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TRANSFORMATION OF OPTIMAL CONTROL PROBLEMS OF DESCRIPTOR SYSTEMS INTO PROBLEMS WITH STATE-SPACE SYSTEMS

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We show how we can transform the $\mathscr{H}_\infty$ and $\mathscr{H}_2$ control problems of descriptor systems with invariant zeros on the extended imaginary into problems with state-space systems without such zeros. Then we present necessary and sufficient conditions for existence of solutions of the original problems.

Numerical algorithm for $\mathscr{H}_\infty$ control is given, based on the Nevanlinna–Pick theorem. Also, we present an explicit formula for the optimal $\mathscr{H}_2$ controller.

Keywords: parametrization of stabilizing controllers, inner matrices, $\mathscr{H}_\infty$ and $\mathscr{H}_2$ control

Classification: 93D15, 49J15

1. INTRODUCTION

Consider a continuous-time descriptor system

$$\boldsymbol{G} = \begin{array}{c} \begin{bmatrix} \overbrace{\boldsymbol{G}_{11}}^{m_1} & \overbrace{\boldsymbol{G}_{12}}^{m_2} \\ \boldsymbol{G}_{21} & \boldsymbol{G}_{22} \end{bmatrix} \begin{matrix} \}p_1 \\ \}p_2 \end{matrix} \end{array} = \left[\begin{array}{c|cc} A - sE & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] , \tag{1.1}$$

where $\boldsymbol{G}$ is an improper plant transfer matrix, with the descriptor realization with $n \times n$-dimensional regular matrix pencil $A - sE$ and matrices $B_1, B_2, C_2, C_2, D_{11}, D_{12}, D_{21}$ and D_{22}, let $\boldsymbol{K}$ be a controller transfer matrix (see Figure 1), and define

$$\mathscr{F}(\boldsymbol{G}, \boldsymbol{K}) = \boldsymbol{G}_{11} + \boldsymbol{G}_{12}\boldsymbol{K}(I - \boldsymbol{G}_{22}\boldsymbol{K})^{-1}\boldsymbol{G}_{21} \ . \tag{1.2}$$

We call stabilizing controller the controller that renders the closed loop system stable and impulse-free.

The $\mathscr{H}_2$ and $\mathscr{H}_\infty$ optimal control problems are to find a controller $\boldsymbol{K}$, such that $\|\mathscr{F}(\boldsymbol{G}, \boldsymbol{K})\|$, where the norm is $\mathscr{H}_2$ and $\mathscr{H}_\infty$, respectively, is minimal in the class of stabilizing controllers. Given $\gamma > 0$, the $\mathscr{H}_\infty$ suboptimal control problem is to find a stabilizing controller $\boldsymbol{K}$, such that $\|\mathscr{F}(\boldsymbol{G}, \boldsymbol{K})\|_\infty < \gamma$.

The $\mathscr{H}_2$ and $\mathscr{H}_\infty$ controls are ones of the most desirable controls, because of their robustness and physical understanding. Concerning the systems with proper $\boldsymbol{G}$, various results and algorithms are presented on the $\mathscr{H}_2$ control [5, 8, 11, 14, 17, 24], and on

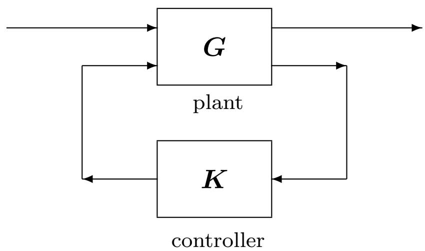

Fig. 1. Standard control structure.

the $\mathscr{H}_\infty$ control [2, 3, 4, 7, 21, 22, 24]. When there are invariant zeros on the extended imaginary axis, the algorithms have numerical problems resulting with stability problems of the closed loop system.

Concerning the systems with improper $\boldsymbol{G}$, $\mathscr{H}_2$ control results and algorithms are given in [9, 20], and $\mathscr{H}_\infty$ control results and algorithms are given in [10, 12, 13, 19, 23].

For systems with improper $\boldsymbol{G}$ having invariant zeros on the extended imaginary axis, there are few results and algorithms in the literature ([9, 13, 23]), but the numerical algorithms are not satisfactory. For instance, it is stated in Section 13 of [9] that the critical part of the transfer-function algorithm of [9] is the final substitution of the optimal parameter matrix into the parametrized controller formula. This operation generically results in common factors that must be cancelled to obtain the optimal controller in reduced form. Another difficulty related to the previous one, is the degree control. Although the algorithms of [13] and [23] theoretically can handle invariant zeros on the imaginary axis, such numerical examples are not presented in the papers. Only examples with invariant zeros at infinity are given.

Exactly the problems with invariant zeros on the extended imaginary axis of descriptor systems are considered in this paper. The used approach is to "regularize" them firstly, i. e. to transform them into problems with state space systems without invariant zeros on the extended imaginary axis. Then the solution of the $\mathscr{H}_2$ control problem is immediate, by a new explicit controller formula (formula (4.2) in Section 4.1). The solution of the $\mathscr{H}_\infty$ control problem is by interpolation with points on the extended imaginary axis, solved by the Nevanlinna–Pick theorem (Section 4.2). Numerical examples are presented.

The preliminary results in Sections 2 and 3, containing a parametization of marginally stabilizing controllers and a parametization of stabilizing controllers, are also new.

Remarks on the notation. The matrix functions of s are rational, with real coefficients, and will be bold-written, and if not ambiguous, without the argument. Poles and zeros of a rational matrix are defined through its McMillan form. If a realization of transfer matrix $\boldsymbol{G}$ is given, the notion invariant zero of $\boldsymbol{G}$ means a zero of the associated matrix pencil to the realization of $\boldsymbol{G}$. The multiplicities of the infinite generalized eigenvalues (IGEs) of this matrix pencil equal the orders of the infinite zeros of $\boldsymbol{G}$ increased by one. An invariant zero of $\boldsymbol{G}$ is simple if the corresponding Kronecker block in the Kroneckar canonical form of the associated matrix pencil to $\boldsymbol{G}$ is diagonal. The invariant zero of $\boldsymbol{G}$ is not always its zero. The notion of (marginal) stability is related to the set ($\Re[s] \leq 0$) $\Re[s] < 0$ in the complex plane $\mathbb{C}$. Analogous is the notion of (marginal) minimum phase. By the superscripts $^{\mathrm{T}}$ and *, we denote respectively matrix transposition and complex conjugation. If $\boldsymbol{\Pi}$ is a rational matrix, by $\boldsymbol{\Pi}^{\sim}$ we denote the matrix $\boldsymbol{\Pi}^{\mathrm{T}}(-s)$. We denote the transfer matrices $D + C(sI - A)^{-1}B$ and $D + C(sE - A)^{-1}B$ by $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ and $\left[\begin{array}{c|c} A - sE & B \\ \hline C & D \end{array}\right]$. If $\boldsymbol{H}$ is a strict proper rational matrix without poles in $\Re[s] \geq 0$, its $\mathscr{H}_2$ norm is defined by $\|\boldsymbol{H}\|_2^2 = \frac{1}{2\pi \mathrm{j}} \int_{-\mathrm{j}\infty}^{\mathrm{j}\infty} \mathrm{Tr}\{\boldsymbol{H}\boldsymbol{H}^{\sim}\}\mathrm{d}s$, where by $\mathrm{Tr}\{\cdot\}$ we denote the trace of square matrices. For given proper matrix $\boldsymbol{H}$ without poles on the imaginary axis, we define $\|\boldsymbol{H}\|_\infty = \sup_{\omega\in\mathbb{R}} \|\boldsymbol{H}(\mathrm{j}\omega)\|$, where the later is spectral norm of complex matrices.

2. MARGINALLY STABILIZING CONTROLLERS AND PARAMETRIZATION

We introduce the following assumptions:

Assumption 2.1. (For system (E, A, B_2, C_1, D_{12}))

- Descriptor system (E, A, B_2) is impulse controllable [6],
- matrix pencil $[A - sE, B_2]$ has full row rank in $\Re[s] \geq 0$,
- matrix $\boldsymbol{G}_{12}$ is left-invertible.

Assumption 2.2. (For system (E, A, B_1, C_2, D_{21}))

- Descriptor system (E, A, C_2) is impulse observable [6],
- matrix pencil $\begin{bmatrix} A - sE \\ C_2 \end{bmatrix}$ has full column rank in $\Re[s] \geq 0$,
- matrix $\boldsymbol{G}_{21}$ is right-invertible.

Under Assumption 2.1, by the algorithm of [15], we can find matrices $\bar{\mathcal{K}}$ and $\bar{\mathcal{L}}$ such that matrix $\bar{\boldsymbol{\Phi}} = \bar{\mathcal{L}} + \bar{\mathcal{K}}\boldsymbol{\Theta}B_2$, where $\boldsymbol{\Theta} = (sE - A)^{-1}$ is nonsingular and marginally minimum phase, with possible zeros at infinity, and satisfies

$$\boldsymbol{G}_{12}^{\sim}\boldsymbol{G}_{12} = \bar{\boldsymbol{\Phi}}^{\sim}\bar{\boldsymbol{\Phi}} \, . \tag{2.1}$$

(See also Appendix.) Analogously, under Assumption 2.2, we can find matrices $\mathcal{K}$ and $\mathcal{L}$ such that matrix $\boldsymbol{\Phi} = \mathcal{L} + C_2\boldsymbol{\Theta}\mathcal{K}$ is nonsingular and marginally minimum phase, with possible zeros at infinity, and satisfies

$$\boldsymbol{G}_{21}\boldsymbol{G}_{21}^{\sim} = \boldsymbol{\Phi}\boldsymbol{\Phi}^{\sim} \, . \tag{2.2}$$

Introduce the following matrices, which are well-defined:

$$\boldsymbol{G}_{\mathrm{i}} = \left[\begin{array}{cc|c} A - sE & B_2 & 0 \\ \bar{\mathcal{K}} & \bar{\mathcal{L}} & -I_{m_2} \\ \hline C_1 & D_{12} & 0 \end{array}\right] , \quad \boldsymbol{G}_{\mathrm{ci}} = \left[\begin{array}{cc|c} A - sE & \mathcal{K} & B_1 \\ C_2 & \mathcal{L} & D_{21} \\ \hline 0 & -I_{p_2} & 0 \end{array}\right] . \tag{2.3}$$

Proposition 2.3. Matrices $\boldsymbol{G}_{\mathrm{i}}$ and $\boldsymbol{G}_{\mathrm{ci}}$ are inner and co-inner respectively. Hence they are proper stable, and $\boldsymbol{G}_{\mathrm{i}}(\infty)$ and $\boldsymbol{G}_{\mathrm{ci}}(\infty)$ are respectively full column and full row rank matrices.

Proof. The proof will be given only for matrix $\boldsymbol{G}_{\mathrm{i}}$. The proof for matrix $\boldsymbol{G}_{\mathrm{ci}}$ follows by analogy.

We have to prove that matrix $\boldsymbol{G}_{\mathrm{i}}$ is stable and satisfies $\boldsymbol{G}_{\mathrm{i}}^{\sim}\boldsymbol{G}_{\mathrm{i}} = I$.

It follows by Proposition 5.2 that matrix

$$\bar{\boldsymbol{\Omega}} = \begin{bmatrix} A - sE & B_2 \\ \bar{\mathcal{K}} & \bar{\mathcal{L}} \end{bmatrix}^{-1} \tag{2.4}$$

is marginally stable (possibly improper), hence matrix $\boldsymbol{G}_{\mathrm{i}}$ is (at least) marginally stable.

