KYBERNETIKA — VOLUME *48* (2012), NUMBER 5, PAGES 977–987

CONSTRUCTING FAMILIES OF SYMMETRIC DEPENDENCE FUNCTIONS

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We construct two pairs $(\mathscr{A}_F^{[1]}, \mathscr{A}_F^{[2]})$ and $(\mathscr{A}_\psi^{[1]}, \mathscr{A}_\psi^{[2]})$ of ordered parametric families of symmetric dependence functions. The families of the first pair are indexed by regular distribution functions F, and those of the second pair by elements ψ of a specific function family $\boldsymbol{\psi}$. We also show that all solutions of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}u} = \frac{\alpha(u)}{u(1-u)}y$ for α in a certain function family $\boldsymbol{\alpha}_\mathrm{s}$ are symmetric dependence functions.

Keywords: archimax copula, copula, dependence function, generator of a dependence function

Classification: 62H20

1. INTRODUCTION

The systematically developed theory of copulas has found applications in statistics, probability theory, theory of stochastic processes and many practical areas, including econometrics, insurance, finance, risk management and survival analysis. The best source of information about copulas and their applications are proceedings of numerous conferences concerning copulas, as well as the monographs of Hutchinson and Lai [6], Joe [7], and Nelsen [8].

In this paper, the interval $[0,1]$ will be denoted by $\mathbb{I}$. Let $\mathbf{X} = (X_1, X_2)$ be a (two-dimensional) random vector defined on a probability space $(\Omega, \mathcal{F}, P)$ with joint distribution $F_\mathbf{X}$ and marginal distributions F_{X_1}, F_{X_2}. We transform $\mathbf{X}$ into $\mathbf{U} = (U_1, U_2)$ according to the formula $U_k = F_{X_k}(X_k)$ for $k = 1, 2$. The random vector $\mathbf{U}$ takes values in the square $\mathbb{I}^2$. The distribution function H of $P^\mathbf{U}$ is called a *copula* if the following conditions are satisfied: $H(u_1, 0) = H(0, u_2) = 0$, $H(u_1, 1) = u_1$ and $H(1, u_2) = u_2$ for all $u_1, u_2 \in \mathbb{I}$. The components of $\mathbf{U}$ have uniform distribution on $\mathbb{I}$.

An important class of copulas is formed by the archimax copulas, determined by two function parameters g and A. This class contains all the archimedean copulas and extreme value copulas.

Define $A^- : \mathbb{I} \to [\frac{1}{2}, 1]$ and $A^+ : \mathbb{I} \to \mathbb{I}$ by $A^-(u) = \max(u, 1-u)$, $A^+(u) \equiv 1$. Let $\mathscr{A}$ be the family of all convex functions $A : \mathbb{I} \to [\frac{1}{2}, 1]$ satisfying

$$A^-(u) \le A(u) \le A^+(u) \qquad \text{for all } u \in \mathbb{I}. \tag{1}$$

Denote by $\mathscr{G}$ the family of all strictly decreasing convex functions $g : \mathbb{I} \to [0, \infty]$ such that $g(1) = 0$. It is understood that $g(0) = \lim_{t \to 0_+} g(t)$. For $g \in \mathscr{G}$ we define the pseudoinverse $g^{\leftarrow}(x) = \inf\{t \in \mathbb{I} : g(t) \leq x, x \in [0, \infty]\}$, $g^{\leftarrow}(\infty) = 0$. The function $g^{\leftarrow}$ coincides with the usual inverse g^{-1} of g if $g(0) = \infty$. For parameters $g \in \mathscr{G}$ and $A \in \mathscr{A}$ we define $H_{g,A} : \mathbb{I}^2 \to \mathbb{I}$ by

$$H_{g,A}(u_1, u_2) = g^{\leftarrow}\left\{[g(u_1) + g(u_2)]A\left[\frac{g(u_1)}{g(u_1) + g(u_2)}\right]\right\} \quad \text{for } (u_1, u_2) \in \mathbb{I}^2. \tag{2}$$

Capéraà et al. [1] proved that (2) is a copula. Setting $A = A^+$ in (2), we obtain the archimedean copula $H_g(u_1, u_2) = g^{\leftarrow}[g(u_1) + g(u_2)]$. The function g is called the *additive generator* of the copula. The generator g is *strict* if $g(0) = \infty$. On the other hand, if we set $g(t) = -\ln(t)$ in (2), we obtain the extreme value copula $H_A(u_1, u_2) = \exp\left\{\ln(u_1 u_2) A\left[\frac{\ln(u_1)}{\ln(u_1 u_2)}\right]\right\}$. The parameter A is called a *dependence function*.

