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NOVEL METHOD FOR GENERALIZED STABILITY ANALYSIS OF NONLINEAR IMPULSIVE EVOLUTION EQUATIONS

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In this paper, we discuss some generalized stability of solutions to a class of nonlinear impulsive evolution equations in the certain piecewise essentially bounded functions space. Firstly, stabilization of solutions to nonlinear impulsive evolution equations are studied by means of fixed point methods at an appropriate decay rate. Secondly, stable manifolds for the associated singular perturbation problems with impulses are compared with each other. Finally, an example on initial boundary value problem for impulsive parabolic equations is illustrated to our theory results.

Keywords: impulsive evolution equations, stabilization, stable manifolds, singularly perturbed problems

Classification: 34G20, 35B40, 35K20

1. INTRODUCTION

In order to describe dynamics of populations subject to abrupt changes as well as other phenomena such as harvesting, diseases, and so forth, some authors have used impulsive differential systems to describe the model since the last century. The theory of impulsive differential equations is emerging as an important area of investigation since it is much richer than the corresponding theory of differential equations. For the basic theory on impulsive differential equations in infinite dimensional spaces or finite dimensional spaces, the reader can see the monographs of Benchohra et al. [4] and Lakshmikantham et al. [11]. In the past decades, existence of (periodic) solutions, controllability and optimal controls for impulsive evolution equations are studied by many researchers such as Abada et al. [1], Ahmed et al. [3, 2], Chang and Nieto [6], Fan and Li [8], Hernández [9], Liang et al. [12], Liu [13], Wang et al. [20, 18], Wei et al. [19, 21], Xiang et al. [22], Yu et al. [26] and so on.

As we known, all kinds of stability including strict stability, exponential stability and asymptotic stability of impulsive differential equations are attracted by many researchers such as Xu et al. [23] Yang and Xu, [25], Zhang and Sun [27] and Dvirnyĭ and Slyn'ko [7]. In the present paper, we reconsider this interesting problem and discuss generalized stability of solutions to nonlinear impulsive evolution equations by utilizing the operator semigroup theory and fixed point methods in a piecewise essentially bounded functions space with specific behavior as time tends to infinity. More precisely, we will combine theses earlier works [5, 10, 18] and extend the study of exponential stability to the generalized stability of solutions to Cauchy problems for nonlinear impulsive evolution equations

$$\begin{cases} u'(t) + (A+B)u(t) = 0, \ t > 0, \ t \neq \tau_k, \\ u(0) = u_0, \\ \triangle u(t) = I_k(u(t)), \ t = \tau_k, \end{cases} \tag{1}$$

where $A : D(A) \subset X \to X$ is a linear unbounded operator in a Banach space X, $B : X \to X$ is a nonlinear operator and u_0 is a given element of X. Impulsive time sequence $\{\tau_k\}_{k=0}^{\infty}$ satisfies $0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_k \ldots$, $\lim_{k\to\infty} \tau_k = \infty$. $\triangle u(\tau_k) = u(\tau_k^+) - u(\tau_k^-)$, $I_k : X \to X$, $u(\tau_k^+) = \lim_{h\to 0^+} u(\tau_k + h)$ and $u(\tau_k^-) = \lim_{h\to 0^-} u(t_k + h)$ represent respectively the right and left limits of $u(t)$ at $t = \tau_k$.

Meanwhile, we study the stable manifolds for parameter dependent problem with impulse

$$\begin{cases} u_\varepsilon'(t) + (\varepsilon A + B)u_\varepsilon(t) = 0, \ t > 0, \ t \neq \tau_k, \\ u_\varepsilon(0) = x, \ \varepsilon \in [0, \varepsilon_0], \ \varepsilon_0 > 0, \ x \in X, \\ \triangle u_\varepsilon(t) = I_k(u_\varepsilon(t)), \ t = \tau_k, \end{cases} \tag{2}$$

and study the stabilization of solutions of the parameter dependent problem with impulse

$$\begin{cases} \varepsilon u_\varepsilon'(t) + (A + B)u_\varepsilon(t) = 0, \ t > 0, \ t \neq \tau_k, \\ u_\varepsilon(0) = x, \ \varepsilon \in [0, \varepsilon_0], \ \varepsilon_0 > 0, \ x \in X, \\ \triangle u_\varepsilon(t) = I_k(u_\varepsilon(t)), \ t = \tau_k. \end{cases} \tag{3}$$

To achieve our purpose, we split the present nonlinear impulsive problem into a linearized impulsive part and nonlinear impulsive perturbation assuming the existence of a stable equilibrium by normalization placed at zero. Then, we invert the linear impulsive part and search for a stabilizing solution as a fixed point of the corresponding integral operator with impulsive terms.

This paper is organized as follows. We start in Section 2 by using the Banach contraction principle in the function space $PL^\infty(0, \infty; X)$, the generalized solution $u : [0, \infty) \to X$ of the system (1) which tending to zero at an appropriate decay rate $w(t)^{-1}$ as $t \to \infty$. The same technique is applied in Section 3 to a singular perturbed problem with impulse (2) making possible to establish at least locally the inclusion $\Omega_0 \subset \Omega_\varepsilon$ between stable manifolds for $\varepsilon = 0$ and $\varepsilon > 0$ small enough. In Section 4, some stability results of the system (1) are proved with the help of growth restriction at infinity for nonlinear perturbation and impulsive terms by using the Schauder fixed point theorem in the space $PL^\infty(0, \infty; X)$. In Section 5, we give the briefly of stabilization of solution to the problem (3). At last, an example is given to demonstrate the applicability of our results.

2. STABILITY RESULTS VIA BANACH CONTRACTION PRINCIPLE

Throughout this paper, we adopt the usual notation $L^p(J; X)$ for the L^p-spaces of functions form a set $J \subset R^n$ into X, $C^k(J; X)$ for the spaces of functions with continuous derivative up to the order k, $L(X, Y)$ for the space of the continuous linear operators from X into Y with $L(X, X) = L(X)$, and so on. By $B_r(0; X)$ we denote the ball centered at zero with the radius r in X.

In order to establish the stability of the stationary point we shall make use of the Banach contraction principle in the space of functions $u:[0,\infty)\to X$ which decrease in an appropriate rate as $t\to\infty$.

Denote $PL^\infty(0,\infty;X)=\{u:[0,\infty)\to X : u\in L^\infty(\tau_k,\tau_{k+1};X)$ and there exist $u(\tau_k^-)$ and $u(\tau_k^+)$ with $u(\tau_k^-)=u(\tau_k),\ k=0,1,2,\ldots\}$, $PL^\infty(0,\infty;X)$ is a Banach space with norm $\|u\|_\infty=\operatorname{ess\,sup}_{t\geq 0}\|u(t)\|<\infty$. For $w\in L^\infty_{loc}(0,\infty)$, we introduce the following work space

$$PL^\infty_w(0,\infty;X)=\left\{u\in PL^\infty(0,\infty;X) : \|u\|_w:=\operatorname*{ess\,sup}_{t\geq 0} w(t)\|u(t)\|<\infty\right\}. \tag{4}$$

It is easy to see that the space $PL^\infty_w(0,\infty;X)$ is a Banach space under the norm $\|\cdot\|_w$.

We make the following assumptions:

(H_1) $-A$ is a generator of a C_0-semigroup $\{T(t),t\geq 0\}$ of bounded linear operators in X.

(H_2) Let $B=D-F$ where $D\in L(X)$, $F:X\to X$ with $F(0)=0$ and $I_k:X\to X$ with $I_k(0)=0,\ k=0,1,2,\ldots$.