Further on, we prove that matrix $\boldsymbol{G}_{12}$ can be factorized as

$$\boldsymbol{G}_{12} = \boldsymbol{G}_{\mathrm{i}}\bar{\boldsymbol{\Phi}} \ , \tag{2.5}$$

where $\bar{\boldsymbol{\Phi}}$ is the matrix in (2.1). For that purpose, note that

$$\bar{\boldsymbol{\Omega}} \begin{bmatrix} 0 \\ I \end{bmatrix} \bar{\mathcal{K}}\boldsymbol{\Theta} = \bar{\boldsymbol{\Omega}} \begin{bmatrix} I \\ 0 \end{bmatrix} + \begin{bmatrix} I \\ 0 \end{bmatrix} \boldsymbol{\Theta} \ . \tag{2.6}$$

We have

$$\begin{aligned}
\boldsymbol{G}_{\mathrm{i}}\bar{\boldsymbol{\Phi}} &= [C_1, D_{12}]\bar{\boldsymbol{\Omega}} \begin{bmatrix} 0 \\ I \end{bmatrix} (\bar{\mathcal{L}} + \bar{\mathcal{K}}\boldsymbol{\Theta}B_2) \\
&= [C_1, D_{12}]\bar{\boldsymbol{\Omega}} \begin{bmatrix} 0 \\ I \end{bmatrix} \bar{\mathcal{L}} + [C_1, D_{12}] \left(\bar{\boldsymbol{\Omega}} \begin{bmatrix} I \\ 0 \end{bmatrix} + \begin{bmatrix} I \\ 0 \end{bmatrix} \boldsymbol{\Theta} \right) B_2 \\
&= [C_1, D_{12}]\bar{\boldsymbol{\Omega}} \begin{bmatrix} B_2 \\ \bar{\mathcal{L}} \end{bmatrix} + C_1\boldsymbol{\Theta}B_2 = [C_1, D_{12}] \begin{bmatrix} 0 \\ I \end{bmatrix} + C_1\boldsymbol{\Theta}B_2 = \boldsymbol{G}_{12} \ .
\end{aligned}$$

A consequence of identity (2.5) is $\boldsymbol{G}_{\mathrm{i}}^{\sim}\boldsymbol{G}_{\mathrm{i}} = \bar{\boldsymbol{\Phi}}^{-\sim}\boldsymbol{G}_{12}^{\sim}\boldsymbol{G}_{12}\bar{\boldsymbol{\Phi}}^{-1} = \bar{\boldsymbol{\Phi}}^{-\sim}\bar{\boldsymbol{\Phi}}^{\sim}\bar{\boldsymbol{\Phi}}\bar{\boldsymbol{\Phi}}^{-1} = I$. Now the stability of $\boldsymbol{G}_{\mathrm{i}}$ follows by its marginal stability and the fact that it has no poles on the imaginary axis. □

The inverse of $\bar{\boldsymbol{\Phi}}$ is the following marginally stable matrix:

$$\bar{\boldsymbol{Q}} = \bar{\boldsymbol{\Phi}}^{-1} = [0, I]\bar{\boldsymbol{\Omega}} \begin{bmatrix} 0 \\ I \end{bmatrix} = \left[\begin{array}{cc|c} A - sE & B_2 & 0 \\ \bar{\mathcal{K}} & \bar{\mathcal{L}} & -I \\ \hline 0 & I & 0 \end{array} \right] , \tag{2.7}$$

and the inverse of $\boldsymbol{\Phi}$ is the following marginally stable matrix:

$$\boldsymbol{Q} = \boldsymbol{\Phi}^{-1} = [0, I]\boldsymbol{\Omega} \begin{bmatrix} 0 \\ I \end{bmatrix} = \left[\begin{array}{cc|c} A - sE & \mathcal{K} & 0 \\ C_2 & \mathcal{L} & -I \\ \hline 0 & I & 0 \end{array} \right] , \tag{2.8}$$

where the introduced matrix

$$\boldsymbol{\Omega} = \begin{bmatrix} A - sE & \mathcal{K} \\ C_2 & \mathcal{L} \end{bmatrix}^{-1} \tag{2.9}$$

is marginally stable. Analogously with (2.5), it holds $\boldsymbol{G}_{21} = \boldsymbol{\Phi}\boldsymbol{G}_{\mathrm{ci}}$.

Starting with these $\bar{\boldsymbol{Q}}$ and $\boldsymbol{Q}$, we define the following marginally stable and marginally minumum phase matrices

$$\begin{bmatrix} \boldsymbol{X} & \boldsymbol{Y} \\ -\boldsymbol{P} & \boldsymbol{Q} \end{bmatrix} = \left[\begin{array}{cc|cc} A - sE & \mathcal{K} & B_2 & 0 \\ C_2 & \mathcal{L} & D_{22} & -I \\ \hline \bar{\mathcal{K}} & 0 & \bar{\mathcal{L}} & 0 \\ 0 & I & 0 & 0 \end{array}\right] , \tag{2.10}$$

$$\begin{bmatrix} \bar{\boldsymbol{Q}} & -\bar{\boldsymbol{Y}} \\ \bar{\boldsymbol{P}} & \bar{\boldsymbol{X}} \end{bmatrix} = \left[\begin{array}{cc|cc} A - sE & B_2 & 0 & \mathcal{K} \\ \bar{\mathcal{K}} & \bar{\mathcal{L}} & -I & 0 \\ \hline 0 & I & 0 & 0 \\ C_2 & D_{22} & 0 & \mathcal{L} \end{array}\right] . \tag{2.11}$$

It is easy to check the following identities:

$$\begin{bmatrix} \bar{\boldsymbol{Q}} & -\bar{\boldsymbol{Y}} \\ \bar{\boldsymbol{P}} & \bar{\boldsymbol{X}} \end{bmatrix} \begin{bmatrix} \boldsymbol{X} & \boldsymbol{Y} \\ -\boldsymbol{P} & \boldsymbol{Q} \end{bmatrix} = \begin{bmatrix} \boldsymbol{X} & \boldsymbol{Y} \\ -\boldsymbol{P} & \boldsymbol{Q} \end{bmatrix} \begin{bmatrix} \bar{\boldsymbol{Q}} & -\bar{\boldsymbol{Y}} \\ \bar{\boldsymbol{P}} & \bar{\boldsymbol{X}} \end{bmatrix} = I , \tag{2.12}$$

$$\boldsymbol{G}_{22} = \bar{\boldsymbol{P}}\bar{\boldsymbol{Q}}^{-1} = \boldsymbol{Q}^{-1}\boldsymbol{P} . \tag{2.13}$$

Define matrices

$$\widehat{\boldsymbol{G}}_{11} = \boldsymbol{G}_{11} - \boldsymbol{G}_{12}\bar{\boldsymbol{Q}}\boldsymbol{Y}\boldsymbol{G}_{21} , \tag{2.14}$$

$$\widehat{\boldsymbol{G}}_{12} = \boldsymbol{G}_{12}\bar{\boldsymbol{Q}} = \boldsymbol{G}_{\mathrm{i}} , \quad \widehat{\boldsymbol{G}}_{21} = \boldsymbol{Q}\boldsymbol{G}_{21} = \boldsymbol{G}_{\mathrm{ci}} .$$

Using (2.10), (2.11), (2.14) and (2.3), we obtain the following descriptor realization

$$\widehat{\boldsymbol{G}} = \begin{bmatrix} \widehat{\boldsymbol{G}}_{11} & \widehat{\boldsymbol{G}}_{12} \\ \widehat{\boldsymbol{G}}_{21} & 0 \end{bmatrix} = \left[\begin{array}{cccc|cc} A - sE & B_2 & 0 & 0 & B_1 & 0 \\ \bar{\mathcal{K}} & \bar{\mathcal{L}} & \bar{\mathcal{K}} & 0 & 0 & -I_{m_2} \\ 0 & 0 & A - sE & \mathcal{K} & -B_1 & 0 \\ 0 & 0 & C_2 & \mathcal{L} & -D_{21} & 0 \\ \hline C_1 & D_{12} & 0 & 0 & D_{11} & 0 \\ 0 & 0 & 0 & I_{p_2} & 0 & 0 \end{array}\right] . \tag{2.15}$$

Proposition 2.4. Matrix $\widehat{\boldsymbol{G}}$ is proper and stable.

Proof. It suffices to prove that matrix $\widehat{\boldsymbol{G}}_{11}$ is proper and stable. By the descriptor realizations of $\boldsymbol{G}_{\mathrm{i}}$ and $\boldsymbol{G}_{\mathrm{ci}}$, both in (2.3), it follows that matrices

$$[C_1, D_{12}]\bar{\boldsymbol{\Omega}} \quad \text{and} \quad \boldsymbol{\Omega}\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix}$$

where matrices $\bar{\boldsymbol{\Omega}}$ and $\boldsymbol{\Omega}$, given in (2.4) and (2.9), are proper and stable, because matrices $\boldsymbol{G}_{\mathrm{i}}$ and $\boldsymbol{G}_{\mathrm{ci}}$ are proper and stable, and matrix pencils

$$\begin{bmatrix} A - sE & B_2 & 0 \\ \bar{\mathcal{K}} & \bar{\mathcal{L}} & -I \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A - sE & \mathcal{K} \\ C_2 & \mathcal{L} \\ 0 & -I \end{bmatrix}$$

have full row and column rank on the imaginary axis, respectively. By

$$\widehat{\boldsymbol{G}}_{11} = D_{11} - [C_1, D_{12}]\bar{\boldsymbol{\Omega}} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} - [C_1, D_{12}]\bar{\boldsymbol{\Omega}} \begin{bmatrix} 0 \\ I \end{bmatrix} \bar{\mathcal{K}}[I, 0]\boldsymbol{\Omega} \begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} , \tag{2.16}$$

we see that matrix $\widehat{\boldsymbol{G}}_{11}$ is proper and stable. □

A controller is marginally stabilizing if the closed loop system on Figure 1 is marginally stable. Impulsive modes are allowed.

Every realization of marginally stabilizing controller $\boldsymbol{K}$ can be presented as a realization without hidden modes in $\Re[s] > 0$ of the following transfer matrix:

$$\boldsymbol{K} = (-\bar{\boldsymbol{Y}} + \bar{\boldsymbol{Q}}\boldsymbol{S})(\bar{\boldsymbol{X}} + \bar{\boldsymbol{P}}\boldsymbol{S})^{-1} \tag{2.17}$$

$$= (\boldsymbol{X} + \boldsymbol{S}\boldsymbol{P})^{-1}(-\boldsymbol{Y} + \boldsymbol{S}\boldsymbol{Q}) , \tag{2.18}$$

where $\boldsymbol{S}$ is a marginally stable parameter matrix satisfying $\det(\bar{\boldsymbol{X}} + \bar{\boldsymbol{P}}\boldsymbol{S}) \not\equiv 0$.

The following two theorems are instrumental in reducing the optimal control problems of descriptor systems into ones with state-space systems.

Theorem 2.5.

$$\mathscr{F}(\boldsymbol{G}, \boldsymbol{K}) = \mathscr{F}(\widehat{\boldsymbol{G}}, \boldsymbol{S}) = \widehat{\boldsymbol{G}}_{11} + \widehat{\boldsymbol{G}}_{12}\boldsymbol{S}\widehat{\boldsymbol{G}}_{21} . \tag{2.19}$$

P r o o f. By direct checking, using (1.2) and (2.17) or (2.18). □

The following theorem gives a necessary condition for existence of a stabilizing solution to the $\mathscr{H}_2$ and $\mathscr{H}_\infty$ control problems.

Theorem 2.6. Let Assumptions 2.1 and 2.2 hold. If there is a stabilizing $\boldsymbol{K}$ such that the $\mathscr{H}_2$ or $\mathscr{H}_\infty$ norm of $\mathscr{F}(\boldsymbol{G}, \boldsymbol{K})$ is finite, then matrix $\boldsymbol{S}$ is proper and stable, where matrix $\boldsymbol{S}$ is defined by the inverse dependence of (2.17) or (2.18).

P r o o f. If $\boldsymbol{K}$ is stabilizing, then it is marginally stabilizing. We apply the parametrization of marginally stabilizing controllers defined by (2.17) or (2.18). Remind that the parameter matrix $\boldsymbol{S}$ is marginally stable and possibly improper. Let

$$\|\mathscr{F}(\boldsymbol{G}, \boldsymbol{K})\| = \left\|\mathscr{F}(\widehat{\boldsymbol{G}}, \boldsymbol{S})\right\| = \left\|\widehat{\boldsymbol{G}}_{11} + \boldsymbol{G}_{\mathrm{i}}\boldsymbol{S}\boldsymbol{G}_{\mathrm{ci}}\right\| < \infty , \tag{2.20}$$

where $\|\cdot\|$ is $\|\cdot\|_2$ or $\|\cdot\|_\infty$ By Proposition 2.4, matrix $\widehat{\boldsymbol{G}}_{11}$ is proper and stable. By the inner and co-inner properties of matrices $\boldsymbol{G}_{\mathrm{i}}$ and $\boldsymbol{G}_{\mathrm{ci}}$, they have no zeros on the extended imaginary axis. Therefore, matrix $\boldsymbol{S}$ must be proper and stable, because the condition (2.20) cannot be satisfied if $\boldsymbol{S}$ have poles on the extended imaginary axis. □

3. STABILIZING CONTROLLERS AND PARAMETRIZATION

We call $\boldsymbol{K}$ a stabilizing controller for the plant (1.1) if the closed loop system on Figure 1 is stable and impulse-free.

