KYBERNETIKA — VOLUME *48* (2012), NUMBER 6, PAGES 1089–1099

SEMIDEFINITE CHARACTERISATION OF INVARIANT MEASURES FOR ONE-DIMENSIONAL DISCRETE DYNAMICAL SYSTEMS

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Using recent results on measure theory and algebraic geometry, we show how semidefinite programming can be used to construct invariant measures of one-dimensional discrete dynamical systems (iterated maps on a real interval). In particular we show that both discrete measures (corresponding to finite cycles) and continuous measures (corresponding to chaotic behavior) can be recovered using standard software.

Keywords: dynamical systems, invariant measures, semidefinite programming

Classification: 37-04, 37L40, 90C22

1. INTRODUCTION

Consider the discrete dynamical system

$$x_{k+1} = f(x_k) \tag{1}$$

where $f : \mathbb{R} \mapsto \mathbb{R}$ is a given polynomial map and $x_k \in \mathbb{R}$ is the system state with initial condition

$$x_0 \in G = \{x \in \mathbb{R} \,:\, g(x) = 1 - x^2 \geq 0\}.$$

It is well-known that for simple nonlinear mappings f, a typical state trajectory x_0, x_1, $x_2, \ldots$ of system (1) may be erratic, or chaotic, and very sensitive to the choice of the initial state x_0. The study of such one-dimensional discrete systems can be greatly facilitated by the use of measure theory, see [1, Chapter 6] and [6] for elementary introductions, and [10, Chapters 1–4] for a more mathematical, yet very accessible treatment. The key idea is the following: if a system operates on a measure as an initial condition, rather than on a single point, then successive measures are given by a linear integral operator called a Markov operator. Fixed points of this operator are called invariant measures, and they convey key information on the long-term behavior of the system, such as the existence of cycles of finite length, or a probabilistic characterization of its chaotic behavior. For example, a chaotic behavior can be chacterized compactly via the knowledge of only a few moments of a measure associated with the dynamical system, in contrast with lengthy computationally demanding time-domain simulations, as illustrated e.g. in [2]. Recently, in the context of linear and nonlinear control systems, it has been realized that there is an insightful duality between occupation measures and Lyapunov functions [12, 13, 14, 15].

In this paper we show that recent results mixing measure theory and algebraic geometry can be used to construct invariant measures numerically, with the help of semidefinite programming, as called optimization over linear matrix inequalities (LMIs), a versatile generalization of linear programming to the cone of positive semidefinite matrices [3, Section 4.6.2]. We hope that such computer-based research efforts may contribute to a better understanding of nonlinear phenomena and may suggest directions for theoretical analysis, along the lines sketched in [4].

2. INVARIANT MEASURES

The material of this introductory section can be found in [9].

Consider dynamical system (1) with its polynomial map f on the interval $G \subset \mathbb{R}$. Let $\mathcal{B}(G)$ be the Borel σ-algebra on G, and let μ be a measure in $\mathcal{B}(G)$. Let $\mathcal{F}(G)$ denote the set of measurable functions on G. Given $x \in G$ and $B \in \mathcal{B}(G)$, define $P(x, B)$ as a Markov operator such that

- for all $x \in G$, $P(x, .) \in \mathcal{B}(G)$ is a probability measure supported on G;
- for all $B \in \mathcal{B}(G)$, $x \mapsto P(x, B) \in \mathcal{F}(G)$ is a measurable function on G.

Markov operator $\mu \to \mu\, P$ acts on measures as follows:

$$(\mu\, P)(B) = \int_G P(x, B)\mu(\mathrm{d}x) \tag{2}$$

and its adjoint operator $\phi \to P\, \phi$ acts on functions as follows:

$$(P\, \phi)(x) = \int_G \phi(y)P(x, \mathrm{d}y) = \int_G \phi(y)\delta_{f(x)}(\mathrm{d}y) = \phi(f(x)) \tag{3}$$

where δ_x is the Dirac measure at x. With respect to dynamical system (1), given $x \in G$ and $B \in \mathcal{B}(G)$, $P(x, B)$ is the probability that $x_{k+1} \in B$, knowing that $x_k = x$.