(H_3) The semigroup $\{\widetilde{T}(t),t\geq 0\}$ generated by $-(A+D)$ satisfies the estimate $\|\widetilde{T}(t)\|\leq \widetilde{w}(t),\ t\geq 0$, with some $\widetilde{w}\in L^\infty_{loc}(0,\infty)$.

(H_4) There exists $r_0>0$ and continuous function $\lambda:[0,r_0)\to R^+$ with $\lambda(0)=0$ such that

$$\|F(u)-F(v)\|\leq\lambda(r)\|u-v\| \text{ and } \|I_k(u)-I_k(v)\|\leq\lambda(r)\|u-v\|$$

for any $r\in(0,r_0)$ and $u,v\in B_r(0;X)$.

(H_5) Let

$$\begin{aligned}\mu(r) &= \sup_{t>0} w(t)\Big[\int_0^t \widetilde{w}(t-s)\lambda(w(s)^{-1}r)w(s)^{-1}\,\mathrm{d}s \\ &\qquad + \sum_{0<\tau_k<t}\widetilde{w}(t-\tau_k)\lambda(w(\tau_k)^{-1}r)w(\tau_k)^{-1}\Big] < \infty\end{aligned}$$

for $r\in(0,r_0]$, with $\limsup_{r\to 0^+}\mu(r)<1$ for some functions $w\in L^\infty_{loc}(0,\infty)$ such that $w(t)\geq 1$ a.e. in $(0,\infty)$ and $\lim_{t\to\infty}w(t)=\infty$.

We say that system (1) has a generalized solution if there is a function $u\in PL^\infty(0,\infty;X)$ such that

$$u(t)=\widetilde{T}(t)u_0+\int_0^t\widetilde{T}(t-s)F(u(s))\,\mathrm{d}s+\sum_{0<\tau_k<t}\widetilde{T}(t-\tau_k)I_k(u(\tau_k)),\ t>0.$$

In the following lemma we prove that the operator

$$\begin{aligned}G(u)(t) &= \int_0^t\widetilde{T}(t-s)F(u(s))\,\mathrm{d}s \\ &\quad+\sum_{0<\tau_k<t}\widetilde{T}(t-\tau_k)I_k(u(\tau_k)),\ u\in B_r(0;PL^\infty_w(0,\infty;X))\end{aligned} \tag{5}$$

is well defined and maps $B_r(0; PL_w^\infty(0,\infty;X))$ into itself if $r > 0$ is sufficiently small.

Lemma 2.1. Let assumptions (H_1)–(H_5) be satisfied. Then there exists $r_1 \in (0, r_0]$ such that for any $r \in (0, r_1]$ the operator G defined by (5) maps $B_r(0; PL_w^\infty(0,\infty;X))$ into itself and

$$\|G(u) - G(v)\|_w \le \mu(r)\|u - v\|_w$$

for $u, v \in B_r(0; PL_w^\infty(0,\infty;X))$, where $\mu(r) < 1$.

Proof. We first check that $G : B_r(0; PL_w^\infty(0,\infty;X)) \to B_r(0; PL_w^\infty(0,\infty;X))$. In fact, for $u, v \in B_r(0; PL_w^\infty(0,\infty;X))$, by (5), (H_3) and (H_4) we have

$$\begin{aligned}
& w(t)\|G(u)(t)\| \\
\le\ & w(t)\int_0^t \|\widetilde{T}(t-s)\|\|F(u(s))\|\,\mathrm{d}s + w(t)\sum_{0<\tau_k<t}\|\widetilde{T}(t-\tau_k)\|\|I_k(u(\tau_k))\| \\
\le\ & w(t)\int_0^t \widetilde{w}(t-s)\lambda(w(s)^{-1}r)w(s)^{-1}\|u\|_w\,\mathrm{d}s \\
& + w(t)\sum_{0<\tau_k<t}\widetilde{w}(t-\tau_k)\lambda(w(\tau_k)^{-1}r)w(\tau_k)^{-1}\|u\|_w \\
\le\ & w(t)\left[\int_0^t \widetilde{w}(t-s)\lambda(w(s)^{-1}r)w(s)^{-1}\,\mathrm{d}s + \sum_{0<\tau_k<t}\widetilde{w}(t-\tau_k)\lambda(w(\tau_k)^{-1}r)w(\tau_k)^{-1}\right] r \\
\le\ & \mu(r)r.
\end{aligned}$$

Analogously we get

$$w(t)\|G(u)(t) - G(v)(t)\| \quad \le \quad \mu(r)\|u - v\|_w.$$

The proof is completed. □

Define an operator H by

$$H(u)(t) = \widetilde{T}(t)u_0 + G(u)(t),\ u \in B_r(0; PL_w^\infty(0,\infty;X)),\ t > 0,\ r \in (0, r_0). \tag{6}$$

If

$$\limsup_{t\to\infty} w(t)\widetilde{w}(t) < \infty \tag{7}$$

then H maps $B_r(0; PL_w^\infty(0,\infty;X))$ into $PL_w^\infty(0,\infty;X)$.

We have the following result.

Theorem 2.2. Let assumptions (H_1)–(H_5) be satisfied and let (7) hold. For $u_0 \in X$, $\|u_0\|$ is sufficiently small, then the operator H defined by (6) has a unique fixed point in the ball $B_r(0; PL_w^\infty(0,\infty;X))$ for $r > 0$ small enough. This fixed point is a generalized solution of (1), and it satisfies

$$\|u(t)\| \le r w(t)^{-1} \ for\ t \ge 0.$$

In another word, we have $\lim_{t\to\infty} u(t) = 0$ in X.

P r o o f. It comes from (7) that there exists $\tilde{c} > 0$ such that $w(t)\widetilde{w}(t) \le \tilde{c}$, for $t \ge 0$. We choose $r > 0$ in Lemma 2.1 so that $\lambda(r) \le \frac{1}{2}$ and $\mu(r) < 1$. Then for $u, v \in B_r(0; PL^\infty_w(0,\infty;X))$ and $\tilde{c}\|u_0\| \le \frac{1}{2}r$ we have

$$\begin{aligned} w(t)\|H(u)(t)\| &\le w(t)\|\widetilde{T}(t)u_0\| + w(t)\|G(u)(t)\| \\ &\le w(t)\widetilde{w}(t)\|u_0\| + \lambda(r)r \\ &\le \tilde{c}\|u_0\| + \frac{1}{2}r \\ &\le \frac{1}{2}r + \frac{1}{2}r \\ &= r, \end{aligned}$$

and

$$\begin{aligned} w(t)\|H(u)(t) - H(v)(t)\| &= w(t)\|G(u)(t) - G(v)(t)\| \\ &\le \mu(r)\|u - v\|_w. \end{aligned}$$

Thus H maps $B_r(0; PL^\infty_w(0,\infty;X))$ into itself and is contractive therein. Thus, we can apply the Banach contraction principle in $B_r(0; PL^\infty_w(0,\infty;X))$ to obtain the result.

□

3. STABLE MANIFOLDS FOR A SINGULAR PERTURBED PROBLEM WITH IMPULSE

In this section we will consider the stable manifolds associated with the problem (2), i. e., the set

$$\Omega_\varepsilon = \{x \in X : u_\varepsilon(t) \to 0 \text{ in } X \text{ as } t \to \infty\},$$

where u_ε is the generalized solution of the problem (2) which dependents on x.

