bigg]. \quad (40)
\end{aligned}
$$

Again from definition in (17), we have

$$
\begin{aligned}
{}_eE(A\cap B) = \frac{1}{n(\sqrt{e}-1)}\sum_{i=1}^{n}\Bigg[\left(\frac{\mu_{A\cap B}(x_i)+1-\nu_{A\cup B}(x_i)}{2}\right)e^{\left(\frac{\nu_{A\cup B}(x_i)+1-\mu_{A\cap B}(x_i)}{2}\right)} \\
+\left(\frac{\nu_{A\cup B}(x_i)+1-\mu_{A\cap B}(x_i)}{2}\right)e^{\left(\frac{\mu_{A\cap B}(x_i)+1-\nu_{A\cup B}(x_i)}{2}\right)}-1\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
= \frac{1}{n(\sqrt{e}-1)}\Bigg[\Bigg\{\sum_{x_i\in X_1}\left(\frac{\mu_A(x_i)+1-\nu_A(x_i)}{2}\right)e^{\left(\frac{\nu_A(x_i)+1-\mu_A(x_i)}{2}\right)} \\
+\left(\frac{\nu_A(x_i)+1-\mu_A(x_i)}{2}\right)e^{\left(\frac{\mu_A(x_i)+1-\nu_A(x_i)}{2}\right)}-1\Bigg\} \\
+\Bigg\{\sum_{x_i\in X_2}\left(\frac{\mu_B(x_i)+1-\nu_B(x_i)}{2}\right)e^{\left(\frac{\nu_B(x_i)+1-\mu_B(x_i)}{2}\right)} \\
+\left(\frac{\nu_B(x_i)+1-\mu_B(x_i)}{2}\right)e^{\left(\frac{\mu_B(x_i)+1-\nu_B(x_i)}{2}\right)}-1\Bigg\}\Bigg]. \quad (41)
\end{aligned}
$$

Now adding (40) and (41), we get

$$
{}_eE(A\cup B)+{}_eE(A\cap B)={}_eE(A)+{}_eE(B).
$$

This proves the theorem. □

Theorem 3.3. For every $A \in AIFS(X)$,

(i) $\Box A @ \Diamond A$ is a fuzzy set;

(ii) $(\Box A @ \Diamond A) = (\Diamond A @ \Box A)$;

(iii) $(\Box A @ \Diamond A) = \tilde{A}^*$;

where $\tilde{A}^*$ is the fuzzy set corresponding to Atanassov's intuitionistic fuzzy set A.

Proof. **(i)** From Definition 2.4, we have

$$\Box A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle \, | x \in X\}; \tag{42}$$
$$\Diamond A = \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle \, | x \in X\}. \tag{43}$$

Now taking @ with (42) and (43), we get

$$\Box A @ \Diamond A = \left\{ \left\langle x, \frac{\mu_A + 1 - \nu_A}{2}, \frac{\nu_A + 1 - \mu_A}{2} \right\rangle | x \in X \right\}. \tag{44}$$

It can be easily observed that,

$$\frac{\mu_A + 1 - \nu_A}{2} + \frac{\nu_A + 1 - \mu_A}{2} = 1 \quad x \in X.$$

(ii) It obviously follows from equation (44).

(iii) From equations (44) and (11), we have

$$\Box A @ \Diamond A = \left\{ \left\langle x, \frac{\mu_A + 1 - \nu_A}{2}, \frac{\nu_A + 1 - \mu_A}{2} \right\rangle | x \in X \right\};$$
$$\tilde{A}^* = \left\{ \left\langle x, \frac{\mu_A + 1 - \nu_A}{2}, \frac{\nu_A + 1 - \mu_A}{2} \right\rangle | x \in X \right\}.$$

This proves the theorem. $\Box$

Theorem 3.4. For every $A \in AIFS(X)$,

$${}_eE(A) = {}_eE(\Box A @ \Diamond A).$$

Proof. From equation (17), we have

$${}_eE(A) = \frac{1}{n(\sqrt{e} - 1)} \sum_{i=1}^{n} \Bigg[\left(\frac{\mu_A(x_i) + 1 - \nu_A(x_i)}{2} \right) e^{\left(\frac{\nu_A(x_i) + 1 - \mu_A(x_i)}{2} \right)} + \left(\frac{\nu_A(x_i) + 1 - \mu_A(x_i)}{2} \right) e^{\left(\frac{\mu_A(x_i) + 1 - \nu_A(x_i)}{2} \right)} - 1 \Bigg]$$

and

$$ {}_eE(\Box A@\Diamond A) = \frac{1}{n(\sqrt{e}-1)} \sum_{i=1}^{n} \Bigg[\left(\frac{\mu_A(x_i) + 1 - \nu_A(x_i)}{2} \right) e^{\left(\frac{\nu_A(x_i)+1-\mu_A(x_i)}{2} \right)} + \left(\frac{\nu_A(x_i) + 1 - \mu_A(x_i)}{2} \right) e^{\left(\frac{\mu_A(x_i)+1-\nu_A(x_i)}{2} \right)} - 1 \Bigg]. \quad (45) $$