Systematic investigation of archimedean copulas was initiated by Genest and MacKay [2]. The reader interested in extreme value copulas is referred to Pickands [9] and Joe's monograph [7, Chap. 6]. In their survey paper, Gudendorf and Segers [3] outlined the theory and a wide spectrum of applications of extreme value copulas. Their article also contains an extensive bibliography.

Example 1. To the strict additive generator $g(t) = \frac{1}{t} - 1$ and the dependence function $A(u) = u^2 - u + 1$ there corresponds the copula

$$H(u_1, u_2) = \frac{(u_1 + u_2 - 2u_1u_2)u_1u_2}{u_1^2 + u_2^2 + u_1^2u_2^2 - 2u_1^2u_2 - 2u_1u_2^2 + u_1u_2}.$$

The theory of archimax copulas is not well developed. Recently some relevant results have been obtained by Hürlimann [5]. He gave formulas for popular dependence measures (Kendall's tau and Spearman's rho) for archimax copulas, and considered certain dependence notions for such copulas. The present paper is a contribution to this theory. The theorems of Section 2 enable one to produce families of symmetric dependence functions. The theorems of Section 3 make use of the notion of generator of a dependence function. This notion is inspired by the formula obtained by Capéraà et al. [1]:

$$P\left(\left\{\frac{g(U_1)}{g(U_1) + g(U_2)} \leq t\right\}\right) - t = t(1 - t)\frac{A^{(1)}(t)}{A(t)},$$

where $A^{(1)}(t)$ is another notation for $\frac{\mathrm{d}A}{\mathrm{d}t}$. In what follows, higher derivatives $\frac{d^k f}{du^k}$ will also sometimes be denoted by $f^{(k)}$.

2. PARAMETRIC FAMILIES OF DEPENDENCE FUNCTIONS

In this section we propose two general methods of constructing parametric families of dependence functions. The first method uses regular distribution functions F as parameters, while the second uses elements ψ of a specific function family $\boldsymbol{\psi}$.

Let $F : \mathbb{R} \to \mathbb{I}$ be a distribution function. For $\vartheta \in [0, \infty)$ we define $f_\vartheta : \mathbb{I} \to \mathbb{I}$ by

$$f_\vartheta(u) = 2\int_0^u F[\vartheta F^{-1}(t)]\,\mathrm{d}t. \tag{3}$$

Theorem 1. Let F be a differentiable distribution function such that:

(i) $F^{(1)}(x) > 0$ for all $x \in \mathbb{R}$,

(ii) $F(-x) = 1 - F(x)$ for all $x \in \mathbb{R}$. (4)

Then all elements of the family $\mathscr{A}_F^{[1]} = \{A_\vartheta : \vartheta \in [0, \infty)\}$ defined by $A_\vartheta(u) = f_\vartheta(u) - u + 1$ belong to $\mathscr{A}$ and have the following properties:

- $A_\vartheta(1-u) = A_\vartheta(u)$ for all $u \in \mathbb{I}$ (symmetry),
- A_ϑ is a strictly convex function,
- A_ϑ is twice differentiable on $(0, 1)$,
- if $\vartheta_1 < \vartheta_2$ then

$$A_{\vartheta_1}(u) > A_{\vartheta_2}(u) \quad \text{for all } u \in (0, 1). \tag{5}$$

Proof. Fix $\vartheta \in [0, \infty)$. First we show the symmetry of A_ϑ, i. e.

$$2\int_0^u F[\vartheta F^{-1}(t)]\,\mathrm{d}t - u + 1 = 2\int_0^{1-u} F[\vartheta F^{-1}(t)]\,\mathrm{d}t + u. \tag{6a}$$

Setting $x = F^{-1}(u)$ in (4) for $u \in (0, 1)$, we obtain $-F^{-1}(u) = F^{-1}(1-u)$. Since the function $F^{(1)}$ is even, it is now easy to check that the second derivatives of both sides of (6a) are equal, which implies

$$2\int_0^u F[\vartheta F^{-1}(t)]\,\mathrm{d}t - u + 1 = 2\int_0^{1-u} F[\vartheta F^{-1}(t)]\,\mathrm{d}t + u + au + b, \tag{6b}$$

where a and b are some real numbers.