The aim of this section is to derive conditions under which there exists $\varepsilon^* > 0$ such that for any $\varepsilon \in (0, \varepsilon^*)$ we have $\Omega_0 \subset \Omega_\varepsilon$ at least locally. To achieve this we shall establish convergence $u_\varepsilon(t) \to u_0(t)$ in X as $\varepsilon \to 0^+$ pointwise on compact intervals, and universal stability of $u = 0$ for $\varepsilon \in (0, \varepsilon^*)$, which means that there exists $r > 0$ such that $u_\varepsilon(t) \to 0$ in X as $t \to \infty$ whenever $\|x\| \le r$ and $\varepsilon \in (0, \varepsilon^*)$.

In what follows we shall make some assumptions.

(H_a) Let assumptions (H_1), (H_2) and (H_5) of Section 2 be satisfied. Further, denote by $\tilde{w}_\varepsilon$ a $L^\infty_{loc}(0,\infty)$ function such that the semigroup $\{\tilde{T}_\varepsilon(t), t \ge 0\}$ generated by $-(\varepsilon A + D)$ satisfies the estimate $\|\tilde{T}_\varepsilon(t)\| \le \tilde{w}_\varepsilon(t),\ t \ge 0,\ \varepsilon \in (0, \varepsilon_0]$, and introduce a family of weight functions $w_\varepsilon \in L^\infty_{loc}(0,\infty)$ such that $w_\varepsilon(t) \ge 1$ a.e. in $(0,\infty)$ and $\lim_{t\to\infty} w_\varepsilon(t) = \infty$, $\varepsilon \in (0, \varepsilon_0]$.

(H_b) There exist $r_1 \in (0, r_0]$ and $\kappa \in (0,1)$ such that we have

$$\begin{aligned} \mu_\varepsilon(r) &= \sup_{t\in R^+} w_\varepsilon(t)\Big[\int_0^t \tilde{w}_\varepsilon(s)(t-s)\lambda(w_\varepsilon(s)^{-1}r)\tilde{w}_\varepsilon(s)^{-1} \\ &\quad + \sum_{0<\tau_k<t} \tilde{w}_\varepsilon(t-\tau_k)\lambda(w_\varepsilon(\tau_k)^{-1}r)w_\varepsilon(\tau_k)^{-1}\Big] \le \kappa \end{aligned}$$

for $r \in (0, r_1]$, $\varepsilon \in (0, \varepsilon_0]$.

Lemma 3.1. Let assumptions (H_1) and (H_2) of Section 2 be satisfied. In addition, let us assume that F, I_k are locally Lipschitz, and there exists a family of generalized solutions $u_\varepsilon(t)$ of the problem (2) which is uniformly bounded in $PL^\infty(0,t_0)$, $t_0>0$ with respect to $\varepsilon\in(0,\varepsilon_0]$, and that there is a constant $\tilde{c}>0$ such that $\tilde{w}_\varepsilon(t)\le\tilde{c}$ for $t\in[0,t_0],\varepsilon\in(0,\varepsilon_0]$. Then

$$u_\varepsilon(t)\to u_0(t) \text{ in } X \text{ as } \varepsilon\to 0^+ \text{ for all } t\in[0,t_0].$$

Proof. Firstly, using the same method in Lemma 3.1 of [10], for any $x\in X$ and any $t>0$ we have $\lim_{\varepsilon\to 0^+}\|\tilde{T}_\varepsilon(t)x-\tilde{T}_0(t)x\|=0$. Similarly, for any $I_k(u_0(\tau_k))\in X$ and any $t>0$, we have $\lim_{\varepsilon\to 0^+}\|\tilde{T}_\varepsilon(t-\tau_k)I_k(u_0(\tau_k))-\tilde{T}_0(t-\tau_k)I_k(u_0(\tau_k))\|=0$. Secondly, the generalized solution $u_\varepsilon(t)$ of the problem (2) satisfies the relation

$$u_\varepsilon(t)=\tilde{T}_\varepsilon(t)x+\int_0^t\tilde{T}_\varepsilon(t-s)F(u_\varepsilon(s))\,\mathrm{d}s+\sum_{0<\tau_k<t}\tilde{T}_\varepsilon(t-\tau_k)I_k(u_\varepsilon(\tau_k)).$$

By subtraction we obtain

$$\begin{aligned}u_\varepsilon(t)-u_0(t) &= \tilde{T}_\varepsilon(t)x-\tilde{T}_0(t)x+\int_0^t[\tilde{T}_\varepsilon(t-s)F(u_\varepsilon(s))-\tilde{T}_0(t-s)F(u_0(s))]\,\mathrm{d}s\\ &\quad+\sum_{0<\tau_k<t}[\tilde{T}_\varepsilon(t-\tau_k)I_k(u_\varepsilon(\tau_k))-\tilde{T}_0(t-\tau_k)I_k(u_0(\tau_k))].\end{aligned}$$

This, the uniform boundedness of u_ε and the fact that F and I_k are locally Lipschitz yield

$$\begin{aligned}\|u_\varepsilon(t)-u_0(t)\| &\le \|\tilde{T}_\varepsilon(t)x-\tilde{T}_0(t)x\|+\int_0^t\|\tilde{T}_\varepsilon(t-s)F(u_0(s))-\tilde{T}_0(t-s)F(u_0(s))\|\,\mathrm{d}s\\ &\quad+\int_0^t\|\tilde{T}_\varepsilon(t-s)\|\|F(u_\varepsilon(s))-F(u_0(s))\|\,\mathrm{d}s\\ &\quad+\sum_{0<\tau_k<t}\|\tilde{T}_\varepsilon(t-\tau_k)I_k(u_0(\tau_k))-\tilde{T}_0(t-\tau_k)I_k(u_0(\tau_k))\|\\ &\quad+\sum_{0<\tau_k<t}\|\tilde{T}_\varepsilon(t-\tau_k)\|\|I_k(u_\varepsilon(\tau_k))-I_k(u_0(\tau_k))\|\\ &\le a_\varepsilon(t)+C_1\int_0^t\|u_\varepsilon(s)-u_0(s)\|\,\mathrm{d}s+C_2\sum_{0<\tau_k<t}\|u_\varepsilon(\tau_k)-u_0(\tau_k)\|\end{aligned}$$

where

$$a_\varepsilon(t):=\|\tilde{T}_\varepsilon(t)x-\tilde{T}_0(t)x\|+\sum_{0<\tau_k<t}\|\tilde{T}_\varepsilon(t-\tau_k)I_k(u_0(\tau_k))-\tilde{T}_0(t-\tau_k)I_k(u_0(\tau_k))\|$$

with $\lim_{\varepsilon\to 0^+}a_\varepsilon(t)=0$, and C_1, C_2 are constants. Applying the well known impulsive Gronwall inequality (see Lemma 1.7.1, Yang [24]), we have $\lim_{\varepsilon\to 0^+}\|u_\varepsilon(t)-u_0(t)\|=0$. □

Next, we will discuss the universal stability of the stationary solution $u=0$.

Theorem 3.2. Let assumptions (H_a) and (H_b) be satisfied and let

$$w_\varepsilon(t)\tilde{w}_\varepsilon(t) \le \tilde{C} < \infty \text{ for all } t \in [0,\infty) \text{ and } \varepsilon \in (0,\varepsilon_0], \tag{8}$$

with a constant $\tilde{C}$ independent of t and ε. Then there exists $R > 0$ such that for any $x \in X$ with $\|x\| \le R$ the generalized solution $u_\varepsilon(t)$ of the problem (2) converges to 0 as $t \to \infty$ for all $\varepsilon \in (0,\varepsilon_0]$.