We can interpret the Markov operator as a generalization to arbitrary probability measures (including singular measures) of the Frobenius–Perron operator [10, Section 3.2] usually restricted to density functions, i. e. to probability measures which are absolutely continuous w.r.t. the Lebesgue measure. Similarly, the adjoint of the Markov operator is a generalization of the Koopmans operator [10, Section 3.3].

For dynamical system (1) we want to characterize fixed points of Markov operator P, that is, measures $\mu \in \mathcal{B}(G)$ satisfying

$$\mu\, P = \mu. \tag{4}$$

Measures satisfying (4) are called invariant measures.

3. SEMIDEFINITE CHARACTERISATION OF INVARIANT MEASURES

An equivalent characterization of invariance condition (4) is that

$$\mu\, P\, \phi = \mu\, \phi$$

for all continuous functions $\phi \in \mathcal{F}(G)$. Recalling (2) and (3), this relation becomes

$$\int_G \phi(f(x))\, \mathrm{d}\mu(x) = \int_G \phi(x)\, \mathrm{d}\mu(x).$$

Since function ϕ is continuous and G is compact, the above relation can be written equivalently

$$\int_G \pi_\alpha(f(x))\, \mathrm{d}\mu(x) = \int_G \pi_\alpha(x)\, \mathrm{d}\mu(x), \quad \alpha = 0, 1, 2, \dots \tag{5}$$

where polynomials $\pi_\alpha(x) \in \mathbb{R}[x]$, $\alpha = 0, 1, \dots, n$ generate a basis for polynomials of degree at most n in indeterminate x. Basis polynomials are gathered in a column vector $\pi(x)$ with entries $\pi_i(x)$, $i = 0, 1, \dots, n$. For example $\pi_\alpha(x) = x^\alpha$, the standard monomial of degree α, or $\pi_\alpha(x) = t_\alpha(x)$, the Chebyshev polynomial of degree α, whose definition is recalled in the Appendix.

Let

$$y_\alpha = \int_G \pi_\alpha(x)\, \mathrm{d}\mu(x) \tag{6}$$

denote the moments of measure μ w.r.t. basis $\pi(x)$. Let y denote the vector with entries y_α.

Notice that $\pi_\alpha(f(x))$ is a polynomial in x, so relation (5) is a linear equality constraint on y for each α. All the constraints can be gathered in an infinite-dimensional linear system of equations

$$Ay = b \tag{7}$$

also including the constraint $y_0 = 1$ indicating that μ is a probability measure.

Since μ is a nonnegative measure supported on G, it follows that

$$\int_G p_0^2(x)\, \mathrm{d}\mu(x) \geq 0, \quad \int_G g(x) p_1^2(x)\, \mathrm{d}\mu(x) \geq 0$$

for all polynomials $p_0(x)$ and $p_1(x)$. By expressing polynomials $p_i(x) = p_i^T \pi(x)$ in basis $\pi(x)$, with p_i a column vector of coefficients, these scalar inequalities can be written as

$$p_0^T (\int_G \pi(x)\pi(x)^T\, \mathrm{d}\mu(x)) p_0 \geq 0, \quad p_1^T (\int_G g(x)\pi(x)\pi(x)^T\, \mathrm{d}\mu(x)) p_1 \geq 0$$

for all vectors p_0 and p_1, or, equivalently, as semidefinite constraints

$$M(y) = \int_G \pi(x)\pi(x)^T\, \mathrm{d}\mu(x) \succeq 0, \quad M(gy) = \int_G g(x)\pi(x)\pi(x)^T\, \mathrm{d}\mu(x) \succeq 0.$$

Symmetric linear mappings $M(y)$ and $M(gy)$ are called moment and localising matrices, respectively. Results from algebraic geometry (namely M. Putinar's sum-of-squares representation of polynomials positive on compact semi-algebraic sets) can be invoked as in [11] to show that these semidefinite conditions are necessary and sufficient for the entries of vector y to correspond to moments of a probability measure μ supported on G.