This proves the theorem. □

Theorem 3.5. For every $A \in AIFS(X)$,

$$ {}_eE(\Box A@\Diamond A) = {}_eE(\Diamond A@\Box A). $$

P r o o f. It readily follows from Theorem 3.3(ii)and equation (45). □

Theorem 3.6. For every $A \in AIFS(X)$,

$$ {}_eE(\Box A@\Diamond A) = {}_eE(((\Box A)^C @(\Diamond A)^C)^C). $$

P r o o f. From equation (45), we have

$$ {}_eE(\Box A@\Diamond A) = \frac{1}{n(\sqrt{e}-1)} \sum_{i=1}^{n} \Bigg[\left(\frac{\mu_A(x_i) + 1 - \nu_A(x_i)}{2} \right) e^{\left(\frac{\nu_A(x_i)+1-\mu_A(x_i)}{2} \right)} + \left(\frac{\nu_A(x_i) + 1 - \mu_A(x_i)}{2} \right) e^{\left(\frac{\mu_A(x_i)+1-\nu_A(x_i)}{2} \right)} - 1 \Bigg] $$

and

$$ {}_eE(((\Box A)^C @(\Diamond A)^C)^C) = \frac{1}{n(\sqrt{e}-1)} \sum_{i=1}^{n} \Bigg[\left(\frac{\mu_A(x_i) + 1 - \nu_A(x_i)}{2} \right) e^{\left(\frac{\nu_A(x_i)+1-\mu_A(x_i)}{2} \right)} + \left(\frac{\nu_A(x_i) + 1 - \mu_A(x_i)}{2} \right) e^{\left(\frac{\mu_A(x_i)+1-\nu_A(x_i)}{2} \right)} - 1 \Bigg]. $$

This proves the theorem. □

In the next section we consider an example to compare our proposed entropy measure on Atanassov's intuitionistic fuzzy set, with others in (12) and (15).

4. NUMERICAL EXAMPLE

Example: Let $A = \{\langle x_i, \mu_A(x_i), \nu_A(x_i)\rangle \,|x_i \in X\}$ be an AIFS in $X = (x_1, \ldots, x_n)$. For any positive real number n, De et al. [6] defined the AIFS A^n as follows:

$$ A^n = \{\langle x_i, [\mu_A(x_i)]^n, 1 - [1 - \nu_A(x_i)]^n\rangle \,|x_i \in X\}. $$

We consider the AIFS A on $X = (x_1, \ldots, x_n)$ defined as:

$$ A = \{\langle 6, 0.1, 0.8\rangle, \langle 7, 0.3, 0.5\rangle, \langle 8, 0.5, 0.4\rangle, \langle 9, 0.9, 0.0\rangle, \langle 10, 1.0, 0.0\rangle\}. $$

By taking into consideration the characterization of linguistic variables, De et al. [6] regarded A as "LARGE" on X. Using the above operations, we have

$A^{1/2}$ for may be treated as "More or less LARGE"
A^2 for may be treated as "Very LARGE"
A^3 for may be treated as "Quite very LARGE"
A^4 for may be treated as "Very very LARGE"

Now we consider these AIFSs to compare the above entropy measures. It may be mentioned that from logical consideration, the entropies of these AIFSs are required to follow the following order pattern:

$$E(A^{1/2}) > E(A) > E(A^2) > E(A^3) > E(A^4). \tag{46}$$

Calculated numerical values of the three entropy functions for these cases are given in the table below:

	$A^{1/2}$	A	A^2	A^3	A^4
E_{ZJ}	0.5819	0.5720	0.4333	0.3321	0.2698
E_{WGG}	0.4545	0.4377	0.3029	0.2159	0.1709
E_e	0.5531	0.5343	0.3772	0.2734	0.2169

Table: Values of the different entropy measures under $A^{1/2}, A, A^2, A^3, A^4$.

Based on the Table, we see that the entropy measures E_{ZJ} and E_{WGG} satisfy (46), and our proposed entropy measure conforms to the same, i. e.

$${}_eE(A^{1/2}) > {}_eE(A) > {}_eE(A^2) > {}_eE(A^3) > {}_eE(A^4).$$

Therefore, the behavior of exponential intuitionistic fuzzy entropy ${}_eE(A)$ is also consistent for the viewpoint of structured linguistic variables.

5. CONCLUSIONS

In this work, we have proposed a new entropy measure called exponential intuitionistic fuzzy entropy in the setting of Atanassov's intuitionistic fuzzy set theory. This measure can be considered as a generalized version of exponential fuzzy entropy proposed by Pal and Pal [10]. This measure is imbued with several properties. A numerical example is given to illustrate the effectiveness of proposed entropy measure. Parametric studies that introduce other flexibility criteria for the same membership functions, of this measure are also under study and will be reported separately.

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