Under Assumptions 2.1 and 2.2, there are matrices $\bar{\mathcal{K}}_\mathrm{s}$ and $\mathcal{K}_\mathrm{s}$ such that matrix pencils $A - B_2\bar{\mathcal{K}}_\mathrm{s} - sE$ and $A - \mathcal{K}_\mathrm{s}C_2 - sE$ are stable and impulse-free, i. e. matrices $\bar{\boldsymbol{\Theta}}_\mathrm{s} = (sE - A + B_2\bar{\mathcal{K}}_\mathrm{s})^{-1}$ and $\boldsymbol{\Theta}_\mathrm{s} = (sE - A + \mathcal{K}_\mathrm{s}C_2)^{-1}$ are proper and stable, and define matrices

$$\begin{bmatrix} \boldsymbol{X}_\mathrm{s} & \boldsymbol{Y}_\mathrm{s} \\ -\boldsymbol{P}_\mathrm{s} & \boldsymbol{Q}_\mathrm{s} \end{bmatrix} = \left[\begin{array}{c|cc} A - \mathcal{K}_\mathrm{s}C_2 - sE & B_2 - \mathcal{K}_\mathrm{s}D_{22} & \mathcal{K}_\mathrm{s} \\ \hline \bar{\mathcal{K}}_\mathrm{s} & I & 0 \\ -C_2 & -D_{22} & I \end{array}\right], \tag{3.1}$$

$$\begin{bmatrix} \bar{\boldsymbol{Q}}_\mathrm{s} & -\bar{\boldsymbol{Y}}_\mathrm{s} \\ \bar{\boldsymbol{P}}_\mathrm{s} & \bar{\boldsymbol{X}}_\mathrm{s} \end{bmatrix} = \left[\begin{array}{c|cc} A - B_2\bar{\mathcal{K}}_\mathrm{s} - sE & B_2 & \mathcal{K}_\mathrm{s} \\ \hline -\bar{\mathcal{K}}_\mathrm{s} & I & 0 \\ C_2 - D_{22}\bar{\mathcal{K}}_\mathrm{s} & D_{22} & I \end{array}\right], \tag{3.2}$$

then matrices $\boldsymbol{X}_\mathrm{s}$, $\boldsymbol{Y}_\mathrm{s}$, $\boldsymbol{P}_\mathrm{s}$, $\boldsymbol{Q}_\mathrm{s}$, $\bar{\boldsymbol{X}}_\mathrm{s}$, $\bar{\boldsymbol{Y}}_\mathrm{s}$, $\bar{\boldsymbol{P}}_\mathrm{s}$ and $\bar{\boldsymbol{Q}}_\mathrm{s}$, are proper and stable, and the following identities analogous to (2.12) and (2.13) hold

$$\begin{bmatrix} \bar{\boldsymbol{Q}}_\mathrm{s} & -\bar{\boldsymbol{Y}}_\mathrm{s} \\ \bar{\boldsymbol{P}}_\mathrm{s} & \bar{\boldsymbol{X}}_\mathrm{s} \end{bmatrix} \begin{bmatrix} \boldsymbol{X}_\mathrm{s} & \boldsymbol{Y}_\mathrm{s} \\ -\boldsymbol{P}_\mathrm{s} & \boldsymbol{Q}_\mathrm{s} \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}_\mathrm{s} & \boldsymbol{Y}_\mathrm{s} \\ -\boldsymbol{P}_\mathrm{s} & \boldsymbol{Q}_\mathrm{s} \end{bmatrix} \begin{bmatrix} \bar{\boldsymbol{Q}}_\mathrm{s} & -\bar{\boldsymbol{Y}}_\mathrm{s} \\ \bar{\boldsymbol{P}}_\mathrm{s} & \bar{\boldsymbol{X}}_\mathrm{s} \end{bmatrix} = I, \tag{3.3}$$

$$\boldsymbol{G}_{22} = \bar{\boldsymbol{P}}_\mathrm{s}\bar{\boldsymbol{Q}}_\mathrm{s}^{-1} = \boldsymbol{Q}_\mathrm{s}^{-1}\boldsymbol{P}_\mathrm{s}\,. \tag{3.4}$$

A criterion for stabilizing controller is that the transfer matrix

$$\begin{bmatrix} I & \boldsymbol{G}_{22} \\ \boldsymbol{K} & I \end{bmatrix}^{-1} \tag{3.5}$$

is proper and stable [9]. The unstable modes (including the impulsive modes) of the closed loop system are the unstable poles of matrix (3.5) (including the poles at infinity).

Proposition 3.1. Under Assumptions 2.1 and 2.2, the unstable modes of the closed loop system with given controller $\boldsymbol{K}$ are the unstable poles of the matrix

$$(\bar{\boldsymbol{Q}}_\mathrm{s} - \boldsymbol{K}\bar{\boldsymbol{P}}_\mathrm{s})^{-1}[-\boldsymbol{K}, I]\,, \tag{3.6}$$

or equivalently, the unstable poles of the matrix

$$\begin{bmatrix} I \\ -\boldsymbol{K} \end{bmatrix} (\boldsymbol{Q}_\mathrm{s} - \boldsymbol{P}_\mathrm{s}\boldsymbol{K})^{-1}\,. \tag{3.7}$$

P r o o f. The following identities are consequences of identities (3.3) and (3.4):

$$\begin{bmatrix} I & \boldsymbol{G}_{22} \\ \boldsymbol{K} & I \end{bmatrix}^{-1} = \begin{bmatrix} \bar{\boldsymbol{P}}_\mathrm{s} & \bar{\boldsymbol{X}}_\mathrm{s} \\ -\bar{\boldsymbol{Q}}_\mathrm{s} & \bar{\boldsymbol{Y}}_\mathrm{s} \end{bmatrix} \begin{bmatrix} \boldsymbol{Y}_\mathrm{s} + (\bar{\boldsymbol{Q}}_\mathrm{s} - \boldsymbol{K}\bar{\boldsymbol{P}}_\mathrm{s})^{-1}\boldsymbol{K} & -(\bar{\boldsymbol{Q}}_\mathrm{s} - \boldsymbol{K}\bar{\boldsymbol{P}}_\mathrm{s})^{-1} \\ \boldsymbol{Q}_\mathrm{s} & 0 \end{bmatrix} \tag{3.8}$$

$$\begin{bmatrix} \boldsymbol{Y}_{\mathrm{s}} & -\boldsymbol{X}_{\mathrm{s}} \\ \boldsymbol{Q}_{\mathrm{s}} & \boldsymbol{P}_{\mathrm{s}} \end{bmatrix} \begin{bmatrix} I & \boldsymbol{G}_{22} \\ \boldsymbol{K} & I \end{bmatrix}^{-1} = \begin{bmatrix} \boldsymbol{Y}_{\mathrm{s}} + (\bar{\boldsymbol{Q}}_{\mathrm{s}} - \boldsymbol{K}\bar{\boldsymbol{P}}_{\mathrm{s}})^{-1}\boldsymbol{K} & -(\bar{\boldsymbol{Q}}_{\mathrm{s}} - \boldsymbol{K}\bar{\boldsymbol{P}}_{\mathrm{s}})^{-1} \\ \boldsymbol{Q}_{\mathrm{s}} & 0 \end{bmatrix} . \tag{3.9}$$

By identity (3.8) follows that unstable modes of the closed loop system are at most the unstable poles of matrix (3.6). By identity (3.9) follows that unstable poles of matrix (3.6) are at most the unstable modes of the closed loop system. Therefore, the unstable modes of the closed loop system are the unstable poles of matrix (3.6).

The proof concerning matrix (3.7) follows by the following two analogous identities:

$$\begin{bmatrix} I & \boldsymbol{G}_{22} \\ \boldsymbol{K} & I \end{bmatrix}^{-1} = \begin{bmatrix} 0 & (\boldsymbol{Q}_{\mathrm{s}} - \boldsymbol{P}_{\mathrm{s}}\boldsymbol{K})^{-1} \\ \bar{\boldsymbol{Q}}_{\mathrm{s}} & -\bar{\boldsymbol{Y}}_{\mathrm{s}} - \boldsymbol{K}(\boldsymbol{Q}_{\mathrm{s}} - \boldsymbol{P}_{\mathrm{s}}\boldsymbol{K})^{-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{Y}_{\mathrm{s}} & \boldsymbol{X}_{\mathrm{s}} \\ \boldsymbol{Q}_{\mathrm{s}} & -\boldsymbol{P}_{\mathrm{s}} \end{bmatrix} ,$$

$$\begin{bmatrix} I & \boldsymbol{G}_{22} \\ \boldsymbol{K} & I \end{bmatrix}^{-1} \begin{bmatrix} \bar{\boldsymbol{P}}_{\mathrm{s}} & \bar{\boldsymbol{X}}_{\mathrm{s}} \\ \bar{\boldsymbol{Q}}_{\mathrm{s}} & -\bar{\boldsymbol{Y}}_{\mathrm{s}} \end{bmatrix} = \begin{bmatrix} 0 & (\boldsymbol{Q}_{\mathrm{s}} - \boldsymbol{P}_{\mathrm{s}}\boldsymbol{K})^{-1} \\ \bar{\boldsymbol{Q}}_{\mathrm{s}} & -\bar{\boldsymbol{Y}}_{\mathrm{s}} - \boldsymbol{K}(\boldsymbol{Q}_{\mathrm{s}} - \boldsymbol{P}_{\mathrm{s}}\boldsymbol{K})^{-1} \end{bmatrix} .$$

□

Since our aim is to obtain a stable and impulse-free closed loop system, we have to find conditions under which a given marginally stabilizing controller satisfies these properties of the closed loop system.

Proposition 3.2. Under Assumptions 2.1 and 2.2, the unstable modes of the closed loop system with applied marginally stabilizing controller (2.17) or (2.18) are the unstable poles of the matrix

$$\bar{\boldsymbol{Z}}^{-1}[-\boldsymbol{S}, I] \begin{bmatrix} \boldsymbol{Q} & -\boldsymbol{P} \\ \boldsymbol{Y} & \boldsymbol{X} \end{bmatrix} , \tag{3.10}$$

or equivalently, the unstable poles of the following matrix

$$\begin{bmatrix} \bar{\boldsymbol{X}} & \bar{\boldsymbol{P}} \\ \bar{\boldsymbol{Y}} & -\bar{\boldsymbol{Q}} \end{bmatrix} \begin{bmatrix} I \\ \boldsymbol{S} \end{bmatrix} \boldsymbol{Z}^{-1} , \tag{3.11}$$

where $\bar{\boldsymbol{Z}}$ and $\boldsymbol{Z}$ are the following proper stable matrices

$$\bar{\boldsymbol{Z}} = \bar{\mathcal{L}} + (\bar{\mathcal{K}} - \bar{\mathcal{L}}\bar{\mathcal{K}}_{\mathrm{s}})\bar{\boldsymbol{\Theta}}_{\mathrm{s}}B_2 , \tag{3.12}$$

$$\boldsymbol{Z} = \mathcal{L} + C_2\boldsymbol{\Theta}_{\mathrm{s}}(\mathcal{K} - \mathcal{K}_{\mathrm{s}}\mathcal{L}) , \tag{3.13}$$

with the following inverses

$$\bar{\boldsymbol{Z}}^{-1} = [\bar{\mathcal{K}}_{\mathrm{s}}, I]\bar{\boldsymbol{\Omega}} \begin{bmatrix} 0 \\ I \end{bmatrix} , \tag{3.14}$$

$$\boldsymbol{Z}^{-1} = [0, I]\boldsymbol{\Omega} \begin{bmatrix} \mathcal{K}_{\mathrm{s}} \\ I \end{bmatrix} . \tag{3.15}$$

The unstable zeros of matrix $\bar{\boldsymbol{Z}}$ are the unstable poles of matrix $\bar{\boldsymbol{\Omega}}$, and the unstable zeros of matrix $\boldsymbol{Z}$ are the unstable poles of matrix $\boldsymbol{\Omega}$.