Lemma 3.1. The moments y of an invariant measure for dynamical system (1) satisfy the linear semidefinite constraints

$$Ay = b, \quad M(y) \succeq 0, \quad M(gy) \succeq 0. \tag{8}$$

The set of moments y satisfying constraints (8) is called the moment set for later reference. It is a convex set with non-smooth boundary.

4. COMPUTATIONAL ISSUES

Since system (1) is defined on the unit interval $[-1, 1]$, an appropriate choice of polynomial basis are Chebyshev polynomials. The explicit construction of moment and localizing matrices is based on a basic property of these polynomials.

Let

$$H_\alpha(y) = \begin{bmatrix} y_{|\alpha|} & y_{|\alpha+1|} & y_{|\alpha+2|} & \\ y_{|\alpha+1|} & y_{|\alpha+2|} & y_{|\alpha+3|} & \\ y_{|\alpha+2|} & y_{|\alpha+3|} & y_{|\alpha+4|} & \\ & & & \ddots \end{bmatrix}, \quad T_\alpha(y) = \begin{bmatrix} y_{|\alpha|} & y_{|\alpha+1|} & y_{|\alpha+2|} & \\ y_{|\alpha+1|} & y_{|\alpha|} & y_{|\alpha+1|} & \\ y_{|\alpha+2|} & y_{|\alpha+1|} & y_{|\alpha|} & \\ & & & \ddots \end{bmatrix}$$

denote symmetric Hankel and Toeplitz matrices, respectively, where α is a possibly negative integer.

From Lemma 6.1 of the Appendix, entry (α, β) in matrix $M(y)$ is equal to

$$\int_G t_\alpha t_\beta \, \mathrm{d}\mu = \frac{1}{2}\Big(\int_G t_{\alpha+\beta} \, \mathrm{d}\mu + \int_G t_{|\alpha-\beta|} \, \mathrm{d}\mu\Big) = \frac{1}{2}(y_{\alpha+\beta} + y_{|\alpha-\beta|}).$$

It follows that

$$M(y) = \frac{1}{2}(T_0(y) + H_0(y))$$

has a mixed Toeplitz–Hankel structure.

Since $g(x) = 1-x^2 = \frac{1}{2}(t_0(x)-t_2(x))$, matrix $M(gy)$ shares a similar mixed Toeplitz–Hankel structure:

$$M(gy) = \frac{1}{8}(2T_0(y) + 2H_0(y) - T_{-2}(y) - T_2(y) - H_{-2}(y) - H_2(y)).$$

The moment set (8) is infinite-dimensional: an infinite number of moments are subject to infinite-dimensional semidefinite constraints. The moment set must be truncated when resorting to numerical computations. Let $y = [1 \; y_1 \; \cdots \; y_{2d}]^T$ now denote the vector of moments up to degree $2d$, and truncate all the data in (8) accordingly, yielding truncated conditions

$$A_d y = b_d, \quad M_d(y) \succeq 0, \quad M_d(gy) \succeq 0. \tag{9}$$

These finite-dimensional conditions are necessary but not sufficient for the entries of y to correspond to moments of an invariant measure. The idea is therefore to reduce the gap between necessity and sufficiency by increasing d, considering a whole hierarchy of semidefinite relaxations of increasing size for moment set (8).

Finally, we may want to characterize a particular invariant measure optimal w.r.t. a linear combination of moments. Given a polynomial $h(x) = \sum_\alpha h_\alpha \pi_\alpha(x)$, the function

$$h(y) = \int_G h(x)\,\mathrm{d}\mu(x) = \sum_\alpha h_\alpha y_\alpha$$

is linear in y. Its minimization over moment set (8) can be achieved via a hierarchy of finite-dimensional convex linear semidefinite programming problems

$$\begin{array}{ll} \min_y & h(y) \\ \text{s.t.} & A_d y = b_d, \\ & M_d(y) \succeq 0, \\ & M_d(gy) \succeq 0 \end{array} \tag{10}$$

that can be solved by interior-point algorithms and off-the-shelf software. When d increases, we obtain a monotonically increasing sequence of lower bounds on the minimum of $h(y)$ achieved over (8). Asymptotic convergence of the sequence to the minimum is proved in [11].