Formula (6b) and its derivative give, for $u = \frac{1}{2}$, the equalities $a + 2b = 0$ and $a = 0$. Hence (6a) holds. Clearly, $A_\vartheta(0) = 1$. To prove $A_\vartheta(1) = 1$, set $\varphi(\vartheta) = A_\vartheta(1)$ for all $\vartheta \in [0, \infty)$. It suffices to check that $\frac{\mathrm{d}\varphi}{\mathrm{d}\vartheta} \equiv 0$ and exhibit a $\vartheta_0 \in (0, \infty)$ such that $\varphi(\vartheta_0) = 1$. By the theorem on differentiating parameter-dependent integrals,

$$\frac{\mathrm{d}\varphi(\vartheta)}{\mathrm{d}\vartheta} = 2\int_0^1 F^{(1)}[\vartheta F^{-1}(t)]F^{-1}(t)\,\mathrm{d}t.$$

Changing variable $z = F^{-1}(t)$, we obtain

$$\frac{\mathrm{d}\varphi(\vartheta)}{\mathrm{d}\vartheta} = 2\int_{-\infty}^{\infty} zF^{(1)}(z)F^{(1)}(\vartheta z)\,\mathrm{d}z = 0,$$

since the integrand is odd. Note that $\vartheta_0 = 1$ yields $\varphi(1) = 1$. By symmetry of A_ϑ, it suffices to prove (1) on the interval $[0, \frac{1}{2}]$, which is equivalent to $f_\vartheta(u) \geq 0$. We see that $A_\vartheta \in \mathscr{A}$. It is obvious that A_ϑ is twice differentiable. As $\frac{\mathrm{d}^2 A_\vartheta(u)}{\mathrm{d}u^2} = 2\vartheta \frac{F^{(1)}[\vartheta F^{-1}(u)]}{F^{(1)}[F^{-1}(u)]} > 0$ on $(0, 1)$, it follows that A_ϑ is strictly convex.

To prove (5) it suffices to show that for each $u \in (0,1)$ the function $\tilde{\varphi}(\vartheta) = 2\int_0^u F[\vartheta F^{-1}(t)]\,\mathrm{d}t - u + 1$ is strictly decreasing on $(0,\infty)$. To do this, note that

$$\begin{aligned}\frac{\mathrm{d}\tilde{\varphi}(\vartheta)}{\mathrm{d}\vartheta} &= 2\int_{-\infty}^{F^{-1}(u)} zF^{(1)}(z)F^{(1)}(\vartheta z)\,\mathrm{d}z \\ &= 2\int_{-\infty}^{-F^{-1}(u)} zF^{(1)}(z)F^{(1)}(\vartheta z)\,\mathrm{d}z + 2\int_{-F^{-1}(u)}^{F^{-1}(u)} zF^{(1)}(z)F^{(1)}(\vartheta z)\,\mathrm{d}z < 0\end{aligned}$$

for all $u \in (\frac{1}{2}, 1)$, because the last integral is zero.

For $u \in (0, \frac{1}{2}]$, the inequality $\frac{\mathrm{d}\tilde{\varphi}(\vartheta)}{\mathrm{d}\vartheta} < 0$ is obvious. □

Corollary 1. The supremum A^+ of $\mathscr{A}_F^{(1)}$, equal to A_0, belongs to $\mathscr{A}_F^{[1]}$, while the infimum A^- of the family, equal to $\lim_{\vartheta\to\infty} A_\vartheta$, belongs to the closure of $\mathscr{A}_F^{[1]}$ in the uniform convergence topology.