P r o o f. A direct consequence of Proposition 3.1 is that the unstable modes of the closed loop system with applied marginally stabilizing controller (2.17) or (2.18) are the unstable poles of the matrix

$$\left([-\boldsymbol{S}, I]\begin{bmatrix}\boldsymbol{Q} & -\boldsymbol{P}\\ \boldsymbol{Y} & \boldsymbol{X}\end{bmatrix}\begin{bmatrix}\bar{\boldsymbol{P}}_{\mathrm{s}}\\ \bar{\boldsymbol{Q}}_{\mathrm{s}}\end{bmatrix}\right)^{-1}[-\boldsymbol{S}, I]\begin{bmatrix}\boldsymbol{Q} & -\boldsymbol{P}\\ \boldsymbol{Y} & \boldsymbol{X}\end{bmatrix}, \tag{3.16}$$

or equivalently, the unstable poles of the following matrix

$$\begin{bmatrix}\bar{\boldsymbol{X}} & \bar{\boldsymbol{P}}\\ \bar{\boldsymbol{Y}} & -\bar{\boldsymbol{Q}}\end{bmatrix}\begin{bmatrix}I\\ \boldsymbol{S}\end{bmatrix}\left([\boldsymbol{Q}_{\mathrm{s}}, \boldsymbol{P}_{\mathrm{s}}]\begin{bmatrix}\bar{\boldsymbol{X}} & \bar{\boldsymbol{P}}\\ \bar{\boldsymbol{Y}} & -\bar{\boldsymbol{Q}}\end{bmatrix}\begin{bmatrix}I\\ \boldsymbol{S}\end{bmatrix}\right)^{-1}. \tag{3.17}$$

Using identities (2.12), (2.13) and (3.4), we obtain

$$[-\boldsymbol{S}, I]\begin{bmatrix}\boldsymbol{Q} & -\boldsymbol{P}\\ \boldsymbol{Y} & \boldsymbol{X}\end{bmatrix}\begin{bmatrix}\bar{\boldsymbol{P}}_{\mathrm{s}}\\ \bar{\boldsymbol{Q}}_{\mathrm{s}}\end{bmatrix} = \bar{\boldsymbol{Z}} = \bar{\boldsymbol{Q}}^{-1}\bar{\boldsymbol{Q}}_{\mathrm{s}},$$

and

$$[\boldsymbol{Q}_{\mathrm{s}}, \boldsymbol{P}_{\mathrm{s}}]\begin{bmatrix}\bar{\boldsymbol{X}} & \bar{\boldsymbol{P}}\\ \bar{\boldsymbol{Y}} & -\bar{\boldsymbol{Q}}\end{bmatrix}\begin{bmatrix}I\\ \boldsymbol{S}\end{bmatrix} = \boldsymbol{Z} = \boldsymbol{Q}_{\mathrm{s}}\boldsymbol{Q}^{-1}.$$

By these identities, the claims with the matrices (3.10) and (3.11) are obvious, and, using the transfer matrices algebra, we obtain the identities (3.14) and (3.15).

To prove the last part of the proposition, we prove that the realizations (3.14) and (3.15) are without hidden modes in $\Re[s] \geq 0$. This follows by the fact that matrix pencils

$$\begin{bmatrix}A - sE & B_2 & 0\\ \bar{\mathcal{K}} & \bar{\mathcal{L}} & I\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}A - sE & \mathcal{K} & \mathcal{K}_{\mathrm{s}}\\ C_2 & \mathcal{L} & I\end{bmatrix} \sim \begin{bmatrix}A - \mathcal{K}_{\mathrm{s}}C_2 - sE & \mathcal{K} - \mathcal{K}_{\mathrm{s}}\mathcal{L} & \mathcal{K}_{\mathrm{s}}\\ 0 & 0 & I\end{bmatrix}$$

have full row ranks in $\Re[s] \geq 0$, the later by the definition of matrix $\mathcal{K}_{\mathrm{s}}$, and matrix pencils

$$\begin{bmatrix}A - sE & B_2\\ \bar{\mathcal{K}} & \bar{\mathcal{L}}\\ \bar{\mathcal{K}}_{\mathrm{s}} & I\end{bmatrix} \sim \begin{bmatrix}A - B_2\bar{\mathcal{K}}_{\mathrm{s}} - sE & 0\\ \bar{\mathcal{K}} - \bar{\mathcal{L}}\bar{\mathcal{K}}_{\mathrm{s}} & 0\\ \bar{\mathcal{K}}_{\mathrm{s}} & I\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}A - sE & \mathcal{K}\\ C_2 & \mathcal{L}\\ 0 & I\end{bmatrix}$$

have full column ranks in $\Re[s] \geq 0$, the first by the definition of matrix $\bar{\mathcal{K}}_{\mathrm{s}}$. □

We introduce the following assumption.

Assumption 3.3. Let matrix $\boldsymbol{G}_{21}$ be without invariant zeros at infinity, and the invariant zeros at infinity of $\boldsymbol{G}_{12}$ be of simple multiplicity. Let $\mathrm{j}\omega_k$, $k = 1, \ldots, L$ be the invariant imaginary axis zeros of matrix $\boldsymbol{G}_{12}$ and $\mathrm{j}\bar{\omega}_i$, $i = 1, \ldots, \bar{L}$ be the invariant imaginary axis zeros of matrix $\boldsymbol{G}_{21}$. Assume that all of them are of simple multiplicity. Assume that $\mathrm{j}\omega_k$ are pairwise distinct of $\mathrm{j}\bar{\omega}_i$.

By (3.12), the invariant zeros on the imaginary axis of matrices $\boldsymbol{G}_{12}$ and $\bar{\boldsymbol{Z}}$ coincide. Analogously, by (3.13), the invariant zeros on the imaginary axis of matrices $\boldsymbol{G}_{21}$ and $\boldsymbol{Z}$ concide.

Define matrices

$$X_0^{\mathrm{T}} = \mathrm{null}\, \bar{\boldsymbol{Z}}(\infty)^{\mathrm{T}} \ , \tag{3.18}$$

$$Y_0 = -X_0(\bar{\mathcal{K}} - \bar{\mathcal{L}}\bar{\mathcal{K}}_{\mathrm{s}})\bar{\boldsymbol{\Theta}}_{\mathrm{s}}(\infty)\mathcal{K} \ , \tag{3.19}$$

$$X_k^* = \mathrm{null}\, \bar{\boldsymbol{Z}}(\mathrm{j}\omega_k)^* \ , \tag{3.20}$$

$$Y_k = -X_k(\bar{\mathcal{K}} - \bar{\mathcal{L}}\bar{\mathcal{K}}_{\mathrm{s}})\bar{\boldsymbol{\Theta}}_{\mathrm{s}}(\mathrm{j}\omega_k)\mathcal{K} \ , \quad k = 1, \ldots, L \ , \tag{3.21}$$

$$\bar{X}_i = \mathrm{null}\, \boldsymbol{Z}(\mathrm{j}\bar{\omega}_i) \ , \tag{3.22}$$

$$\bar{Y}_i = \bar{\mathcal{K}}\boldsymbol{\Theta}_{\mathrm{s}}(\mathrm{j}\bar{\omega}_i)(\mathcal{K} - \mathcal{K}_{\mathrm{s}}\mathcal{L})\bar{X}_i \ , \quad i = 1, \ldots, \bar{L} \ , \tag{3.23}$$

where by null (H) we denote a matrix whose columns span a unitary basis of the right kernel of the matrix H.

Theorem 3.4. Let Assumptions 2.1, 2.2 and 3.3 hold and the norm ($\mathscr{H}_2$ or $\mathscr{H}_\infty$) of $\mathscr{F}(\boldsymbol{G}, \boldsymbol{K})$ be finite. The controller (2.17) or (2.18) is stabilizing if and only if matrix $\boldsymbol{S}$ satisfies the following left and right interpolation conditions

$$Y_0 = -X_0\boldsymbol{S}(\infty) \ , \tag{3.24}$$

$$Y_k = -X_k\boldsymbol{S}(\mathrm{j}\omega_k) \ , \quad k = 1, \ldots, L \ , \tag{3.25}$$

$$\bar{Y}_i = \boldsymbol{S}(\mathrm{j}\bar{\omega}_i)\bar{X}_i \ , \quad i = 1, \ldots, \bar{L} \ . \tag{3.26}$$

Proof. Note that by the theorem's assumptions, matrix $\boldsymbol{S}$ is without poles on the extended imaginary axis. By Proposition 3.2, matrices $\boldsymbol{Z}$ and $\bar{\boldsymbol{Z}}$ are independent of $\boldsymbol{S}$ and their zeros are independent of $\bar{\mathcal{K}}_{\mathrm{s}}$ and $\mathcal{K}_{\mathrm{s}}$. The closed loop system with controller (2.17) or (2.18) is stable if and only if $\mathrm{j}\omega_k$, $k = 1, \ldots, L$, which are the unstable poles of $\bar{\boldsymbol{Z}}^{-1}$, are not poles of matrix (3.10), and $\mathrm{j}\bar{\omega}_i$, $i = 1, \ldots, \bar{L}$, which are the unstable poles of $\boldsymbol{Z}^{-1}$, are not poles of matrix (3.11).

Necessity. This happens only if for every $\mathrm{j}\omega_k$ (which can be infinity):

$$X_k[-\boldsymbol{S}, I]\begin{bmatrix} \boldsymbol{Q} & -\boldsymbol{P} \\ \boldsymbol{Y} & \boldsymbol{X} \end{bmatrix} = 0 \ , \quad X_k[-\boldsymbol{S}, I]\begin{bmatrix} \boldsymbol{Q} & -\boldsymbol{P} \\ \boldsymbol{Y} & \boldsymbol{X} \end{bmatrix}\begin{bmatrix} \bar{\boldsymbol{P}}_{\mathrm{s}} \\ \bar{\boldsymbol{Q}}_{\mathrm{s}} \end{bmatrix} = X_k\bar{\boldsymbol{Z}} = 0 \ , \tag{3.27}$$

and for every $\mathrm{j}\bar{\omega}_i$:

$$\begin{bmatrix} \bar{\boldsymbol{X}} & \bar{\boldsymbol{P}} \\ \bar{\boldsymbol{Y}} & -\bar{\boldsymbol{Q}} \end{bmatrix}\begin{bmatrix} I \\ \boldsymbol{S} \end{bmatrix}\bar{X}_i = 0 \ , \quad [\boldsymbol{Q}_{\mathrm{s}}, \boldsymbol{P}_{\mathrm{s}}]\begin{bmatrix} \bar{\boldsymbol{X}} & \bar{\boldsymbol{P}} \\ \bar{\boldsymbol{Y}} & -\bar{\boldsymbol{Q}} \end{bmatrix}\begin{bmatrix} I \\ \boldsymbol{S} \end{bmatrix}\bar{X}_i = \boldsymbol{Z}\bar{X}_i = 0 \ . \tag{3.28}$$

The conditions (3.24), (3.25) and (3.26) we derive using the realizations (2.10) and (2.11).

Sufficiency. By noting that the second identities in (3.27) and (3.28) are consequences of the first, and that $\mathrm{j}\omega_k$ are pairwise distinct of $\mathrm{j}\bar{\omega}_i$, by assumption. □

Remark 1. If matrix pencil

$$\begin{bmatrix} A - sE & B_2 \\ C_1 & D_{12} \end{bmatrix}$$

has no zeros on the extended imaginary axis, i. e. its finite generalized eigenvalues (FGEs) are not on the imaginary axis and its IGEs are of multiplicity ≤ 1, and if matrix pencil

$$\begin{bmatrix} A - sE & B_1 \\ C_2 & D_{21} \end{bmatrix}$$

has no zeros on the extended imaginary axis,[1] then all stabilizing controllers $\boldsymbol{K}$ such that the norm of $\mathscr{F}(\boldsymbol{G}, \boldsymbol{K})$ is finite are given by (2.17) or (2.18), where $\boldsymbol{S}$ is a proper stable parameter matrix satisfying $\det(\bar{\boldsymbol{X}} + \bar{\boldsymbol{P}}\boldsymbol{S}) \not\equiv 0$. Indeed, matrices $\bar{\boldsymbol{\Omega}}$ and $\boldsymbol{\Omega}$ are proper and stable. This claim is a consequence of Proposition 3.2.

4. OPTIMAL CONTROL OF DESCRIPTOR SYSTEMS

4.1. $\mathscr{H}_2$-optimal control

Theorem 4.1. Under Assumptions 2.1 and 2.2, there exists a solution of the $\mathscr{H}_2$ control problem if and only if there is a solution S_∞ of the matrix equation:

$$\widehat{\boldsymbol{G}}_{11}(\infty) + \boldsymbol{G}_{\mathrm{i}}(\infty) S_\infty \boldsymbol{G}_{\mathrm{ci}}(\infty) = 0 \ , \tag{4.1}$$

and the controller

$$\boldsymbol{K} = \left[\begin{array}{ccc|c} A - sE & B_2 & \mathcal{K} & 0 \\ \bar{\mathcal{K}} & \bar{\mathcal{L}} & -S_\infty & 0 \\ C_2 & D_{22} & \mathcal{L} & -I \\ \hline 0 & I & 0 & 0 \end{array}\right] \tag{4.2}$$

exists and satisfies $\det(I - \boldsymbol{G}_{22}\boldsymbol{K}) \not\equiv 0$ and is stabilizing. In that case controller (4.2) is $\mathscr{H}_2$ optimal and unique.

Proof. *Necessity.* We apply the parametrization of marginally stabilizing controllers (2.17) or (2.18). The necessity of (4.1) follows by the finiteness of the norm $\|\mathscr{F}(\boldsymbol{G}, \boldsymbol{K})\|_2$, and by Theorems 2.5 and 2.6.

Introduce matrix $\boldsymbol{M} = \boldsymbol{G}_{\mathrm{i}}^{\sim} \widehat{\boldsymbol{G}}_{11} \boldsymbol{G}_{\mathrm{ci}}^{\sim}$. By Proposition 2.4, it is proper and its poles are not on the imaginary axis.