As will be shown below, for some $h(y)$, the optimal measure can be discrete (corresponding to cycles of finite length) or continuous (corresponding to chaotic behavior), and in general, different choices of objective functions $h(y)$ yield different invariant measures. Finally, the optimal measure can be a linear combination of several measures which are invariant for system (1).

5. EXAMPLE

Consider the logistic map $\bar{x}_{k+1} = 4\bar{x}_k(1-\bar{x}_k)$ defined on the unit interval $[0,1]$. Applying the affine change of variables $\bar{x}_k = \frac{1}{2}(1-x_k)$, we obtain a discrete dynamical system (1) defined on the symmetric interval $G = [-1,1]$ with

$$f(x) = 2x^2 - 1.$$

Note that $f(x) = t_2(x)$, the second Chebyshev polynomial.

Choosing a Chebyshev polynomial basis $\pi_\alpha(x) = t_\alpha(x)$, and using Lemma 6.2 of the Appendix, it follows that relation (5) satisfied by moments of an invariant measure becomes

$$y_{2\alpha} = y_\alpha, \quad \alpha = 0, 1, 2, \ldots \tag{11}$$

Therefore only odd moments y_1, y_3, y_5, ... are linearly independent.

For illustration, when $d = 3$, the matrices defining the truncated moment set (9) are given by

$$M_3(y) = \frac{1}{2}\left[\begin{array}{cccc} 2 & 2y_1 & 2y_1 & 2y_3 \\ 2y_1 & 1+y_1 & y_1+y_3 & 2y_1 \\ 2y_1 & y_1+y_3 & 1+y_1 & y_1+y_5 \\ 2y_3 & 2y_1 & y_1+y_5 & 1+y_3 \end{array}\right]$$

and

$$M_3(gy) = \frac{1}{8}\left[\begin{array}{ccc} 4-4y_1 & 2y_1-2y_3 & -2+2y_1 \\ 2y_1-2y_3 & 1-y_1 & y_3-y_5 \\ -2+2y_1 & y_3-y_5 & 2-y_1-y_3 \end{array}\right].$$

5.1. Continuous measure

Now suppose we want to find a vector y in the truncated moment set (9). Using the semidefinite programming solver SeDuMi 1.1 for Matlab without specifying an objective function, we obtain an interior point y for which $M_d(y) \succ 0$ and $M_d(gy) \succ 0$. For example, when $d = 3$ (truncation to 6 moments), we obtain (to five significant digits)

$$y_1 = -0.012851,\ y_3 = -0.057926,\ y_5 = 0.020568.$$

The other moments are $y_0 = 1$, $y_2 = y_4 = y_1$, $y_6 = y_3$.

By solving a semidefinite programming problem (not described here for conciseness) we can find a polynomial density $q(x)$ generating these moments (up to numerical round-off errors), i. e. $\int_G t_\alpha(x) q(x)\,\mathrm{d}x = y_\alpha$, $\alpha = 0, 1, \ldots, 6$. We obtain $q(x) = \sum_{\alpha=0}^{6} q_\alpha t_\alpha(x)$ with $q_0 = 0.77395$, $q_1 = -0.084370$, $q_2 = 0.72051$, $q_3 = -0.10286$, $q_4 = 0.42402$, $q_5 = -0.023627$, $q_6 = 0.19274$. The approximate density $q(x)$ obtained from the 6 moments and the exact density $\pi^{-1}(1-x^2)^{-\frac{1}{2}}$ corresponding to the continuous measure invariant for the logistic map are represented on Figure 1.