Example 2. The distribution function $F(x) = \frac{\exp(x)}{1+\exp(x)}$ satisfies the assumptions of Theorem 1. Elements of the corresponding family $\mathscr{A}_F^{[1]}$ are defined by

$$A_\vartheta(u) = 2\int_0^u \frac{t^\vartheta}{t^\vartheta + (1-t)^\vartheta}\,\mathrm{d}t - u + 1.$$

We now construct another parametric family of dependence functions. Let $\boldsymbol{\psi}$ be the family of all functions $\psi : [-\frac{1}{2}, \frac{1}{2}] \to [-\frac{1}{2}, \frac{1}{2}]$ satisfying:

(1) $\psi(-\frac{1}{2}) = \psi(0) = \psi(\frac{1}{2}) = 0$,

(2) $\psi(t) < 0$ for all $t \in (-\frac{1}{2}, 0)$ and $\psi(t) > 0$ for all $t \in (0, \frac{1}{2})$,

(3) $\psi(-t) = -\psi(t)$ for all $t \in [-\frac{1}{2}, \frac{1}{2}]$,

(4) ψ is differentiable on $[-\frac{1}{2}, \frac{1}{2}]$,

(5) $\inf[\psi^{(1)}(t), 0] \geq -1$ for all $t \in (-\frac{1}{2}, \frac{1}{2})$.

We define $f_\psi : \mathbb{I} \to [-1, 0]$ by

$$f_\psi(u) = 2\int_{-\frac{1}{2}}^{u-\frac{1}{2}} \psi(t)\,\mathrm{d}t. \tag{7}$$

Theorem 2. Let $\psi \in \boldsymbol{\psi}$. Then the elements of the family $\mathscr{A}_\psi^{[1]} = \{A_\vartheta^\psi : \vartheta \in \mathbb{I}\}$, defined by $A_\vartheta^\psi(u) = \vartheta f_\psi(u) + u^2 - u + 1$, all belong to $\mathscr{A}$. Moreover, they are twice differentiable, convex symmetric functions such that if $\vartheta_1 < \vartheta_2$ then

$$A_{\vartheta_1}^\psi(u) > A_{\vartheta_2}^\psi(u) \quad \text{for all } u \in (0,1). \tag{8}$$

P r o o f. Fix $\vartheta \in \mathbb{I}$. We first prove the symmetry of A_ϑ^ψ, i. e.

$$2\vartheta \int_{-\frac{1}{2}}^{u-\frac{1}{2}} \psi(t)\,\mathrm{d}t + u^2 - u + 1 = 2\vartheta \int_{-\frac{1}{2}}^{-(u-\frac{1}{2})} \psi(t)\,\mathrm{d}t + u^2 - u + 1$$

for all $u \in \mathbb{I}$. The derivatives of both sides of this equality are equal, so

$$2\vartheta \int_{-\frac{1}{2}}^{u-\frac{1}{2}} \psi(t)\,\mathrm{d}t + u^2 - u + 1 = 2\vartheta \int_{-\frac{1}{2}}^{-(u-\frac{1}{2})} \psi(t)\,\mathrm{d}t + u^2 - u + 1 + c$$

for all $u \in \mathbb{I}$. Setting $u = \frac{1}{2}$ yields $c = 0$, which proves the symmetry of A_ϑ^ψ.

To prove that A_ϑ^ψ is convex we note that

$$\frac{\mathrm{d}^2 A_\vartheta^\psi(u)}{\mathrm{d}u^2} = 2[\vartheta\psi^{(1)}(u - \tfrac{1}{2}) + 1] \geq 0 \quad \text{for all } u \in (0,1).$$

Clearly, $A_\vartheta^\psi(0) = A_\vartheta^\psi(1) = 1$. By symmetry of A_ϑ^ψ, it suffices to check the validity of $A_\vartheta^\psi(u) \geq 1 - u$ on $[0, \frac{1}{2}]$. This inequality is equivalent to

$$2\vartheta \int_{-\frac{1}{2}}^{u-\frac{1}{2}} \psi(t)\,\mathrm{d}t + u^2 \geq 0. \tag{9}$$

The left hand side is strictly convex. Hence the function $2[\vartheta\psi(u - \frac{1}{2}) + u]$ is strictly increasing and equal to zero at $u = 0$, which guarantees its nonnegativity on $[0, \frac{1}{2}]$. We infer that the left hand side of (9) is nonnegative on $[0, \frac{1}{2}]$. We have thus shown that $A_\vartheta^\psi \in \mathscr{A}$.