P r o o f. As in Section 2, we intend to apply the Banach contraction principle to the operator H_ε given by

$$H_\varepsilon(u)(t) = \tilde{T}_\varepsilon(t)x + \int_0^t \tilde{T}_\varepsilon(t-s)F(u_\varepsilon(s))\,\mathrm{d}s + \sum_{0<\tau_k<t} \tilde{T}_\varepsilon(t-\tau_k)I_k(u_\varepsilon(\tau_k)). \tag{9}$$

It suffices to show that there is $R > 0$ such that if $x \in X, \|x\| \le R$ and $\varepsilon \in (0,\varepsilon_0]$, then there exists $r > 0$ such that H_ε maps the ball $B_r(0; PL^\infty_w(0,\infty;X))$ into itself and is a contraction in that ball.

So, let $R > 0$, $\|x\| \le R$, $r \in (0, r_1]$ and $u, v \in B_r(0; PL^\infty_w(0,\infty;X))$. Then, for $R \le \frac{(1-\kappa)r}{\tilde{C}}$, we have

$$\begin{aligned} & w_\varepsilon(t)\|H_\varepsilon(u)(t)\| \\ \le\ & w_\varepsilon(t)\tilde{w}_\varepsilon(t)R + w_\varepsilon(t)\left[\int_0^t \tilde{w}_\varepsilon(t-s)\lambda(w_\varepsilon(s)^{-1}r)w_\varepsilon(s)^{-1}\,\mathrm{d}s\right]\|u\|_{w_\varepsilon} \\ & + w_\varepsilon(t)\left[\sum_{0<\tau_k<t} \tilde{w}_\varepsilon(t-\tau_k)\lambda(w_\varepsilon(\tau_k)^{-1}r)w_\varepsilon(\tau_k)^{-1}\right]\|u\|_{w_\varepsilon} \\ \le\ & \tilde{C}R + \mu_\varepsilon(r)r \\ \le\ & \tilde{C}R + \kappa r \\ \le\ & r, \end{aligned}$$

and consequently $\|H_\varepsilon(u)\|_{w_\varepsilon} \le r$.

Similarly we have

$$w_\varepsilon(t)\|H_\varepsilon(u)(t) - H_\varepsilon(v)(t)\| \quad \le \quad \kappa\|u-v\|_{w_\varepsilon}$$

and the assertion easily follows. □

The following main theorem of this section gives a local comparison result.

Theorem 3.3. Let assumptions (H_a), (H_b) and (8) be satisfied. In addition, let F and I_k be locally Lipschitz and for any $x \in X$ and any $t_0 > 0$ let there exist a family of generalized solutions $u_\varepsilon(t)$ of the problem (2) which is uniformly bounded in $PL^\infty(0,t_0)$ with respect to $\varepsilon \in (0,\varepsilon_0]$. Then for any $x \in \Omega_0$ there exists an $\varepsilon^* \in (0,\varepsilon_0]$ such that $\lim_{\varepsilon\to 0^+} u_\varepsilon(t) = 0$, $\varepsilon \in (0,\varepsilon^*)$, $t \in [0,t_0]$ for corresponding solutions $u_\varepsilon(t)$ of the problem (2).

Proof. If $x \in \Omega_0$ then $\lim_{t\to\infty} u_0(t) = 0$ in X, where $u_0(t)$ is a generalized solution of the problem (2) with $\varepsilon = 0$. Take $R > 0$ whose existence is guaranteed by Theorem 3.2 and find $t_0 > 0$ such that $\|u_0(t)\| \le \frac{1}{2}R$ for all $t \ge t_0$. Let $u_\varepsilon(t)$, $\varepsilon \in (0, \varepsilon_0]$ be the family of generalized solutions of (2). In view of our assumptions Lemma 3.1 may be applied on the interval $(0, t_0]$ to obtain that, in particular, there exists $\varepsilon^* \in (0, \varepsilon_0]$ such that $\|u_\varepsilon(t_0) - u_0(t_0)\| \le \frac{1}{2}R$ for $\varepsilon \in (0, \varepsilon^*)$. Thus we get $\|u_\varepsilon(t_0)\| \le \|u_0(t_0)\| + \|u_\varepsilon(t_0) - u_0(t_0)\| \le \frac{1}{2}R + \frac{1}{2}R = R$. But taking $x = u_\varepsilon(t_0)$ by Theorem 3.2 we can construct a solution $u_\varepsilon(t_0)$ of the problem (2), which converges to 0 as $t \to \infty$ for all $\varepsilon \in (0, \varepsilon^*)$. Since F and I_k are locally Lipschitz continuous, one can the uniqueness of the generalized solution. Further, by uniqueness, we get $v_\varepsilon(t) = u_\varepsilon(t_0 + t)$ so that $\lim_{t\to\infty} u_\varepsilon(t) = 0$ in X for $\varepsilon \in (0, \varepsilon^*)$ as well. □

4. STABILITY RESULTS VIA SCHAUDER FIXED POINT THEOREM

We recall that system (1) has a generalized solution if there is a function $u \in PL^\infty(0, \infty; X)$ such that

$$u(t) = T(t)u_0 - \int_0^t T(t-s)Bu(s)\,\mathrm{d}s + \sum_{0<\tau_k<t} T(t-\tau_k)I_k(u(\tau_k)),\ t > 0. \tag{10}$$

To show that under appropriate assumptions $u(t) \to 0$ as $t \to \infty$ in X we shall make use of the Schauder fixed point theorem to find the solution of (10) in an appropriate Banach space of functions $u : R^+ \to X$ for which $u(t) \to 0$ in X as $t \to \infty$.

In the sequel we shall need the compactness criterion in $PL^\infty(0, \infty; X)$, which we prove for reader's convenience.

Lemma 4.1. A set $\mathcal{K} \subset PL^\infty(0, \infty; X)$ is relatively compact in $PL^\infty_w(0, \infty; X)$ if the following conditions hold:
(i) there is a set $M \subset (\tau_k, \tau_{k+1})$ of measure zero such that for any $t \in (\tau_k, \tau_{k+1}) \setminus M$ the orbit $X_t = \{f(t) : f \in \mathcal{K}\}$ is relatively compact in X; $X_{\tau_k^+} = \{f(\tau_k + 0) : f \in \mathcal{K}\}$ and $X_{\tau_k^-} = \{f(\tau_k - 0) : f \in \mathcal{K}\}$ are relatively compact in X;
(ii) for any $\varepsilon > 0$ there is a finite partition of (τ_k, τ_{k+1}) into measurable sets $A_1, \ldots, A_n$ such that $\operatorname{ess\,sup}_{s,t\in A_j} \|w(s)f(s) - w(t)f(t)\| < \varepsilon$ for all $j \in 1, \ldots, n$ and all $f \in \mathcal{K}$.