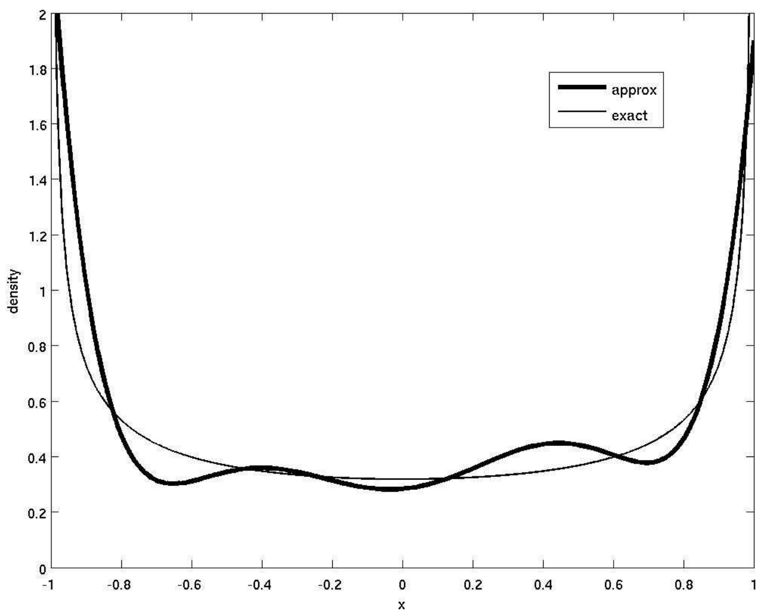

Fig. 1. Approximate density (thick) and exact density (thin) of the continuous measure invariant for the logistic map.

5.2. Discrete measures

Our semidefinite characterization of invariant measures also captures discrete measures corresponding to finite cycles. Such measures can be recovered by optimizing a particular linear combination of moments $h(y) = \sum_\alpha h_\alpha y_\alpha$. In general, moment matrix $M_d(y)$ is rank deficient, since the optimum is at the boundary of the semidefinite cone. The corresponding measure is atomic. The discrete support of the measure can be computed via eigenvalue decomposition (not described here for conciseness).

Here are some examples solved with SeDuMi for $d = 3$ (truncation to 6 moments):

- minimizing $h(y) = y_1$ returns the moments

  $$y_1 = -0.50000,\ y_3 = 1.0000,\ y_5 = -0.50000$$

  corresponding to a rank-one moment matrix $M_3(y)$ of the Dirac measure δ_x at $x = -\frac{1}{2}$. This is a fixed point for our dynamical systems since it is a root of the polynomial $f(x) - x = 2x^2 - x - 1 = (2x + 1)(x - 1)$;

- maximizing y_1 yields

  $$y_1 = y_3 = y_5 = 1.0000$$

  and the Dirac measure at $x = 1$, corresponding to the other root of $f(x) - x$;

- a cycle of length three can be found by minimizing y_3. We obtain

  $$y_1 = 0.0000,\ y_3 = -0.50000,\ y_5 = 0.0000$$

  and a rank-three moment matrix corresponding to an atomic measure supported at $\{-0.93969,\ 0.17365\ ,0.76604\}$. The cycle corresponds to the three roots of the polynomial $8x^3 - 6x + 1$ dividing the degree-8 polynomial $f(f(f(x))) - x = 128x^8 - 256x^6 + 160x^4 - 32x^2 - x + 1 = (2x+1)(x-1)(8x^3-6x+1)(8x^3+4x^2-4x-1)$;

- by maximizing y_3 we obtain

  $$y_1 = -0.095852,\ y_3 = 1.0000,\ y_5 = -0.095852$$

  and a rank-two moment matrix corresponding to an atomic measure supported at $\{-0.50000,\ 1.0000\}$. This measure is a linear combination of the two invariant Dirac measures at $x = -\frac{1}{2}$ and $x = 1$, already found above;

- by minimizing y_5 we obtain

  $$y_1 = 0.20414,\ y_3 = -0.025163,\ y_5 = -1.1323$$

  and a rank-three moment matrix. This moment matrix corresponds to an atomic measure supported at $\{-1.0439,\ -0.25265{,}0.86800\}$, but these values do not correspond to an invariant measure and a finite cycle since $x = -1.0439$ does not belong to G;

- by maximizing y_5 we obtain

$$y_1 = -0.20658,\ y_3 = -0.018795,\ y_5 = 1.04521$$

and a rank-three moment matrix. This moment matrix corresponds to an atomic measure supported at $\{-0.77048,\ 0.29058, 1.0450\}$, but these values do not correspond to an invariant measure and a finite cycle since $x = 1.0450$ does not belong to G.