Lemma 4.2. We have

$$\boldsymbol{M} = \boldsymbol{G}_{\mathrm{i}}^{\sim} D_{11} \boldsymbol{G}_{\mathrm{ci}}^{\sim} + \bar{\mathcal{K}}[X_1^{\mathrm{T}}, X_2^{\mathrm{T}}]\boldsymbol{\Omega}^{\sim} \begin{bmatrix} 0 \\ I \end{bmatrix} + [0, I]\bar{\boldsymbol{\Omega}}^{\sim} \begin{bmatrix} \bar{X}_1^{\mathrm{T}} \\ \bar{X}_2^{\mathrm{T}} \end{bmatrix} B_1 [B_1^{\mathrm{T}}, D_{21}^{\mathrm{T}}] \boldsymbol{\Omega}^{\sim} \begin{bmatrix} 0 \\ I \end{bmatrix} \ , \tag{4.3}$$

for some matrices X_1, X_2, $\bar{X}_1$, and $\bar{X}_2$.

[1]These conditions, together with Assumptions 2.1 and 2.2 we call regularity conditions for the $\mathscr{H}_2$ and $\mathscr{H}_\infty$ control problems of descriptor systems. They generalize the regularity conditions for the corresponding state-space problems (conditions i), ii), iii) iv) in Section 14.5 and conditions A1, A2, A3 and A4 in Section 17.1 of [24].)

P r o o f . We start with the realizations of $\boldsymbol{G}_{\mathrm{i}}$ in (2.3) and $\widehat{\boldsymbol{G}}_{11}$ in (2.16), and the following identity, proved in Proposition 5.3

$$\bar{\boldsymbol{\Omega}}^{\sim} \begin{bmatrix} C_1^{\mathrm{T}} \\ D_{12}^{\mathrm{T}} \end{bmatrix} [C_1, D_{12}]\bar{\boldsymbol{\Omega}} = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} \bar{X}_1 & \bar{X}_2 \\ 0 & 0 \end{bmatrix} \bar{\boldsymbol{\Omega}} - \bar{\boldsymbol{\Omega}}^{\sim} \begin{bmatrix} \bar{X}_1^{\mathrm{T}} & 0 \\ \bar{X}_2^{\mathrm{T}} & 0 \end{bmatrix}, \tag{4.4}$$

for some matrices $\bar{X}_1$ and $\bar{X}_2$. We obtain

$$\boldsymbol{G}_{\mathrm{i}}^{\sim}\widehat{\boldsymbol{G}}_{11} = \boldsymbol{G}_{\mathrm{i}}^{\sim} D_{11} - \bar{\mathcal{K}}[I,0]\boldsymbol{\Omega} \begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} + [I,0]\bar{\boldsymbol{\Omega}}^{\sim} \begin{bmatrix} \bar{X}_1^{\mathrm{T}} \\ \bar{X}_2^{\mathrm{T}} \end{bmatrix} B_1 \ , \tag{4.5}$$

We use the following identity, analogous to (4.4):

$$\boldsymbol{\Omega} \begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} [B_1^{\mathrm{T}}, D_{21}^{\mathrm{T}}]\boldsymbol{\Omega}^{\sim} = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} X_1^{\mathrm{T}} & X_2^{\mathrm{T}} \\ 0 & 0 \end{bmatrix} \boldsymbol{\Omega}^{\sim} - \boldsymbol{\Omega} \begin{bmatrix} X_1 & 0 \\ X_2 & 0 \end{bmatrix} \ , \tag{4.6}$$

for some matrices X_1 and X_2. Using also the realization of $\boldsymbol{G}_{\mathrm{ci}}$ in (2.3), we find the realization (4.3) of $\boldsymbol{M}$. □

By this lemma and Proposition 2.4, matrix $\boldsymbol{M}$ is proper and its poles are in $\Re[s] > 0$. Introduce the stable and strict proper matrix:

$$\boldsymbol{G}_1 = \widehat{\boldsymbol{G}}_{11} - \widehat{\boldsymbol{G}}_{11}(\infty) \ , \tag{4.7}$$

the matrix with all its poles in $\Re[s] > 0$:

$$\boldsymbol{M}_1 = \boldsymbol{G}_{\mathrm{i}}^{\sim}\boldsymbol{G}_1\boldsymbol{G}_{\mathrm{ci}}^{\sim} + \boldsymbol{G}_{\mathrm{i}}^{\sim}\widehat{\boldsymbol{G}}_{11}(\infty)\boldsymbol{G}_{\mathrm{ci}}^{\sim} + S_{\infty} \ , \tag{4.8}$$

which is strict proper by (4.1), and the new unknown matrix $\boldsymbol{S}_1 = \boldsymbol{S} - S_{\infty}$. By Theorem 2.5 we obtain

$$\|\mathscr{F}(\boldsymbol{G},\boldsymbol{K})\|_2^2$$
$$= \frac{1}{2\pi \mathrm{j}} \int_{-\mathrm{j}\infty}^{\mathrm{j}\infty} (\mathrm{Tr}\{\boldsymbol{G}_1\boldsymbol{G}_1^{\sim}\} - \mathrm{Tr}\{\boldsymbol{M}_1\boldsymbol{M}_1^{\sim}\} + \mathrm{Tr}\{(\boldsymbol{S}_1 + \boldsymbol{M}_1)(\boldsymbol{S}_1 + \boldsymbol{M}_1)^{\sim}\})\,\mathrm{d}s \ .$$

By the finiteness of both sides of this identity and the strict properness of $\boldsymbol{M}_1$ follows that $\boldsymbol{S}_1$ must be strict proper and stable. Since $\int_{-\mathrm{j}\infty}^{\mathrm{j}\infty} \boldsymbol{S}_1\boldsymbol{M}_1^{\sim}\mathrm{d}s = 0$ (both $\boldsymbol{S}_1$ and $\boldsymbol{M}_1^{\sim}$ are strict proper and stable), we obtain

$$\|\mathscr{F}(\boldsymbol{G},\boldsymbol{K})\|_2^2 = \frac{1}{2\pi \mathrm{j}} \int_{-\mathrm{j}\infty}^{\mathrm{j}\infty} \mathrm{Tr}\{\boldsymbol{G}_1\boldsymbol{G}_1^{\sim}\}\mathrm{d}s + \frac{1}{2\pi \mathrm{j}} \int_{-\mathrm{j}\infty}^{\mathrm{j}\infty} \mathrm{Tr}\{\boldsymbol{S}_1\boldsymbol{S}_1^{\sim}\}\mathrm{d}s \ . \tag{4.9}$$

By this identity, the minimum is achieved for $\boldsymbol{S}_1 = 0$ i. e. for $\boldsymbol{S} = S_{\infty}$. By

$$\boldsymbol{K} = (-\bar{\boldsymbol{Y}} + \bar{\boldsymbol{Q}}S_{\infty})(\bar{\boldsymbol{X}} + \bar{\boldsymbol{P}}S_{\infty})^{-1} \tag{4.10}$$

and by (2.11), optimal controller $\boldsymbol{K}$ is given with (4.2).
Sufficiency. Obvious by the formulation of Theorem 4.1. □

A drawback of Theorem 4.1 is that the necessary and sufficient condition is given by the stability of a rational matrix. In the following theorem the necessary and sufficient conditions are identities with constant matrices.

Theorem 4.3. Under Assumptions 2.1, 2.2 and 3.3 there exists a solution of the $\mathscr{H}_2$ control problem if and only if equation (4.1) is solvable and conditions (3.24), (3.25) and (3.26) with $\boldsymbol{S} = S_\infty$ hold.

Proof. We combine the necessary condition $\boldsymbol{S} = S_\infty$ of Theorem 4.1 with Theorem 3.4. $\square$

Remark 2. Not only the property being stabilizing controller, but also the properness of the controller can be checked by formula (4.2).

Example 1. Let be given the following matrices

$$A = \begin{bmatrix} 1 & -1 & 0 & 3 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 3 & 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -3 & 3 & 0 & 0 \\ 0 & 0 & 2 & 2 & 5 & 3 & 0 & 0 \\ 1 & -2 & 3 & 0 & -2 & -2 & 3 & 0 \\ 1 & -2 & 0 & 1 & 0 & -1 & 0 & -2 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} -4 & -2 & 4 & 0 \\ 1 & -1 & -2 & 1 \\ 1 & -3 & -1 & -1 \\ -1 & 2 & 1 & 2 \\ 1 & 0 & 4 & 2 \\ -2 & 5 & -2 & 4 \\ -1 & -2 & 1 & 2 \\ 2 & 0 & 0 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.5 & 4 & 3 \\ 0.5 & 2 & -2 \\ 0 & 0 & 3 \\ 0 & 0 & -1 \\ 0 & 2 & 1 \\ 0 & 6 & 2 \\ -2 & 0 & -1 \\ -3 & -2 & -1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & -1 & 4 & 0 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} -3 & 4 & -1 & 3 & 3 & -2 & 2 & 1 \\ 0 & -2 & 3 & -1 & 1 & 3 & -1 & -3 \end{bmatrix},$$

$$D_{12} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

and $D_{11} = 0$, $D_{22} = 0$. Matrix pencil $A - sE$ has an IGE of multiplicity two, hence transfer matrix $\boldsymbol{G}$ is improper. The system (E, A, B_2, C_1, D_{12}) has four invariant zeros on the extended imaginary axis: ∞, 0 and $\pm$j, of single multiplicity (equal to one).

We obtain

$$\bar{\mathcal{K}} = \begin{bmatrix} -0.2662 & 1.9459 & -1.2857 & -0.7321 & 0.2934 & -1.7361 & 0 & 0.5324 \\ -0.3153 & -0.7660 & -0.8870 & -0.6539 & 0.1404 & -1.1984 & 0 & 0.6306 \\ -0.2823 & 0.7918 & 2.5467 & 0.5352 & -0.9626 & 3.4356 & 0 & 0.5646 \end{bmatrix},$$

$$\bar{\mathcal{L}} = \begin{bmatrix} 1.5053 & 0 & -0.1997 \\ 0.2476 & 0 & -0.0233 \\ 0.9605 & 0 & 1.0998 \end{bmatrix},$$

$$\mathcal{K} = \begin{bmatrix} 24.5960 & 2.8776 \\ 12.6357 & 5.8103 \\ 32.0949 & -4.2224 \\ 11.5744 & 0.3992 \\ 2.6290 & 2.6601 \\ 2.6323 & 5.9269 \\ 22.4456 & -10.0850 \\ -1.2867 & 1.8288 \end{bmatrix}, \quad \mathcal{L} = \begin{bmatrix} 3.7348 & -1.7467 \\ -1.2867 & 1.8288 \end{bmatrix}.$$

A stable state space realization of matrix $\widehat{\boldsymbol{G}}$ is given by the following A, B, C and D matrices,

$$\begin{bmatrix} -0.255 & -0.6364 & -0.4787 & -0.4283 & -0.1995 & 0.5224 & -0.1684 & 0.1155 \\ 0.5704 & -0.393 & -1.881 & -1.35 & -0.4869 & 1.345 & -0.4889 & 0.1345 \\ 0.2683 & 1.535 & -1.463 & -6.912 & -2.113 & 4.003 & -1.47 & 0.04061 \\ -0.1864 & -0.1608 & 5.835 & -6.823 & -1.487 & 11.4 & -4.726 & 0.2229 \\ 0.08937 & -0.4478 & 1.805 & -5.294 & -2.619 & 6.722 & -2.435 & 1.1 \\ -0.1541 & -0.08031 & -1.493 & 1.317 & 0.2431 & -12.87 & 13.3 & -1.737 \\ -0.264 & 0.1368 & 0.4602 & -1.92 & -0.513 & -8.855 & -9.737 & 1.853 \\ -0.05425 & -0.08103 & 0.8516 & -0.6323 & -0.2249 & 1.055 & -0.5682 & -4.816 \end{bmatrix}$$

$$\begin{bmatrix} 1.655 & 2.942 & 1.691 & 2.951 & -0.00797 & -0.009156 & -0.006704 \\ -3.146 & -1.55 & 1.099 & -1.609 & -0.03064 & -0.0352 & -0.02577 \\ 1.384 & -0.5284 & -5.476 & 0.5997 & -0.02468 & -0.02836 & -0.02076 \\ -4.099 & 6.266 & 3.307 & -2.234 & -0.1247 & -0.1433 & -0.1049 \\ -3.184 & 2.073 & 0.9913 & -1.956 & 0.3888 & 0.4467 & 0.3271 \\ -1.456 & -3.486 & -0.6751 & 6.627 & 0.0001315 & 0.000151 & 0.0001106 \\ -1.259 & 1.888 & 0.5497 & 2.572 & 0.00137 & 0.001573 & 0.001152 \\ 0.3408 & -0.2769 & 1.829 & -0.1502 & -0.007225 & -0.0083 & -0.006077 \end{bmatrix}$$

$$\begin{bmatrix} -4.468 & -2.641 & -2.103 & -1.369 & -0.8468 & 2.547 & -0.6262 & 1.3 \\ -0.07305 & 0.163 & -0.1957 & 0.5162 & 3.171 & -0.8142 & 0.001223 & -1.109 \\ -1.729 & -3.009 & -5.298 & -8.208 & -2.947 & 6.815 & -2.798 & -0.3143 \\ -0.009756 & -0.04692 & 0.1029 & -1.584 & -0.4871 & 2.157 & -1.573 & 0.7276 \\ -0.0815 & 0.1896 & -0.07571 & 0.1507 & 0.05845 & 0.6255 & -1.171 & -0.1211 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0.7067 & -0.6983 & 0.1136 \\ 1 & 0 & 0 & 0.5 & -0.5324 & -0.6306 & -0.5646 \\ 0 & 0 & 0 & 0 & -0.4659 & -0.3386 & 0.8175 \\ -0.4349 & 0.7981 & 0 & -0.417 & 0 & 0 & 0 \\ 0.7877 & 0.5616 & 0 & 0.2534 & 0 & 0 & 0 \end{bmatrix}$$