Twice differentiability of A_ϑ^ψ and the ordering (8) is clear. □

Corollary 2. The element $A_0^\psi(u) = u^2 - u + 1$ is greatest in the sense of the ordering (8).

Example 3. The function $\psi(t) = \frac{1}{2\pi}\sin(2\pi t)$ satisfies the assumptions of Theorem 2. Elements of the family $\mathscr{A}_\psi^{[1]}$ have the form

$$A_\vartheta^\psi(u) = \frac{\vartheta}{4\pi^2}\cos(2\pi u) + u^2 - u + 1 - \frac{\vartheta}{4\pi^2}.$$

3. GENERATORS OF DEPENDENCE FUNCTIONS

Let A be a dependence function. We define a function α by

$$\alpha(t) = t(1-t)\frac{A^{(1)}(t)}{A(t)}. \tag{10a}$$

This function leads to the following linear differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}u} = \frac{\alpha(u)}{u(1-u)}y, \quad \text{where } y = A. \tag{10b}$$

A natural question is for which functions α every solution of (10b) is a dependence function. It turns out that under some natural assumptions the answer is fairly easy. We start by studying the properties of functions (10a).

Lemma 1. Let A be a strictly convex symmetric dependence function such that:

- A is twice differentiable on $(0,1)$,
- $\lim_{u\to 0_+} A^{(1)}(u) = -1$.

Then the function $\alpha : \mathbb{I} \to [-\frac{1}{2}, \frac{1}{2}]$ has the following properties:

(a_1) $\alpha(0) = \alpha(\frac{1}{2}) = \alpha(1) = 0$,

(a_2) $\alpha(t) < 0$ for all $t \in (0, \frac{1}{2})$ and $\alpha(t) > 0$ for all $t \in (\frac{1}{2}, 1)$,

(a_3) $\alpha(t) \ge -t$ for all $t \in [0, \frac{1}{2}]$ and $\alpha(t) \le 1 - t$ for all $t \in (\frac{1}{2}, 1]$,

(a_4) $\alpha(1-t) = -\alpha(t)$ for all $t \in \mathbb{I}$,

(a_5) α is differentiable on $(0,1)$,

(a_6) $\alpha^{(1)}(t) > \dfrac{\alpha(t)[1-2t-\alpha(t)]}{t(1-t)}$ for all $t \in (0,1)$,

(a_7) $\lim_{t\to 0_+} \alpha^{(1)}(t) = -1$.

Proof. Property (a_4) follows from the symmetry of A. Clearly, $\alpha(0) = \alpha(1) = 0$. The equality $\alpha(\frac{1}{2}) = 0$ is a consequence of the fact that $u = \frac{1}{2}$ is an absolute minimum point for A. Hence (a_1) holds. Property (a_2) is obvious. By (a_4), it suffices to prove (a_3) on $[0, \frac{1}{2}]$, which follows from the obvious inequalities $\alpha(t) \ge t(1-t)A^{(1)}(t) \ge -t(1-t) \ge -t$. Property ($a_5$) is also evident, and ($a_6$) follows from

$$\frac{\mathrm{d}^2 y}{\mathrm{d}u^2} = \left\{ \frac{u(1-u)\alpha^{(1)}(u) - (1-2u)\alpha(u)}{[u(1-u)]^2} + \left[\frac{\alpha(u)}{u(1-u)}\right]^2 \right\} y > 0.$$

The equality (a_7) follows by a direct calculation. □

We introduce two function families, $\mathscr{A}_{\mathrm{s}}^{-}$ and $\boldsymbol{\alpha}_{\mathrm{s}}$. The first consists of all dependence functions satisfying the assumptions of Lemma 1, and the second consists of all functions $\alpha : \mathbb{I} \to [-\frac{1}{2}, \frac{1}{2}]$ satisfying (a_1)–(a_7).