The two latter cases illustrate that the moment set (8), when truncated, may contain vectors y which do not correspond to moments of an invariant measure. However, invariant measures and finite cycles can be recovered at the price of introducing more moments:

- with $d = 4$ (truncation to 8 moments) by maximizing y_5 we obtain

$$y_1 = -0.022753,\ y_3 = -0.022753,\ y_5 = 1.0000,\ y_7 = -0.022753$$

and a rank-three moment matrix. This moment matrix corresponds to an atomic measure supported at $\{-0.80902,\ 0.30902, 1.0000\}$. This measure is a linear combination of two invariant measures: the Dirac measure at the fixed point $x = 1$ and the two-cycle invariant measure supported at the points $\{\frac{1}{4}(-1-\sqrt{5}),\ \frac{1}{4}(-1+\sqrt{5})\}$. These points correspond to three roots of the degree-four polynomial $f(f(x)) - x = 8x^4 - 8x^2 - x + 1 = (2x+1)(x-1)(4x^2+2x-1)$;

- with $d = 5$ (truncation to 10 moments) by minimizing y_5 we obtain

$$y_1 = 0.12500,\ y_3 = -0.25000,\ y_5 = -0.50000,\ y_7 = 0.12500,\ y_9 = -0.25000$$

and a rank-five moment matrix corresponding to an atomic measure supported at $\{-0.97815,\ -0.50000,\ -0.10453,\ 0.66913,\ 0.91355\}$. This measure is a linear combination of two invariant measures: the Dirac measure at the fixed point $x = -\frac{1}{2}$ and the four-cycle invariant measure supported at the roots of the polynomial $16x^4 - 8x^3 - 16x^2 + 8x + 1$ dividing the degree-16 polynomial $f(f(f(f(x)))) - x$.

Minimizing or maximizing higher order moments can result in longer finite cycles. For example, with $d = 20$ (truncation to 40 moments), minimizing y_9 yields $y_9 = -0.50000$, $y_{27} = 1.0000$ and $y_i = 0.0000$ for other values of i between 1 and 40. The 21-by-21 moment matrix has rank 9, and it corresponds (to machine precision) to a cycle of length 9.

Our numerical experiments with the solver SeDuMi reveal that semidefinite problems with a few hundreds moments can be solved routinely, to a relative accuracy of the order of 10^{-8}, in a few seconds on a standard computer.

On Figure 2 we represent the projections of the truncated moment set (9) on the plane (y_1, y_3) of first and third order moments, when truncated to 4, 6 and 40 moments ($d = 2, 3, 20$) respectively. We have not represented projections for intermediate truncations, or for a higher number of moments, because they could not be distinguished on the figure. It seems that the shape of the moment set (8) is quickly captured by low-order relaxations.