Proof. Without loss of generality, we only need to show that $\|f - f_k\|_w < \varepsilon$ where $\{f_k(t)\}$ is a subsequence of $\mathcal{K}$ for $t \in (\tau_k, \tau_{k+1})$ and $f \in \mathcal{K}$. For $\varepsilon > 0$, there is a measurable partition $A_1 \ldots A_n$ of (τ_k, τ_{k+1}) and a set $\mathcal{B} \subset (\tau_k, \tau_{k+1})$ of measure zero such that $\|w(s)f(s) - w(t)f(t)\| < \varepsilon$ whenever $t, s \in A_j \setminus \mathcal{B}$ for some j and $f \in \mathcal{K}$. We may assume that $A_j \setminus (M \cup \mathcal{B})$ is nonempty for all j. Choose points $t_j \in A_j \setminus (M \cup \mathcal{B})$ for $j \in 1, \ldots, n$ and define a map $P : \mathcal{K} \to X^n$ by $P(f) = (w(t_1)f(t_1), \ldots, w(t_n)f(t_n)$ for all $f \in \mathcal{K}$. The set $P(\mathcal{K})$ is relatively compact in X being a subset of a relatively compact set $w(t_1)X_{t_1} \times \cdots \times w(t_n)X_{t_n}$. Let $P(f_1), \ldots, P(f_p)$ be a ε-net for $P(\mathcal{K})$ with respect to the norm $\|(x_1, \ldots, x_n)\| = \max(\|x_1\|, \ldots, \|x_n\|)$. We show that $f_1, \ldots, f_p$ is an ε-net for $\mathcal{K}$. Let $f \in \mathcal{K}$. There is $k \in 1, \ldots, p$ such that $\|P(f) - P(f_k)\| < \varepsilon$. Given $t \in (0, \infty) \setminus (M \cup \mathcal{B})$, there is $j \in 1, \ldots, n$ such that $t \in A_j \setminus (M \cup \mathcal{B})$. For $t \in (\tau_k, \tau_{k+1})$, we have

$$\begin{aligned}\|w(t)f(t)-w(t)f_k(t)\| &\le \|w(t)f(t)-w(t_j)f(t_j)\|+\|w(t_j)f(t_j)-w(t_j)f_k(t_j)\|\\ &\quad+\|w(t_j)f_k(t_j)-w(t)f_k(t)\|\\ &\le \varepsilon+\|P(f)-P(f_k)\|+\varepsilon\\ &< 3\varepsilon,\end{aligned}$$

which shows that $\|f-f_k\|_w<3\varepsilon$. The proof is completed. □

Assume that

(E_1) $B=D-F$ where $D\in L(X)$, F and I_k are locally Lipschitz with $F(0)=0$ and $I_k(0)=0$.

Lemma 4.2. Let (E_1) holds and let the weight function w be such that

$$\limsup_{t\to\infty} w(t)\left[\int_0^t \widetilde{w}(t-s)w(s)^{-1}\,\mathrm{d}s+\sum_{0<\tau_k<t}\widetilde{w}(t-\tau_k)w(\tau_k)^{-1}\right]<\infty. \tag{11}$$

Then the operator G defined by (5) maps $PL^\infty_w(0,\infty;X)$ into itself and is locally Lipschitz.

P r o o f. By (11) we have

$$w(t)\left[\int_0^t \widetilde{w}(t-s)w(s)^{-1}\,\mathrm{d}s+\sum_{0<\tau_k<t}\tilde{w}(t-\tau_k)w(s)^{-1}\right]\le \text{const}<\infty$$

for $t\ge 0$. Since $u\in PL^\infty_w(0,\infty;X)$ we have $\|u(s)\|\le\text{const}<\infty$. By (E_1) we have $\|F(u(s))\|\le K\|u(s)\|$ and $\|I_k(u(\tau_k))\|\le K\|u(\tau_k)\|$ with some $K>0$. Hence

$$\begin{aligned}\|\widetilde{T}(t-s)F(u(s))\| &\le K\|u\|_w\widetilde{w}(t-s)w(s)^{-1},\\ \|\widetilde{T}(t-\tau_k)I_k(u(\tau_k))\| &\le K\|u\|_w\tilde{w}(t-\tau_k)w(\tau_k)^{-1}.\end{aligned}$$

Consequently,

$$\begin{aligned}\|w(t)G(u)(t)\| &\le Kw(t)\left[\int_0^t \widetilde{w}(t-s)w(s)^{-1}\,\mathrm{d}s+\sum_{0<\tau_k<t}\tilde{w}(t-\tau_k)w(\tau_k)^{-1}\right]\|u\|_\omega\\ &\le \text{const}\|u\|_\omega,\end{aligned}$$

which yields the first result.

Now, having $u,v\in PL^\infty_w(0,\infty;X)$, $\|u\|_w,\|u\|_w\le R(R>0)$, we have u,v bounded, and by (E_1) there is $K=K(R)>0$ such that $\|Fu(s)-Fv(s)\|\le K\|u(s)-v(s)\|$ and $\|I_k(u(\tau_k))-I_k(v(\tau_k))\|\le K\|u(\tau_k)-v(\tau_k)\|$. Thus we obtain

$$\begin{aligned}&w(t)\|G(u)(t)-G(v)(t)\|\\ \le\ & Kw(t)\left[\int_0^t \widetilde{w}(t-s)w(s)^{-1}\,\mathrm{d}s+\sum_{0<\tau_k<t}\widetilde{w}(t-\tau_k)w(\tau_k)^{-1}\right]\|u-v\|_w,\end{aligned}$$

which yields the Lipschitz continuity of G in the ball $B_R(0; PL_w^\infty(0,\infty;X))$. $\square$

In the following lemma we give a sufficient condition for G to maps a ball in $PL_w^\infty(0,\infty;X)$ into itself.

Lemma 4.3. Let there exist a nondecreasing function φ such that

$$\|F(u)\| \le \varphi(\|u\|) \text{ and } \|I_k(u)\| \le \varphi(\|u\|) \text{ for all } u \in X, \tag{12}$$

$$\kappa = \sup_{\sigma>0}(\sigma^{-1}\varphi(\sigma)) \sup_{t>0} w(t) \left[\int_0^t \widetilde{w}(t-s)w(s)^{-1}\,\mathrm{d}s + \sum_{0<\tau_k<t} \widetilde{w}(t-\tau_k)w(\tau_k)^{-1}\right] < 1, \tag{13}$$

$$\widetilde{S} \;=\; \sup_{s>0} w(s)\widetilde{w}(s) < \infty. \tag{14}$$

Then for any $R>0$ we have $\|G(u)\|_w \le \kappa R$ for all $u \in B_R(0; PL_w^\infty(0,\infty;X))$; for any $x \in X$ there exists $R>0$ such that the mapping H defined by (9) maps $B_R(0; PL_w^\infty(0,\infty;X))$ into itself where R can be chosen independently of $x \in B_r(0,X)$ for any fixed $r>0$.

Proof. Take R so large that $\widetilde{S}r + \kappa R \le R$ for any fixed $r>0$.

Let $u \in B_R(0; PL_w^\infty(0,\infty;X))$. Then we have

$$\begin{aligned}
& w(t)\|G(u)(t)\| \\
\le\;& w(t)\int_0^t \widetilde{w}(t-s)\varphi(\|u(s)\|)\,\mathrm{d}s + \sum_{0<\tau_k<t} \widetilde{w}(t-\tau_k)\varphi(\|u(\tau_k)\|) \\
\le\;& w(t)\int_0^t \widetilde{w}(t-s)w(s)^{-1}w(s)R^{-1}\varphi(w(s)^{-1}w(s)\|u(s)\|)\,\mathrm{d}sR \\
& + w(t)\sum_{0<\tau_k<t} \widetilde{w}(t-\tau_k)w(\tau_k)^{-1}w(\tau_k)R^{-1}\varphi(w(\tau_k)^{-1}w(\tau_k)\|u(\tau_k)\|)R \\
\le\;& \sup_{\sigma>0}(\sigma^{-1}\varphi(\sigma))w(t)\left[\int_0^t \widetilde{w}(t-s)w(s)^{-1}\,\mathrm{d}s + \sum_{0<\tau_k<t} \widetilde{w}(t-\tau_k)w(\tau_k)^{-1}\right]R \\
\le\;& \kappa R.
\end{aligned}$$

Hence, if $r>0$ and $x \in B_r(0,X)$ then

$$w(t)\|H(u)(t)\| \le \sup_{s>0}(w(s)\widetilde{w}(s))\|x\| + \kappa R \le \widetilde{S}r + \kappa R \le R.$$

This completes the proof. $\square$

Lemma 4.4. Let $\mathcal{K} \subset PL_w^\infty(0,\infty;X)$ be bounded and let $Y \hookrightarrow\hookrightarrow X$ be a Banach space such that either

(i) $\widetilde{T} \in L^1_{loc}(0,\infty; L(X,Y))$, $F : X \to X$ and $I_k : X \to X$ are locally bounded or

(ii) $\widetilde{T} \in L^1_{loc}(0,\infty; L(Y))$, $F : X \to Y$ and $I_k : X \to Y$ are locally bounded.