We find

$$S_\infty = \begin{bmatrix} -0.3425 & 0.4868 \\ -0.4057 & 0.5766 \\ -0.3633 & 0.5163 \end{bmatrix}.$$

The infimal value of the $\mathscr{H}_2$ norm of the closed-loop system is 47.6043. It is achieved by the following state-space realization of $\mathscr{H}_2$ controller $\boldsymbol{K}$

$$\left[\begin{array}{ccccccc|cc}
-14.48 & 38.58 & -32.54 & -14.87 & -8.448 & 134.3 & 104.9 & 1025 & 2092 \\
-0.7943 & -3.58 & -5.085 & -4.494 & -4.213 & 6.946 & 13.55 & 88.1 & 143.4 \\
4.744 & 17.8 & -14.02 & -10.8 & -13.11 & 62.38 & 39.65 & 458.3 & 904.4 \\
-8.443 & -9.237 & 13.5 & 10.56 & 14.46 & -52.61 & -34.43 & -376.4 & -760.3 \\
4.172 & 7.301 & -3.692 & -7.407 & -9.089 & 22.73 & 12.97 & 195.4 & 322.5 \\
11.24 & 21.63 & -13 & -19.75 & -19.08 & 71.95 & 43.71 & 555.8 & 1057 \\
-7.529 & -3.061 & 10.5 & 10.58 & 8.085 & -41.1 & -34.35 & -313.5 & -666.6 \\
\hline
-0.009692 & 0.01139 & -0.0125 & -0.005731 & 0.002471 & -0.007388 & -0.008867 & 0 & 0 \\
-0.002822 & -0.0002391 & 0.002482 & -0.001076 & -0.002065 & 0.002953 & -0.00383 & 0 & 0 \\
-0.01718 & -0.01966 & -0.01043 & 0.008739 & -0.01208 & -0.01247 & 0.03778 & 0 & 0
\end{array}\right]$$

This controller is not stabilizing, it is only marginally stabilizing. Another criterion to check the existence of a stabilizing $\mathscr{H}_2$ controller is given in Theorem 4.3. Assumptions 2.1, 2.2 and 3.3 of that theorem are satisfied, but the conditions (3.25) do not hold. Indeed,

$$X_1 S_\infty + Y_1$$
$$= [-0.6201 + 0.0122\mathrm{j},\ 0.8813 - 0.0174\mathrm{j}] + [-2.8406 + 2.5078\mathrm{j},\ -2.0570 + 0.0565\mathrm{j}] \neq 0$$

$$X_2 S_\infty + Y_2$$
$$= [-0.6201 - 0.0122\mathrm{j},\ 0.8813 + 0.0174\mathrm{j}] + [-2.8406 - 2.5078\mathrm{j},\ -2.0570 - 0.0565\mathrm{j}] \neq 0$$

$$X_3 S_\infty + Y_3 = [-0.6107,\ 0.8680] + [19.9826,\ -3.1029] \neq 0\ .$$

4.2. $\mathscr{H}_\infty$ control

Motivated by Theorem 2.6 (the part with $\mathscr{H}_\infty$ norm), we define γ_{is}, the infimum over stabilizing controllers, and γ_{ims}, the infimum over marginally stabilizing controllers. In general, $\gamma_{\text{is}} \geq \gamma_{\text{ims}}$. γ_{ims} can be found by iterative solving in $\boldsymbol{S}$ the regular state-space problem $\left\|\mathscr{F}(\widehat{\boldsymbol{G}}, \boldsymbol{S})\right\|_\infty < \gamma$ for decreasing γ's.

Let there exist a solution of that regular problem, and let by

$$\boldsymbol{S} = (\boldsymbol{\Psi}_{22} - \boldsymbol{U}\boldsymbol{\Psi}_{12})^{-1}(-\boldsymbol{\Psi}_{21} + \boldsymbol{U}\boldsymbol{\Psi}_{11}) \tag{4.11}$$

$$= (\bar{\boldsymbol{\Psi}}_{21} + \bar{\boldsymbol{\Psi}}_{22}\boldsymbol{U})(\bar{\boldsymbol{\Psi}}_{11} + \bar{\boldsymbol{\Psi}}_{12}\boldsymbol{U})^{-1} \tag{4.12}$$

be given all its solutions, where $\boldsymbol{U}$ is a proper stable matrix satisfying $\|\boldsymbol{U}\|_\infty < 1$, and

$$\boldsymbol{\Psi} = \begin{bmatrix} \boldsymbol{\Psi}_{11} & \boldsymbol{\Psi}_{12} \\ \boldsymbol{\Psi}_{21} & \boldsymbol{\Psi}_{22} \end{bmatrix} = \bar{\boldsymbol{\Psi}}^{-1} = \begin{bmatrix} \bar{\boldsymbol{\Psi}}_{11} & \bar{\boldsymbol{\Psi}}_{12} \\ \bar{\boldsymbol{\Psi}}_{21} & \bar{\boldsymbol{\Psi}}_{22} \end{bmatrix}^{-1} \tag{4.13}$$

is a biproper minimum phase and stable matrix (Theorem 4.6 in [3]).

Introduce the marginally minimum phase and marginally stable matrix:

$$\boldsymbol{\Gamma} = \boldsymbol{\Psi} \begin{bmatrix} -\boldsymbol{P} & \boldsymbol{Q} \\ \boldsymbol{X} & \boldsymbol{Y} \end{bmatrix}\ . \tag{4.14}$$

Denote $\boldsymbol{\Gamma}^{-1} = \bar{\boldsymbol{\Gamma}}$, and introduce the corresponding partitions of matrices $\boldsymbol{\Gamma}$ and $\bar{\boldsymbol{\Gamma}}$. By composing the mappings (2.17) and (2.18) with the mappings (4.11) and (4.12), we obtain a parametrization on $\boldsymbol{U}$ of the controllers

$$\boldsymbol{K} = (\boldsymbol{\Gamma}_{21} - \boldsymbol{U}\boldsymbol{\Gamma}_{11})^{-1}(-\boldsymbol{\Gamma}_{22} + \boldsymbol{U}\boldsymbol{\Gamma}_{12}) \tag{4.15}$$

$$= (\bar{\mathbf{\Gamma}}_{11} + \bar{\mathbf{\Gamma}}_{12}\boldsymbol{U})(\bar{\mathbf{\Gamma}}_{21} + \bar{\mathbf{\Gamma}}_{22}\boldsymbol{U})^{-1} \ .$$

Proposition 4.4. There is a proper stable matrix $\boldsymbol{S}$ satisfying (3.24), (3.25) and (3.26) if and only if there is a proper stable matrix $\boldsymbol{U}$ such that

$$U_0 = -V_0\boldsymbol{U}(\infty) \ , \tag{4.16}$$

$$U_k = -V_k\boldsymbol{U}(\mathrm{j}\omega_k) \ , \quad k = 1, \ldots, L \tag{4.17}$$

$$\bar{V}_i = \boldsymbol{U}(\mathrm{j}\bar{\omega}_i)\bar{U}_i \ , \quad i = 1, \ldots, \bar{L} \tag{4.18}$$

where

$$U_0 = Y_0\bar{\boldsymbol{\Psi}}_{11}(\infty) + X_0\bar{\boldsymbol{\Psi}}_{21}(\infty) \tag{4.19}$$

$$V_0 = Y_0\bar{\boldsymbol{\Psi}}_{12}(\infty) + X_0\bar{\boldsymbol{\Psi}}_{22}(\infty) \tag{4.20}$$

$$U_k = Y_k\bar{\boldsymbol{\Psi}}_{11}(\mathrm{j}\omega_k) + X_k\bar{\boldsymbol{\Psi}}_{21}(\mathrm{j}\omega_k) \tag{4.21}$$

$$V_k = Y_k\bar{\boldsymbol{\Psi}}_{12}(\mathrm{j}\omega_k) + X_k\bar{\boldsymbol{\Psi}}_{22}(\mathrm{j}\omega_k) \ , \quad k = 1, \ldots, L \tag{4.22}$$

$$\bar{U}_i = \boldsymbol{\Psi}_{11}(\mathrm{j}\bar{\omega}_i)\bar{X}_i + \boldsymbol{\Psi}_{12}(\mathrm{j}\bar{\omega}_i)\bar{Y}_i \tag{4.23}$$

$$\bar{V}_i = \boldsymbol{\Psi}_{21}(\mathrm{j}\bar{\omega}_i)\bar{X}_i + \boldsymbol{\Psi}_{22}(\mathrm{j}\bar{\omega}_i)\bar{Y}_i \ , \quad i = 1, \ldots, \bar{L} \tag{4.24}$$

Proof. Follows by identities (3.24), (3.25) and (3.26) and identities (4.11) and (4.12). □

By Proposition 4.4, the $\mathscr{H}_\infty$ suboptimal control problem of descriptor systems becomes a left and right boundary interpolation problem, where the interpolant matrix is $\boldsymbol{U}$, and the left and right conditions are (4.16), (4.17) and (4.18). The following theorem gives necessary and sufficient conditions.

Theorem 4.5. Let Assumptions 2.1, 2.2 and 3.3 hold. The $\mathscr{H}_\infty$ control problem $\|\mathscr{F}(\boldsymbol{G},\boldsymbol{K})\|_\infty < \gamma$ has a stabilizing solution $\boldsymbol{K}$ if and only if the regular state-space problem $\|\mathscr{F}(\widehat{\boldsymbol{G}},\boldsymbol{S})\|_\infty < \gamma$ has a stabilizing solution $\boldsymbol{S}$, and

$$V_kV_k^* - U_kU_k^* > 0 \ , \quad k = 0, 1, \ldots, L \ , \tag{4.25}$$

$$\bar{U}_i^*\bar{U}_i - \bar{V}_i^*\bar{V}_i > 0 \ , \quad i = 1, \ldots, \bar{L} \ . \tag{4.26}$$

If these conditions are satisfied, a parametrization of controllers $\boldsymbol{K}$ is given by

$$\boldsymbol{K} = (\boldsymbol{H}_{21} + \boldsymbol{H}_{22}\boldsymbol{V})(\boldsymbol{H}_{11} + \boldsymbol{H}_{12}\boldsymbol{V})^{-1} \tag{4.27}$$

$$= (\bar{\boldsymbol{H}}_{22} - \boldsymbol{V}\bar{\boldsymbol{H}}_{12})^{-1}(-\bar{\boldsymbol{H}}_{21} + \boldsymbol{V}\bar{\boldsymbol{H}}_{11}) \tag{4.28}$$

for some matrices

$$\begin{bmatrix} \boldsymbol{H}_{11} & \boldsymbol{H}_{12} \\ \boldsymbol{H}_{21} & \boldsymbol{H}_{22} \end{bmatrix} = \begin{bmatrix} \bar{\boldsymbol{H}}_{11} & \bar{\boldsymbol{H}}_{12} \\ \bar{\boldsymbol{H}}_{21} & \bar{\boldsymbol{H}}_{22} \end{bmatrix}^{-1}$$

and $\boldsymbol{V}$, given in the proof.

Proof. The proof is based on the following lemma.

Lemma 4.6. Under Assumption 2.1, 2.2 and 3.3, there exists proper stable matrix $\boldsymbol{U}$ satisfying $\|\boldsymbol{U}\|_\infty < 1$, (4.16), (4.17) and (4.18) if and only if conditions (4.25) and (4.26) hold. If these conditions are satisfied, there is a parametrization of $\boldsymbol{U}$.