Theorem 3. If $\alpha \in \boldsymbol{\alpha}_{\mathrm{s}}$, then the equation (10b) with the initial condition

$$\lim_{u \to 0_+} y^{(1)}(u) = -1 \tag{10c}$$

has a unique solution given by

$$y(u) = \exp\left[\int_0^u \frac{\alpha(t)}{t(1-t)}\,\mathrm{d}t\right] \quad \text{for all } u \in \mathbb{I}. \tag{11}$$

This solution is a dependence function in the family $\mathscr{A}_{\mathrm{s}}^{-}$.

P r o o f. By l'Hospital's rule, $\lim_{t\to 0_+} \frac{\alpha(t)}{t(1-t)} = -1$, and using ($a_4$) we obtain $\lim_{t\to 1_-} \frac{\alpha(t)}{t(1-t)} = 1$. Hence the continuity of α yields $\int_0^u \frac{\alpha(t)}{t(1-t)}\,\mathrm{d}t < \infty$ for all $u \in \mathbb{I}$. It is easy to check that that the function (11) satisfies (10b) with the initial condition (10c). To prove uniqueness, suppose that y_1 and y_2 are solutions of (10b) satisfying (10c). The function $\frac{y_1}{y_2}$ has zero derivative on $(0,1)$, so $y_1 = cy_2$ for some constant c. From (10c) we see that $c = 1$.

To prove that y is symmetric, we write

$$y^{(1)}(u) = \frac{\alpha(u)}{u(1-u)} y(u) \quad \text{and} \quad y^{(1)}(1-u) = -\frac{\alpha(u)}{u(1-u)} y(1-u),$$

which implies

$$-\frac{y^{(1)}(1-u)}{y(1-u)} = \frac{y^{(1)}(u)}{y(u)}.$$

This can be rewritten as $\frac{\mathrm{d}}{\mathrm{d}u}\{\ln[y(1-u)]\} = \frac{\mathrm{d}}{\mathrm{d}u}\{\ln[y(u)]\}$, leading to $\ln[y(1-u)] = \ln[y(u)] + c'$ for some constant c'. Setting $u = \frac{1}{2}$ we find $c' = 0$, proving the symmetry of y.

We now show that y is a dependence function. Clearly, $y(1) = 1$. We have to check (1). It sufices to do that on $[0, \frac{1}{2}]$. We divide both sides of $\alpha(t) \ge -t$ by $t(1-t)$ and integrate over $[0,u]$ to obtain $\int_0^u \frac{\alpha(t)}{t(1-t)}\,\mathrm{d}t \ge \ln(1-u)$. Hence $\exp\left[\int_0^u \frac{\alpha(t)}{t(1-t)}\,\mathrm{d}t\right] \ge 1-u$. Thus, the solution (11) is twice differentiable. The strict convexity of y follows by differentiating (10b) and taking into account (a_6). We have thus shown that $y \in \mathscr{A}_{\mathrm{s}}^{-}$. $\square$

Remark 1. The map $\alpha \mapsto \exp\left[\int_0^u \frac{\alpha(t)}{t(1-t)}\,\mathrm{d}t\right]$ is a bijection of $\boldsymbol{\alpha}_{\mathrm{s}}$ onto $\mathscr{A}_{\mathrm{s}}^{-}$.

The elements of the family $\boldsymbol{\alpha}_{\mathrm{s}}$ will be called *generators of dependence functions.*

Example 4. The family $\left\{\alpha_\vartheta(t) = \frac{(1-t)t^\vartheta - t(1-t)^\vartheta}{t^\vartheta + (1-t)^\vartheta} : \vartheta > 1\right\}$ is contained in $\boldsymbol{\alpha}_{\mathrm{s}}$. By Theorem 3 it yields the family $\{A_\vartheta(u) = [u^\vartheta + (1-u)^\vartheta]^{\frac{1}{\vartheta}} : \vartheta > 1\}$ of dependence functions, generating the family of Gumbel copulas [4] (without the independent copula) defined by $H_\vartheta(u_1, u_2) = \exp\{-[(-\ln u_1)^\vartheta + (-\ln u_2)^\vartheta]^{\frac{1}{\vartheta}}\}$ for $\vartheta > 1$.