Then for any $t \geq 0$ the set $\{G(u)(t) : u \in \mathcal{K}\}$ is relatively compact in X.

P r o o f. Suppose (i) hold. For any $u \in \mathcal{K}$ we have

$$\begin{aligned} \left\| \int_0^t \widetilde{T}(t-s)F(u(s))\,\mathrm{d}s \right\|_Y &\leq \text{const} \int_0^t \|\widetilde{T}(s)\|_{L(X,Y)}\,\mathrm{d}s, \\ \left\| \sum_{0<\tau_k<t} \widetilde{T}(t-\tau_k)I_k(u(\tau_k)) \right\|_Y &\leq \text{const} \sum_{0<\tau_k<t} \|\widetilde{T}(t-\tau_k)\|_{L(X,Y)}. \end{aligned}$$

Since $\|u(s)\| \leq \text{const}\omega(s)^{-1} \leq \text{const}$ for almost all $s \in (0,t)$ and all $u \in \mathcal{K}$. So, for all $t \geq 0, \{G(u)(t) : u \in \mathcal{K}\}$ is bounded in Y and, by the compactness of the imbedding, relatively compact in X.

If (ii) holds, then similarly we have

$$\begin{aligned} \left\| \int_0^t \widetilde{T}(t-s)F(u(s))\,\mathrm{d}s \right\|_Y &\leq \text{const} \int_0^t \|\widetilde{T}(s)\|_{L(Y)}\,\mathrm{d}s, \\ \left\| \sum_{0<\tau_k<t} \widetilde{T}(t-\tau_k)I_k(u(\tau_k)) \right\|_Y &\leq \text{const} \sum_{0<\tau_k<t} \|\widetilde{T}(t-\tau_k)\|_{L(Y)}. \end{aligned}$$

So we obtain the same conclusion. □

The next lemma provides a sufficient condition for (ii) of Lemma 4.1.

Lemma 4.5. Let $Y \hookrightarrow X$ be a Banach space with the norm $\|\cdot\|_Y$, $w \in C([0,\infty))$ and let there exist $r_0 > 0, \delta_0 > 0$ and functions $\eta \geq 0, \psi \in L^1_{loc}([0,\infty); R^+)$ such that the following relations are satisfied:

$$\left\{ \begin{array}{l} \|F(u)\| \leq \text{const}\|u\|^{1+\delta_0} \; for \; u \in B_{r_0}(0,X); \\ \|I_k(u)\| \leq \text{const}\|u\|^{1+\delta_0} \; for \; u \in B_{r_0}(0,X); \\ \|\widetilde{T}(\tau)y - y\| \leq \eta(\tau)\|y\|_Y \; for \; y \in Y \; and \; \tau \geq 0; \\ \|\widetilde{T}(\tau)x\|_Y \leq \psi(\tau)\|u_0\| \; for \; u_0 \in X \; and \; \tau > 0; \\ \limsup_{\tau\to 0^+} \eta(\tau) = 0; \end{array} \right. \tag{15}$$

and

$$\limsup_{t\to\infty} w(t) \left[\int_0^t \widetilde{w}(t-\tau)w(\tau)^{-1-\delta_0}\,\mathrm{d}\tau + \sum_{0<\tau_k<t} \widetilde{w}(t-\tau_k)w(\tau_k)^{-1-\delta_0} \right] = 0. \tag{16}$$

If $M \subset PL^\infty_w(0,\infty;X)$ is bounded, then the set $\mathcal{K} = G(M) \subset PL^\infty_w(0,\infty;X)$ satisfies condition (ii) of Lemma 4.1.

Proof. First, let $t^* > 0$ be arbitrary and assume that $t^* < \tau_k < s < t < \tau_{k+1} < \infty$. If $u \in M$ and $v = G(u)$, we can write

$$\begin{aligned}
& w(t)v(t) - w(s)v(s) \\
= & \int_0^t w(t)\widetilde{T}(t-\tau)F(u(\tau))\,\mathrm{d}\tau - \int_0^s w(s)\widetilde{T}(s-\tau)F(u(\tau))\,\mathrm{d}\tau \\
& + w(t)\sum_{0<\tau_k<t}\widetilde{T}(t-\tau_k)I_k(u(\tau_k)) - w(s)\sum_{0<\tau_k<s}\widetilde{T}(s-\tau_k)I_k(u(\tau_k)) \\
= & \int_0^s (w(t)\widetilde{T}(t-s) - w(s)I)\widetilde{T}(s-\tau)F(u(\tau))\,\mathrm{d}s \\
& + \int_s^t w(t)\widetilde{T}(t-\tau)F(u(\tau))\,\mathrm{d}\tau \\
& + \sum_{0<\tau_k<s}(w(t)\widetilde{T}(t-s) - w(s)I)\widetilde{T}(s-\tau_k)I_k(u(\tau_k)).
\end{aligned}$$

Since $\|u(\tau)\| \le \mathrm{const} w(\tau)^{-1}$ and F and I_k are locally Lipschitz continuous, then $\|Fu(\tau)\| \le \mathrm{const}\|u(\tau)\|^{1+\delta_0}$ and $\|I_k u(\tau_k)\| \le \mathrm{const}\|u(\tau_k)\|^{1+\delta_0}$, and consequently

$$\begin{aligned}
& \left\|\int_0^s (w(t)\widetilde{T}(t-s) - w(s)I)\widetilde{T}(s-\tau)F(u(\tau))\,\mathrm{d}\tau\right\| \\
\le\ & \mathrm{const}\left(w(s)\int_0^s \widetilde{w}(s-\tau)w(\tau)^{-1-\delta_0}\,\mathrm{d}\tau + w(t)\int_0^t \widetilde{w}(t-\tau)w(\tau)^{-1-\delta_0}\,\mathrm{d}\tau\right)\|u\|_w^{1+\delta_0},
\end{aligned} \tag{17}$$

and

$$\begin{aligned}
& \left\|\sum_{0<\tau_k<s}(w(t)\widetilde{T}(t-s) - w(s)I)\widetilde{T}(s-\tau_k)I_k(u(\tau_k))\right\| \\
\le\ & \left(w(s)\sum_{0<\tau_k<s}\widetilde{w}(s-\tau_k)w(\tau_k)^{-1-\delta_0} + w(t)\sum_{0<\tau_k<t}\widetilde{w}(t-\tau_k)w(\tau_k)^{-1-\delta_0}\right)\|u\|_w^{1+\delta_0}.
\end{aligned} \tag{18}$$

Similarly we get

$$w(t)\left\|\int_s^t \widetilde{T}(t-\tau)F(u(\tau))\,\mathrm{d}\tau\right\| \le \mathrm{const}\ w(t)\int_{t^*}^t \widetilde{w}(t-\tau)w(\tau)^{-1-\delta_0}\,\mathrm{d}\tau\|u\|_w^{1+\delta_0}. \tag{19}$$

So by (15), for any $\varepsilon > 0$, t^* can be chosen so that

$$\|w(t)v(t) - w(s)v(s)\| < \frac{\varepsilon}{2} \text{ for } \tau_{k+1} > t > s > \tau_k > t^*.$$

Let $k' \in N$ and put $t_j = \frac{jt^*}{k'-1}$, $j = 0, 1, \ldots, k'-1$. Choose a particular $j \in \{1, \ldots, k'-1\}$ and estimate $\|w(t)v(t) - w(s)v(s)\|$ for $\tau_k < t_{j-1} \le s \le t \le t_j < \tau_{k+1}$.