Proof. The necessity of (4.25) and (4.26) is obvious by (4.16), (4.17) and (4.18), and by $\|\boldsymbol{U}\|_\infty < 1$.

Now we prove the sufficiency. The problem given by (4.16), (4.17) and (4.18) is a left and right interpolation problem. If the interpolation points are finite and are in $\Re[s] > 0$, the Nevanlinna–Pick theorem can be applied, in the form of Theorem 18.5.3 in [1]. In our case the interpolation points are on the extended imaginary axis. Consider temporarily that there are no interpolation points at infinity, i. e. that condition (4.16) is absent. For some $\alpha > 0$ we can introduce the change of variables:

$$\boldsymbol{U}(s) = \boldsymbol{U}_\alpha(s+\alpha) \ , \tag{4.29}$$

for some new unknown matrix $\boldsymbol{U}_\alpha$. By the introduced transformation, the points $\mathrm{j}\omega_k$ map into the points $z_k = \alpha + \mathrm{j}\omega_k$, and the points $\mathrm{j}\bar{\omega}_i$ map into the points $\bar{z}_i = \alpha + \mathrm{j}\bar{\omega}_i$. The points $z_1, \ldots, z_L, \bar{z}_1, \ldots, \bar{z}_{\bar{L}}$ are finite and are in $\Re[s] > 0$, hence Theorem 18.5.3 of [1] can be applied. Define the matrices

$$C_+ = [\bar{V}_1, \ldots, \bar{V}_{\bar{L}}] \ , \quad C_- = [\bar{U}_1, \ldots, \bar{U}_{\bar{L}}] \ , \quad A_\pi = \mathrm{diag}\{\bar{z}_1 I, \ldots, \bar{z}_{\bar{L}} I\} \ ,$$

$$A_\zeta = \begin{bmatrix} z_1 I & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & z_L I \end{bmatrix}, \quad B_+ = \begin{bmatrix} V_1 \\ \vdots \\ V_L \end{bmatrix}, \quad B_- = \begin{bmatrix} U_1 \\ \vdots \\ U_L \end{bmatrix} .$$

It is easy to prove the properties that matrix pair (A_ζ, B_+) is controllable and matrix pair (A_π, C_-) is observable, properties required in Theorem 18.5.3 of [1].

The Pick matrix for this new problem (matrix (18.5.3) in [1]):

$$\Lambda = \begin{bmatrix} \left\{ \frac{\bar{U}_m^* \bar{U}_i - \bar{V}_m^* \bar{V}_i}{\bar{z}_m^* + \bar{z}_i} \right\} & \left\{ \frac{\bar{U}_m^* U_\ell^* + \bar{V}_m^* V_\ell^*}{z_\ell^* - \bar{z}_m^*} \right\} \\ \left\{ \frac{U_k \bar{U}_i + V_k \bar{V}_i}{z_k - \bar{z}_i} \right\} & \left\{ \frac{V_k V_\ell^* - U_k U_\ell^*}{z_k + z_\ell^*} \right\} \end{bmatrix} , \quad \begin{matrix} m, i = 1, \ldots, \bar{L} \\ k, \ell = 1, \ldots, L \end{matrix}$$

has block-diagonal matrices

$$\frac{1}{2\alpha}(\bar{U}_i^* \bar{U}_i - \bar{V}_i^* \bar{V}_i) > 0 \ , \quad i = 1, \ldots, \bar{L} \ ,$$

$$\frac{1}{2\alpha}(V_k V_k^* - U_k U_k^*) > 0 \ , \quad k = 1, \ldots, L \ .$$

Under Assumption 3.3, all other blocks of the Pick matrix Λ remain finite as $\alpha \to 0$. Therefore, there is a sufficiently small α such that the Pick matrix is positive definite, and there is proper stable matrix $\boldsymbol{U}_\alpha(z)$ satisfying the conditions (4.17) and (4.18) in the points $z_1, \ldots, z_L, \bar{z}_1, \ldots, \bar{z}_{\bar{L}}$. It is given in terms of matrix

$$\boldsymbol{\Theta}_\alpha(z) = I + \begin{bmatrix} C_+ & -B_+^* \\ C_- & B_-^* \end{bmatrix} \begin{bmatrix} (zI - A_\pi)^{-1} & 0 \\ 0 & (zI + A_\zeta^*)^{-1} \end{bmatrix} \Lambda^{-1} \begin{bmatrix} -C_+^* & C_-^* \\ B_+ & B_- \end{bmatrix} ,$$

as:

$$\boldsymbol{U}_\alpha(z) = (\boldsymbol{\Theta}_{\alpha 11}\boldsymbol{V}_\alpha + \boldsymbol{\Theta}_{\alpha 12})(\boldsymbol{\Theta}_{\alpha 21}\boldsymbol{V}_\alpha + \boldsymbol{\Theta}_{\alpha 22})^{-1}\ , \quad \boldsymbol{\Theta}_\alpha = \begin{bmatrix} \boldsymbol{\Theta}_{\alpha 11} & \boldsymbol{\Theta}_{\alpha 12} \\ \boldsymbol{\Theta}_{\alpha 21} & \boldsymbol{\Theta}_{\alpha 22} \end{bmatrix}\ , \tag{4.30}$$

where $\boldsymbol{V}_\alpha(z)$ ranges in the set of all proper matrices, without poles in $\Re[z] \geq 0$, and satisfies $\|\boldsymbol{V}_\alpha\|_\infty < 1$.

Then we find real rational $\boldsymbol{U}(s)$ by (4.29):

$$\boldsymbol{U}(s) = (\boldsymbol{\Theta}_{11}\boldsymbol{V} + \boldsymbol{\Theta}_{12})(\boldsymbol{\Theta}_{21}\boldsymbol{V} + \boldsymbol{\Theta}_{22})^{-1}\ , \tag{4.31}$$

where $\boldsymbol{\Theta}_{ij}$ and $\boldsymbol{V}$ are the corresponding matrices to $\boldsymbol{\Theta}_{\alpha ij}$ and $\boldsymbol{V}_\alpha$. Matrix $\boldsymbol{U}(s)$ has no poles in $\Re[s] \geq -\alpha$, and $\|\boldsymbol{U}(-\alpha + \mathrm{j}\omega)\| < 1\ \ (\forall\omega \in \mathbb{R})$. The parameter matrix $\boldsymbol{V}$ is proper and stable in $\Re[s] \geq -\alpha$ and satisfies $\|\boldsymbol{V}(-\alpha + \mathrm{j}\omega)\| < 1$, for all $\omega \in \mathbb{R}$.

If there is an infinite interpolation point, instead of (4.29), for some $\beta > 0$, we can apply the following transformation

$$s \to z = \frac{s+\beta}{1+\beta s}\ .$$

By this transformation, the imaginary axis maps into the circle with center on the real axis, which intersects the real axis at the points β and $\frac{1}{\beta}$. Therefore, the interpolation points on the imaginary axis map into some points on that circle (the infinity point into the point $\frac{1}{\beta}$), and the right half-plane into the interior of the circle. Under the conditions (4.25) and (4.26), we can choose β sufficiently small positive number, such that the Pick matrix for the transformed problem is positive definite. □

By composition of mappings (4.31), (4.12) and (2.17) we obtain controller (4.27) with

$$\begin{bmatrix} \boldsymbol{H}_{11} & \boldsymbol{H}_{12} \\ \boldsymbol{H}_{21} & \boldsymbol{H}_{22} \end{bmatrix} = \begin{bmatrix} \bar{\boldsymbol{X}} & \bar{\boldsymbol{P}} \\ -\bar{\boldsymbol{Y}} & \bar{\boldsymbol{Q}} \end{bmatrix} \bar{\boldsymbol{\Psi}} \begin{bmatrix} \boldsymbol{\Theta}_{22} & \boldsymbol{\Theta}_{21} \\ \boldsymbol{\Theta}_{12} & \boldsymbol{\Theta}_{11} \end{bmatrix}. \tag{4.32}$$

□

Remark 3. The realization of $\boldsymbol{K}$ in (4.15) has hidden modes on the extended imaginary axis, which by (4.16), (4.17) and (4.18), can and must be cancelled, for the stability and impulse-free closed loop system.

Example 2. Let be given the descriptor system of Example 1.

By solving the regular $\mathscr{H}_\infty$ state-space problem $\left\|\mathscr{F}(\widehat{\boldsymbol{G}}, \boldsymbol{S})\right\|_\infty < \gamma$, we obtain the infimum over marginally stabilizing controllers $\gamma_{\mathrm{ims}} = 35.8272$. Take $\gamma = 36.5$.

We find matrices U_0, U_1, U_2, U_3 and V_0, V_1, V_2, V_3 - in this example vector-rows, corresponding to the invariant zeros ∞, j, $-$j and 0:

$$\begin{bmatrix} U_0 \\ U_1 \\ U_2 \\ U_3 \end{bmatrix} = 10^2 \begin{bmatrix} -0.0036 & 0.0061 \\ 0.0774 - 0.2014\mathrm{j} & 0.2898 - 0.6422\mathrm{j} \\ 0.0774 + 0.2014\mathrm{j} & 0.2898 + 0.6422\mathrm{j} \\ -1.1218 & -3.4629 \end{bmatrix}$$

$$\begin{bmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \end{bmatrix} = 10^2 \begin{bmatrix} -0.0320 & -0.2170 & -0.2918 \\ -0.1498 + 0.1178\mathrm{j} & 0.3211 + 0.0720\mathrm{j} & 0.2932 - 0.6673\mathrm{j} \\ -0.1498 - 0.1178\mathrm{j} & 0.3211 - 0.0720\mathrm{j} & 0.2932 + 0.6673\mathrm{j} \\ 0.7076 & 0.7105 & -3.5181 \end{bmatrix}$$

They satisfy conditions (4.25). The Pick matrix for the transformed problem is positive definite, hence we can find a stable state-space realization of matrix $\boldsymbol{U}$, given by

$$\boldsymbol{U} = \left[\begin{array}{ccc|cc} -0.6673 & 0.8372 & -1.531 & 8.528 & 32.56 \\ -1.126 & -0.2777 & -0.8221 & 4.484 & 11.41 \\ -9.933 & -2.714 & -37.5 & 224.4 & 720.7 \\ \hline 0.004507 & -0.001736 & 0.009661 & -0.000865 & 0.001468 \\ -0.008415 & 0.008227 & 0.00781 & -0.005869 & 0.009964 \\ -0.01316 & -0.004124 & -0.04838 & -0.007894 & 0.0134 \end{array}\right], \quad \|\boldsymbol{U}\|_\infty = 0.9953\,.$$

The central (i. e. for $\boldsymbol{V} = 0$) $\mathscr{H}_\infty$ controller is given by

$$\boldsymbol{K} = \left[\begin{array}{cccccc|cc} -6.202 & -5.793 & 11.29 & 11.23 & -6.018 & -3.295 & 24.25 & 33.4 \\ -13.86 & -13.56 & -2.415 & 15.87 & -17.09 & -30.41 & 17.25 & 62.46 \\ -0.5101 & 29.16 & -28.55 & -21.88 & -3.998 & -4.685 & -74.62 & -147 \\ 4.677 & -9.491 & 14.17 & 1.309 & 5.146 & 14.82 & 29.23 & 41.27 \\ 2.621 & -1.057 & -0.7126 & -2.926 & 0.7362 & 2.533 & 1.449 & 1.709 \\ -1.886 & -0.7898 & 2.911 & 3.106 & 1.34 & -0.2246 & 2.94 & 5.477 \\ \hline 0.0453 & 0.009607 & 0.03394 & -0.01091 & -0.03785 & 0.1442 & 0.1315 & 0.2631 \\ -0.03668 & -0.1871 & 0.2123 & 0.1518 & 0.0659 & 0.04718 & 0.4347 & 0.8699 \\ -0.2276 & 0.0981 & -0.1022 & 0.1539 & -0.3574 & -0.4972 & -0.1608 & -0.3205 \end{array}\right]$$

The norm of the closed loop transfer matrix with this controller is $36.4305 < 36.5$, and the closed loop system is stable and impulse-free. Its modes are

$$\left\{\begin{array}{c} -75.6893 \\ -10.6966 \pm 9.9305\mathrm{j} \\ -9.2478 \\ -4.4070 \pm 5.4469\mathrm{j} \\ -5.7810 \\ -4.8434 \\ -0.2201 \pm 0.9973\mathrm{j} \\ -1.1228 \pm 0.0752\mathrm{j} \\ -0.1243 \end{array}\right\}$$

CONCLUSIONS

In this paper the problems of $\mathscr{H}_2$ and $\mathscr{H}_\infty$ control of descriptor systems with invariant zeros on the extended imaginary axis are transformed into problems with state space systems, without such zeros. Concerning the $\mathscr{H}_\infty$ case, the transformed problem is not equivalent, unless we add interpolation conditions on the invariant zeros of the plant on the extended imaginary axis. Then the $\mathscr{H}_\infty$ control is found by the Nevanlinna–Pick theorem.