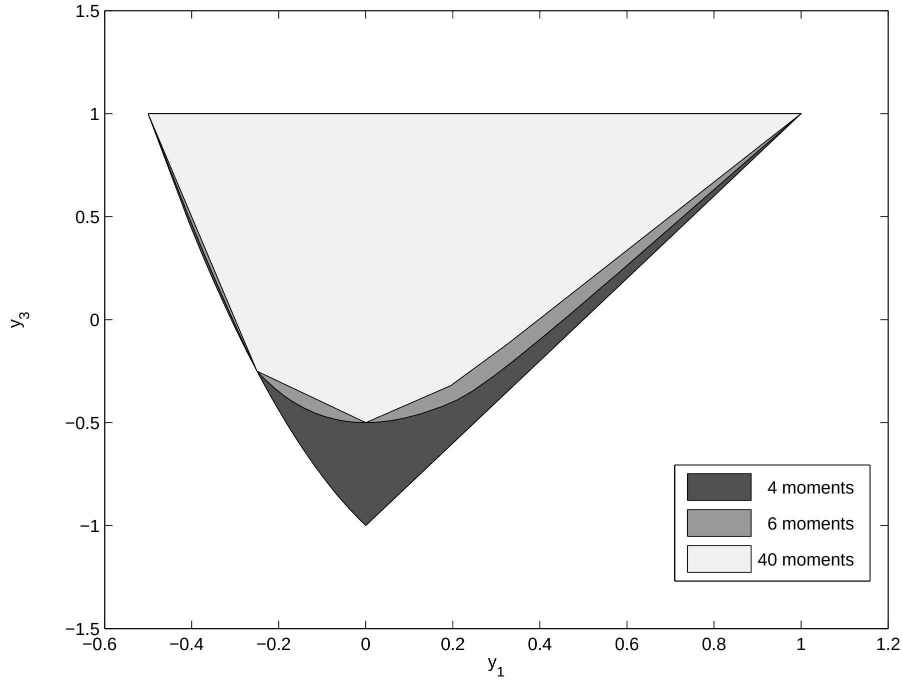

Fig. 2. Projection on the plane (y_1, y_3) of the moment set (8) truncated to 4, 6 and 40 moments.

6. CONCLUSION

Our objective in this paper was to illustrate that semidefinite programming could be an interesting setup for a computer-aided study of nonlinear dynamical systems, and especially of their long-term behavior characterized by invariant measures.

Measures are characterized by their moments with the help of linear constraints in the cone of positive semi-definite matrices. As explained in [7], due to numerical round-off errors, histograms of computer simulations of dynamical systems display only the invariant measure that is absolutely continuous w.r.t. to the Lebesgue measure, discarding periodic orbits of finite lengths. Examples for which computer-generated trajectories land on periodic orbits, and for which the absolutely continuous invariant measure is not observable, are exceptional [8]. It turns out that optimization of a linear function of the moments subject to semidefinite constraints, as proposed in this paper, can produce computationally both types of measures on a unifying ground. Note however that a comparison of our semidefinite programming approach with other numerical techniques lies out of the scope of this short contribution. See e. g. [5] or [15] for numerical approximations of invariant measures based on discretizations of the Frobenius–Perron operator.

On the negative side, it seems difficult to establish a correspondence between finite cycles and objective functions $h(y)$. For example, in Section 5 if one wants to recover a 9-cycle of the logistic map, what is a suitable choice of $h(y)$?

Also unclear is the numerical conditioning of the semidefinite programming problem (10). In the logistic map example of Section 5, thanks to the interpretation of the mapping f as a Chebyshev polynomial, the linear system of equations $Ay = b$ became the particularly simple relations (11) which allowed for a straightforward substitution of even degree moments. In general, for an arbitrary mapping f, such an explicit substitution is not possible, and it may happen that the linear system of equations $Ay = b$ is poorly scaled, and/or that the semidefinite problem (10) is ill-conditioned.

Finally, and maybe most importantly, the methodology allows for (almost direct) extensions to higher dimensional dynamical systems (using e. g. multivariate Chebyshev polynomials) and iterated functional systems (IFS). In the absence of further numerical experiments, it is however unclear whether this approach can be effective in higher dimension.

APPENDIX

Given a non-negative integer α, let $t_\alpha(x)$ denote the Chebyshev polynomial of first kind satisfying the three-term recurrence relation $t_{\alpha+1}(x) = 2xt_\alpha(x) - t_{\alpha-1}(x)$ with $t_0(x) = 1$ and $t_1(x) = x$. For example $t_2(x) = 2x^2 - 1$, $t_3(x) = 4x^3 - 3x$, $t_4(x) = 8x^4 - 8x^2 + 1$, $t_5(x) = 16x^5 - 20x^3 + 5x$.

Lemma 6.1. $t_\alpha(x)t_\beta(x) = \frac{1}{2}(t_{\alpha+\beta}(x) + t_{|\alpha-\beta|}(x))$.

Proof. By induction. □

Lemma 6.2. $t_\alpha(t_\beta(x)) = t_{\alpha\beta}(x)$.

Proof. Follows from the trigonometric relation $t_\alpha(x) = \cos(\alpha \arccos(x))$. Indeed $t_\alpha(t_\beta(x)) = \cos(\alpha \arccos(\cos(\beta \arccos(x)))) = \cos(\alpha\beta \arccos(x)) = t_{\alpha\beta}(x)$. □

ACKNOWLEDGMENTS

This work was initiated by Jean-Bernard Lasserre, whose technical input is gratefully acknowledged. This work was partly supported by research project No. 103/10/0628 of the Grant Agency of the Czech Republic.

(Received February 3, 2012)

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