Denote

$$\delta_{k'} = \sup\left\{\|w(\theta_1) - w(\theta_2)\| : |\theta_1 - \theta_2| \le \frac{t^*}{k'-1}, \theta_1, \theta_2 \in [0, t^*]\right\}.$$

Then using (15) we can estimate (17) and (18) as follows:

$$\begin{aligned}
&\left\| \int_0^s (w(t)\widetilde{T}(t-s) - w(s)I)\widetilde{T}(s-\tau)F(u(\tau))\,\mathrm{d}\tau \right\| \\
\le\;& \text{const}\left[w(t)\eta(t-s)\int_0^s \psi(s-\tau)w(\tau)^{-1-\delta_0}\,\mathrm{d}\tau + \delta_{k'}\int_0^s \widetilde{w}(s-\tau)w(\tau)^{-1-\delta_0}\,\mathrm{d}\tau \right], \\
&\left\| \sum_{0<\tau_k<s} (w(t)\widetilde{T}(t-s) - w(s)I)\widetilde{T}(s-\tau_k)I_k(u(\tau_k)) \right\| \\
\le\;& \text{const}\left[\sum_{0<\tau_k<s} w(t)\eta(t-s)\psi(s-\tau_k)w(\tau_k)^{-1-\delta_0} + \sum_{0<\tau_k<s} \delta_{k'}\psi(s-\tau_k)w(\tau_k)^{-1-\delta_0} \right].
\end{aligned}$$

By (15) and the uniform continuity of w on $[0,t^*]$ the last expression is less than ε for a sufficiently large $k' \in N$.

Finally, (19) is estimated in the following way:

$$w(t)\left\| \int_0^t \widetilde{T}(t-\tau)F(u(\tau))\,\mathrm{d}\tau \right\| \;\le\; \text{const} \sup_{\tau\in[0,t^*]} w(\tau) \int_0^{\frac{t^*}{k'-1}} \widetilde{w}(\sigma)\,\mathrm{d}\sigma,$$

the last expression may be made less than ε when k' is chosen appropriately large. So the system of intervals $[t_{j-1},t_j] \subset (\tau_k,\tau_{k+1})$, $j=1,\dots,k'-1$, $[t_{k'-1},\infty)$ is the desired measurable partition of R^+ corresponding to the given $\varepsilon>0$ as required in condition (ii) of Lemma 4.1. The proof is complete. $\square$

Now we are ready to present the main result in this section.

Theorem 4.6. Let the following assumptions be satisfied:

(i) $-A$ is a generator of a C_0-semigroup $\{T(t), t\ge 0\}$ in X;

(ii) $B = D - F$, $D \in L(X)$ and $\{\widetilde{T}(t), t\ge 0\}$ satisfies $\|\widetilde{T}(t)\| \le \widetilde{w}(t)$ for $t>0$, where $\widetilde{w} \in L^1_{loc}([0,\infty))$;

(iii) $F : X \to X$ and $I_k : X \to X$ are locally Lipschitz continuous with $F(0)=0$, $I_k(0)=0$ and there exist constants $\delta_0>0, r_0>0, k_0>0$ such that

$$\|F(u)\| \le k_0\|u\|^{1+\delta_0} \textit{ and } \|I_k(u)\| \le k_0\|u\|^{1+\delta_0}$$

for $u \in B_{r_0}(0;X)$, and a nondecreasing function φ such that (12) and (13) hold;

(iv) there exists a Banach space $Y \hookrightarrow\hookrightarrow X$ such that either

(a) $\widetilde{T} \in L^1_{loc}(0,\infty; L(X,Y))$, $F : X \to X$ and $I_k : X \to X$ are locally bounded or

(b) $\widetilde{T} \in L^1_{loc}(0,\infty; L(Y))$, $F : X \to Y$ and $I_k : X \to Y$ are locally bounded;

in both cases we assume that

$$\begin{aligned}
\|\widetilde{T}(\tau)y - \widetilde{T}(0)y\| &\le \eta(\tau)\|y\|_Y,\ y\in Y,\ \tau\ge 0,\\
\|\widetilde{T}(\tau)\xi\|_Y &\le \psi(\tau)\|\xi\|,\ \xi\in X,\ \tau>0,
\end{aligned}$$

where $\eta \ge 0$, $\limsup_{\tau\to 0^+}\eta(\tau)=0$, $\psi \in L^1_{loc}([0,\infty);R^+)$;

(v) there is a positive function $w \in C([0,\infty))$ with $\lim_{t\to\infty} w(t) = \infty$ and such that $\limsup_{t\to\infty} w(t)\widetilde{w}(t) = 0$, and (16) hold.

Then for any $u_0 \in X$, the problem (1) has at least a generalized solution $u \in PL^\infty_w(0,\infty;X)$.

Proof. We shall make use of the Schauder fixed point theorem in $B_R(0; PL^\infty_\omega(0,\infty;X))$ with $R > 0$ sufficiently large. To this end, consider the operator H defined by (6) in Section 2. By Lemma 4.2 the operator H maps $PL^\infty_\omega(0,\infty;X)$ continuously into itself. By assumption (v) and Lemma 4.3, for any u in the ball $B_R(0; PL^\infty_w(0,\infty;X))$ with $R > 0$ sufficiently large we find

$$\|H(u)\|_w \le \|\widetilde{T}(t)u_0\|_w + \|G(u)\|_w \le \widetilde{S}\|u_0\| + \kappa R \le R.$$

Finally, by Lemmas 4.1, 4.4 and 4.5, the mapping H is compact in $PL^\infty_w(0,\infty;X)$. Thus, H satisfies the assumptions of the Schauder fixed point theorem in $B_R(0; PL^\infty_w(0,\infty;X))$ for a sufficiently large $R > 0$ and hence it has a fixed point $u \in PL^\infty_w(0,\infty;X)$, that is $u = H(u)$. Thus u is a solution of (10). □

5. STABILIZATION FOR A SINGULARLY PERTURBED PROBLEM WITH IMPULSE

In this section, we turn to consider the problem (3). It is easy to see that the generalized solution of the problem (3) is $u_0(t) \equiv 0$ for $\varepsilon = 0$ and a function $u_\varepsilon \in PL^\infty(0,\infty;X)$ such that

$$u_\varepsilon(t) \;=\; \widetilde{T}(\frac{t}{\varepsilon})x + \frac{1}{\varepsilon}\int_0^t \widetilde{T}(\frac{t-s}{\varepsilon})F(u_\varepsilon(s))\,\mathrm{d}s + \frac{1}{\varepsilon}\sum_{0<\tau_k<t}\widetilde{T}(\frac{t-\tau_k}{\varepsilon})I_k(u_\varepsilon(\tau_k)),\ t \ge 0, \tag{20}$$

if $\varepsilon \in (0,\varepsilon_0]$. Introducing new variables

$$\theta = \frac{t}{\varepsilon},\quad v(\theta) = u_\varepsilon(\varepsilon\theta), \tag{21}$$

we transform the problem (3) into the problem

$$\begin{cases} v'(\theta) + (A+B)v(\theta) = 0,\ \theta > 0,\ \theta \neq \frac{\tau_k}{\varepsilon}, \\ v(0) = x,\ \varepsilon \in [0,\varepsilon_0],\ \varepsilon_0 > 0,\ x \in X, \\ \triangle v(\theta) = I_k(v(\theta)),\ \theta = \frac{\tau_k}{\varepsilon}. \end{cases} \tag{22}$$

Now we can apply the results of Section 2 and Section 4 to the problem (3). If we succeed in finding a weight function $w(t)$ such that the hypotheses of Theorems 2.2 or 4.6, respectively, are satisfied, then we get

$$\|v(\tau)\| \le Cw(\tau)^{-1}, \tau \ge 0 \tag{23}$$

with a constant C depending only on the radius r of the ball $B_r(0;X)$ the initial datum $x \in X$ is taken from. The relation (23) can be translated to

$$\|u_\varepsilon(t)\| \le Cw\big(\frac{t}{\varepsilon}\big)^{-1},\ t \ge 0,\ \varepsilon \in (0,\varepsilon_0].$$

This yields not only the stabilization of the solutions for $x \in B(0; X)$ to the zero stationary solution as $t \to \infty$ but also the pointwise convergence of $u_\varepsilon(t)$ to 0 as $\varepsilon \to 0^+$ for $t > 0$, and the rate of convergence in terms of $\frac{t}{\varepsilon}$.