Using the general results in the appendix, the assumption of left-invertiblity of $\boldsymbol{G}_{12}$ and right-invertiblity of $\boldsymbol{G}_{21}$ can be omitted.

5. APPENDIX

Let us be given descriptor system (E, A, B, C, D) satisfying the following assumption.

Assumption 5.1.

- Matrix pencil $A - sE$ is regular,
- Descriptor system (E, A, B) is impulse controllable,
- Matrix $\boldsymbol{G} = D + C(sE - A)^{-1}B$ is left-invertible and matrix pencil $[A - sE, B]$ has full row rank in $\Re[s] \geq 0$, or matrix $\boldsymbol{G}$ isn't left invertible but matrix pencil $[A - sE, B]$ has full row rank for all $s \in \mathbb{C}$.

Consider the following linear matrix inequality (LMI) ([12, 13]):

$$\begin{bmatrix} A^{\mathrm{T}}X_1 + X_1^{\mathrm{T}}A + C^{\mathrm{T}}C & A^{\mathrm{T}}X_2 + X_1^{\mathrm{T}}B + C^{\mathrm{T}}D \\ X_2^{\mathrm{T}}A + B^{\mathrm{T}}X_1 + D^{\mathrm{T}}C & B^{\mathrm{T}}X_2 + X_2^{\mathrm{T}}B + D^{\mathrm{T}}D \end{bmatrix} \geq 0 \ , \tag{5.1}$$

in the unknown matrices X_1 and X_2 satisfying

$$E^{\mathrm{T}}X_1 = X_1^{\mathrm{T}}E \geq 0 \ , \quad E^{\mathrm{T}}X_2 = 0 \ . \tag{5.2}$$

We are interested to find a rank minimizing solution, i.e. to find matrices K and L such that the row rank of $[K, L]$ is full and minimal, equal to the normal rank of $\boldsymbol{G}$, and

$$\begin{bmatrix} A^{\mathrm{T}}X_1 + X_1^{\mathrm{T}}A + C^{\mathrm{T}}C & A^{\mathrm{T}}X_2 + X_1^{\mathrm{T}}B + C^{\mathrm{T}}D \\ X_2^{\mathrm{T}}A + B^{\mathrm{T}}X_1 + D^{\mathrm{T}}C & B^{\mathrm{T}}X_2 + X_2^{\mathrm{T}}B + D^{\mathrm{T}}D \end{bmatrix} = \begin{bmatrix} K^{\mathrm{T}} \\ L^{\mathrm{T}} \end{bmatrix} [K, L] \ . \tag{5.3}$$

We shall prove that, under Assumption 5.1, a solution exists and present a method to solve the LMI (5.1).

Under the introduced assumptions, there is a matrix F such that matrix pencil $A - BF - sE$ is stable and impulse-free, i.e. its inverse is proper stable.

Define matrices $\widehat{A} = A - BF$, $\widehat{C} = C - DF$, $\widehat{K} = K - LF$ and $\widehat{X}_1 = X_1 - X_2F$. Then identity (5.3) becomes

$$\begin{bmatrix} \widehat{A}^{\mathrm{T}}\widehat{X}_1 + \widehat{X}_1^{\mathrm{T}}\widehat{A} + \widehat{C}^{\mathrm{T}}\widehat{C} & \widehat{A}^{\mathrm{T}}X_2 + \widehat{X}_1^{\mathrm{T}}B + \widehat{C}^{\mathrm{T}}D \\ X_2^{\mathrm{T}}\widehat{A} + B^{\mathrm{T}}\widehat{X}_1 + D^{\mathrm{T}}\widehat{C} & B^{\mathrm{T}}X_2 + X_2^{\mathrm{T}}B + D^{\mathrm{T}}D \end{bmatrix} = \begin{bmatrix} \widehat{K}^{\mathrm{T}} \\ L^{\mathrm{T}} \end{bmatrix} [\widehat{K}, L] \ , \tag{5.4}$$

while the first condition in (5.2) becomes

$$E^{\mathrm{T}}\widehat{X}_1 = \widehat{X}_1^{\mathrm{T}}E \geq 0 \ . \tag{5.5}$$

Let P and Q be nonsingular matrices such that

$$PEQ = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \ , \tag{5.6}$$

and define

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = P\widehat{A}Q \ ,$$

and introduce the partitioned matrices

$$[C_1, C_2] = \widehat{C}Q \ , \quad \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = PB \ .$$

By the choice of F, matrix A_{22} is nonsingular ([6], Theorem 6).

Introduce matrices

$$\bar{A} = A_{11} - A_{12}A_{22}^{-1}A_{21} \ , \quad \bar{B} = B_1 - A_{12}A_{22}^{-1}B_2 \ , \tag{5.7}$$

$$\bar{C} = C_1 - C_2A_{22}^{-1}A_{21} \ , \quad \bar{D} = D - C_2A_{22}^{-1}B_2 \ . \tag{5.8}$$

From the relation

$$[A - sE, B] \sim [\widehat{A} - sE, B] \sim \begin{bmatrix} A_{11} - sI & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \end{bmatrix}$$
$$\sim \begin{bmatrix} \bar{A} - sI & 0 & \bar{B} \\ A_{21} & A_{22} & B_2 \end{bmatrix} , \tag{5.9}$$

where by $\sim$ the strict equivalence of matrix pencils is denoted, we see that $(\bar{A}, \bar{B})$ is a stabilizable pair, if $[A - sE, B]$ has a full row rank in $\Re[s] \geq 0$, and that $(\bar{A}, \bar{B})$ is a controllable pair, if $[A - sE, B]$ has a full row rank for all $s \in \mathbb{C}$.

From the relation

$$\begin{bmatrix} A - sE & B \\ C & D \end{bmatrix} \sim \begin{bmatrix} \widehat{A} - sE & B \\ \widehat{C} & D \end{bmatrix} \sim \begin{bmatrix} A_{11} - sI & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ C_1 & C_2 & D \end{bmatrix}$$
$$\sim \begin{bmatrix} \bar{A} - sI & 0 & \bar{B} \\ 0 & A_{22} & 0 \\ \bar{C} & 0 & \bar{D} \end{bmatrix} , \tag{5.10}$$

we see that system $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ is left-invertible if descriptor system (E, A, B, C, D) is left-invertible.

The stabilizability of $(\bar{A}, \bar{B})$ and left-invertibility of system $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$, or the controllability of the pair $(\bar{A}, \bar{B})$ ([14], Section 4, or [18], Lemma 1) guarrant the existence of matrix $Y_{11} \geq 0$, and matrices $\bar{K}$ and L such that the row rank of $[\bar{K}, L]$ is full and minimal, equal to the normal rank of matrix $\bar{\boldsymbol{G}} = \bar{D} + \bar{C}(sI - \bar{A})^{-1}\bar{B}$, and

$$\begin{bmatrix} \bar{A}^{\mathrm{T}}Y_{11} + Y_{11}\bar{A} + \bar{C}^{\mathrm{T}}\bar{C} & Y_{11}\bar{B} + \bar{C}^{\mathrm{T}}\bar{D} \\ \bar{B}^{\mathrm{T}}Y_{11} + \bar{D}^{\mathrm{T}}\bar{C} & \bar{D}^{\mathrm{T}}\bar{D} \end{bmatrix} = \begin{bmatrix} \bar{K}^{\mathrm{T}} \\ L^{\mathrm{T}} \end{bmatrix} [\bar{K}, L] \ . \tag{5.11}$$

Moreover, matrix pencil $\begin{bmatrix} \bar{A} - sI & \bar{B} \\ \bar{K} & L \end{bmatrix}$ is of full row normal rank and its zeros are in $\Re[s] \leq 0$.

Define matrices

$$Y_{21} = -A_{22}^{-\mathrm{T}}A_{12}^{\mathrm{T}}Y_{11} + \frac{1}{2}A_{22}^{-\mathrm{T}}C_2^{\mathrm{T}}C_2A_{22}^{-1}A_{21} - A_{22}^{-\mathrm{T}}C_2^{\mathrm{T}}C_1 \ , \tag{5.12}$$

$$Y_{22} = \frac{1}{2} A_{22}^{-\mathrm{T}} C_2^{\mathrm{T}} C_2 \ , \tag{5.13}$$

$$Z = \frac{1}{2} A_{22}^{-\mathrm{T}} C_2^{\mathrm{T}} C_2 A_{22}^{-1} B_2 - A_{22}^{-\mathrm{T}} C_2^{\mathrm{T}} D \ . \tag{5.14}$$

Now the solution of (5.3) is given by matrix L in (5.11) and by the matrices:

$$X_1 = P^{\mathrm{T}} \begin{bmatrix} Y_{11} & 0 \\ Y_{21} & Y_{22} \end{bmatrix} Q^{-1} + P^{\mathrm{T}} \begin{bmatrix} 0 \\ Z \end{bmatrix} F \ , \tag{5.15}$$

$$X_2 = P^{\mathrm{T}} \begin{bmatrix} 0 \\ Z \end{bmatrix} \ , \tag{5.16}$$

$$K = [\bar{K}, 0] Q^{-1} + LF \ . \tag{5.17}$$

The conditions (5.2) are also satisfied, because

$$E^{\mathrm{T}} X_1 = X_1^{\mathrm{T}} E = Q^{-\mathrm{T}} \begin{bmatrix} Y_{11} & 0 \\ 0 & 0 \end{bmatrix} Q^{-1} \geq 0 \ , \quad E^{\mathrm{T}} X_2 = Q^{-\mathrm{T}} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ Z \end{bmatrix} = 0 \ . \tag{5.18}$$

Remark 3. Matrices K and L coincides with the same matrices in [15].

Proposition 5.2. Matrix pencil $\begin{bmatrix} A - sE & B \\ K & L \end{bmatrix}$ has full row normal rank and its zeros are in $\Re[s] \leq 0$.

P r o o f. Obvious by the following strict equivalence relations:

$$\begin{bmatrix} A - sE & B \\ K & L \end{bmatrix} \sim \begin{bmatrix} \widehat{A} - sE & B \\ \widehat{K} & L \end{bmatrix} \sim \begin{bmatrix} P\widehat{A}Q - sPEQ & PB \\ \widehat{K}Q & L \end{bmatrix}$$

$$\sim \begin{bmatrix} A_{11} - sI & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \bar{K} & 0 & L \end{bmatrix} \sim \begin{bmatrix} A_{11} - A_{12} A_{22}^{-1} A_{21} - sI & 0 & B_1 - A_{12} A_{22}^{-1} B_2 \\ A_{21} & A_{22} & B_2 \\ \bar{K} & 0 & L \end{bmatrix}$$

$$\sim \begin{bmatrix} \bar{A} - sI & 0 & \bar{B} \\ 0 & A_{22} & 0 \\ \bar{K} & 0 & L \end{bmatrix} .$$

□

Now let matrix $\boldsymbol{G}$ be left-invertible, and introduce the following marginally stable rational matrix

$$\boldsymbol{\Omega} = \begin{bmatrix} A - sE & B \\ K & L \end{bmatrix}^{-1} .$$

Proposition 5.3. We have

$$\boldsymbol{\Omega}^{\sim} \begin{bmatrix} C^{\mathrm{T}} \\ D^{\mathrm{T}} \end{bmatrix} [C, D] \boldsymbol{\Omega} = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} X_1 & X_2 \\ 0 & 0 \end{bmatrix} \boldsymbol{\Omega} - \boldsymbol{\Omega}^{\sim} \begin{bmatrix} X_1^{\mathrm{T}} & 0 \\ X_2^{\mathrm{T}} & 0 \end{bmatrix} . \tag{5.19}$$

Proof. Rewrite identity (5.3) as

$$\begin{bmatrix} A^{\mathrm{T}} + sE^{\mathrm{T}} & K^{\mathrm{T}} \\ B^{\mathrm{T}} & L^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} X_1 & X_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} X_1^{\mathrm{T}} & 0 \\ X_2^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} A - sE & B \\ K & L \end{bmatrix}$$

$$+ \begin{bmatrix} C^{\mathrm{T}} \\ D^{\mathrm{T}} \end{bmatrix} [C, D] = \begin{bmatrix} K^{\mathrm{T}} \\ L^{\mathrm{T}} \end{bmatrix} [K, L] \ .$$

Left and right-multiplying this identity by $\mathbf{\Omega}^{\sim}$ and $\mathbf{\Omega}$, having in mind that $[K, L]\mathbf{\Omega} = [0, I]$, we obtain identity (5.19). □

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