Of course, one can easily formulate the corresponding theorems for the problem (3) by just modifying Theorems 2.2 or 4.6, respectively.

6. EXAMPLE

As an illustration of application of the results of Section 4 let us consider a semilinear impulsive parabolic equation and formulate an explicit condition that guarantees the assumptions of Theorem 4.6.

Let $\Omega \subset R^n$ be a bounded domain with uniformly C^2-boundary. Consider the the following initial boundary value problem with impulse

$$\begin{cases} u_t = \sum_{i,j=1}^n (a_{ij} u_{x_j})_{x_i} + g(u), \ x \in R^n, \ t > 0, \ t \neq \tau_k, \\ u(x,t) = 0, x \in \partial\Omega, t > 0, \\ u(x,0) = u^0(x), x \in R^n, \\ \triangle u(x,t) = l_k u(x,t), \ x \in R^n, \ t = \tau_k, \ l_k > 0, \ k = 1, 2, \ldots, \end{cases} \tag{24}$$

where

$$\begin{cases} a_{ij} \in C^2(\overline{\Omega}), \ a_{ji} = a_{ij}, \ i,j = 1, \ldots n, \\ \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq c_0|\xi|^2 \text{ for } \xi \in R^n, \ \xi \neq 0 \text{ with } c_0 > 0, \end{cases} \tag{25}$$

and

$$g \in C^1(R), \ g = \mathrm{d}u + f(u), u \in R, \ g(0) = 0. \tag{26}$$

Let $s \in (0,1)$, $p > \frac{n}{s}$, and let $X := \overline{W}^s_p(\Omega)$, where $W^r_q (0 < r < \infty, 1 \leq q \leq \infty)$ stands for the usual Sobolev space, $\overline{W}^r_q(\Omega)$ being the closure in $W^r_q(\Omega)$ of $C^\infty_0(\Omega)$. Also, denote by $\|\cdot\|_q$ the norm in $L^q(\Omega)$ and by $\|\cdot\|_{r,q}$ in W^r_q. Let us note that if $r \in (0,1)$ then for $v \in W^r_q(\Omega)$ we use the norm

$$\|v\|_{r,q} := \left\{ \int_\Omega |v(x)|^q \, \mathrm{d}x + \int_\Omega \int_\Omega \frac{|v(x) - v(y)|^q}{|x-y|^{n+rq}} \, \mathrm{d}x \, \mathrm{d}y \right\}^{1/q}.$$

Let (25) be satisfied. A is defined by

$$\begin{cases} D(A) := W^2_p(\Omega) \cap \overline{W}^1_p(\Omega), \\ (Av)(x) := -\sum_{i,j=1}^n (a_{i,j}(x) v_{x_j}(x))_{x_i} \text{ for } v \in D(A). \end{cases} \tag{27}$$

By Proposition 6.1 of [10], the operator $-A$ is a generator of an analytic exponentially decreasing semigroup $\{T(t), t \geq 0\}$ of continuous linear operators in $L^p(\Omega)$ which is an exponentially decreasing C_0-semigroup of contractions in X. That is, there exists $c = \frac{4c_0 m(p-1)}{p^2} > 0$, $c_0 > 0$ such that $\|T(t)\| \leq e^{-ct} := w(t)$, $t \geq 0$.

Further, let $d < c$ and $\nu \in (\frac{s}{p}, 1)$, set $\widetilde{T}(t) = A^\nu e^{-(d+A)t}$, $x = A^{-\nu}u^0$, $Dv = dv$, $Fv = A^{-\nu} f(v)$, $I_k(v) = l_k(v)$ for $v \in X$. We have $\|\widetilde{T}(t)\|_{L(X)} \leq c(\nu) t^{-\nu} e^{-\gamma t} := \widetilde{w}(t)$ for $t > 0$, $\gamma = \frac{c-d}{2}$.

Let $f \in C^1(R), f(0) = 0$, with f' locally Lipschitz continuous. By Lemma 6.2 of [10], f maps $\overline{W}_p^s(\Omega)$ into itself and is locally Lipschitz continuous. It is obvious that l_k maps $\overline{W}_p^s(\Omega)$ into itself and is locally Lipschitz continuous. Suppose that there exist δ_0, K, r_1 such that $\|f(v)\| \leq K\|v\|^{1+\delta_0}$, $\|l_k(v)\| \leq K\|v\|^{1+\delta_0}$, $\|f'(v)\| \leq K\|v\|^{\delta_0}$ for $\|v\| \leq r_1$, $S = \sup_{v\neq 0} \|\frac{f(v)}{v}\| < \kappa_0^{-1}K(\nu,p)^{-1}$, $K(\nu,p) = C(\nu)(cp)^{s-p\nu}\Gamma(p\nu - s)$ and $C(\nu) := \sup_{t>0}\{t^{\nu-1}e^{-ct}\|A^{1-\nu}T(t)\|\}$. Let $Y := D(A) = W_p^2(\Omega) \cap \overline{W}_p^1(\Omega)$. Then by the Sobolev imbedding theorem Y is compactly imbedded in X. Define $\eta(\tau) = \tau^\theta$, $\theta \in (0, \frac{s}{p})$ and $\psi(\tau) = \tau^{\theta-1}$. Suppose that there exist the suitable positive numbers c, ν and γ such that

$$\limsup_{t\to\infty} w(t)\left[\int_0^t \widetilde{w}(t-\tau)w(\tau)^{-1-\delta_0}\,\mathrm{d}\tau + \sum_{0<\tau_k<t} \widetilde{w}(t-\tau_k)w(\tau_k)^{-1-\delta_0}\right] = 0.$$

Then all the assumptions in Theorem 4.6, our results can be used to solve the problem (24).

7. FINAL REMARKS

Recently, generating complex multiscroll chaotic attractors via simple electronic circuits has drawn the attention of many researchers (see for example, Lü and Chen [14], Lü et al. [15]). The current novel method for generalized stability analysis of nonlinear impulsive evolution equations also contain some potential applications in the complex multiscroll chaotic systems. In real-world, we try to seek some suitable function $w(t)$ to guarantee the solution of some certain complex multiscroll chaotic systems tending to zero at an appropriate decay rate $w(t)^{-1}$ as $t \to \infty$. Of course, how to find a simple or the best function w is a very interesting problem. It is worthwhile mentioning that Wang et al. [16, 17] apply the idea here to discuss the stability of some fractional integral equations whose solutions are tending to zero at an appropriate rate $t^{-\nu}(\nu > 0)$ as $t \to \infty$.

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