Example 5. The functions $\alpha_1(t) = t(1-t)(2t-1)$ and $\alpha_2(t) = -t(1-t)\cos(\pi t)$ belong to $\boldsymbol{\alpha}_{\mathrm{s}}$. By Theorem 3 the corresponding dependence functions are

$$A_1(u) = \exp[-u(1-u)], \tag{12a}$$

$$A_2(u) = \exp\left[-\frac{1}{\pi}\sin(\pi u)\right]. \tag{12b}$$

The generators of the dependence functions (12a) and (12b) can be written in the form $\alpha_1(t) = t(1-t)v_1^{(1)}(t)$ and $\alpha_2(t) = t(1-t)v_2^{(1)}(t)$, where $v_1(t) = -t(1-t)$, $v_2(t) = -\cos(\pi t)$. It turns out that the functions v_1 and v_2 have the following properties:

(b_1) $v(0) = v(1) = 0$ and $v(t) < 0$ for all $t \in (0,1)$,

(b_2) $v(t) \geq A^-(t) - 1$ for all $t \in \mathbb{I}$,

(b_3) $v(1-t) = v(t)$ for all $t \in \mathbb{I}$,

(b_4) v is strictly convex,

(b_5) v is twice differentiable on $(0,1)$,

(b_6) $\lim_{t\to 0_+} v^{(1)}(t) = -1$.

We denote by $\boldsymbol{v}$ the family of all functions $v : \mathbb{I} \to [-\frac{1}{2}, 0]$ satisfying (b_1)–(b_6). Since v_1 and v_2 are in $\boldsymbol{v}$, one may suspect that to elements of $\boldsymbol{v}$ correspond dependence functions. This is discussed below.

Theorem 4.

(a) If $v \in \boldsymbol{v}$, then the function

$$\alpha_v(t) = t(1-t)v^{(1)}(t) \tag{13a}$$

belongs to $\boldsymbol{\alpha}_{\mathrm{s}}$, and the corresponding dependence function is

$$A_v(u) = \exp[v(u)]. \tag{13b}$$

(b) If $v_1, v_2 \in \boldsymbol{v}$ and $v_1(t) > v_2(t)$ for all $t \in (0,1)$, then also $A_{v_1}(u) > A_{v_2}(u)$ on this interval.

Proof. (a) We have to show that (13a) satisfies (a_1)–(a_7). This is easy. To illustrate, we prove (a_6). The left hand side of (a_6) for (13a) is equal to $(1-2t)v^{(1)}(t) + t(1-t)v^{(2)}(t)$, and the right hand side is $(1-2t)v^{(1)}(t) - t(1-t)[v^{(1)}(t)]^2$. Hence $v^{(2)}(t) > -[v^{(1)}(t)]^2$.

(b) follows from (13b). □

To give an application of Theorem 4, set $v_\vartheta(t) = f_\vartheta(t) - t$ and $v_\vartheta^\psi(t) = \vartheta f_\psi(t) + t^2 - t$, where f_ϑ and f_ψ are given by (3) and (7) respectively.

It is easy to check that the families $\{v_\vartheta : \vartheta \in (0,\infty)\}$ and $\{v_\vartheta^\psi : \vartheta \in \mathbb{I}\}$ are contained in $\boldsymbol{v}$. By Theorem 4 the corresponding ordered families of symmetric dependence functions are

$$\mathscr{A}_F^{[2]} = \left\{A_\vartheta(u) = \exp\left[2\int_0^u F[\vartheta F^{-1}(t)]\,\mathrm{d}t - u\right] : \vartheta \in [0,\infty)\right\},$$
$$\mathscr{A}_\psi^{[2]} = \left\{A_\vartheta^\psi(u) = \exp\left[2\vartheta\int_{-\frac{1}{2}}^{u-\frac{1}{2}} \psi(t)\,\mathrm{d}t + u^2 - u\right] : \vartheta \in \mathbb{I}\right\},$$

4. CONCLUDING REMARKS

The families of extreme value copulas $\mathcal{H}_F^{[1]}$, $\mathcal{H}_F^{[2]}$ and $\mathcal{H}_\psi^{[1]}$, $\mathcal{H}_\psi^{[2]}$ corresponding to the families of dependence functions $\mathscr{A}_F^{[1]}$, $\mathscr{A}_F^{[2]}$ and $\mathscr{A}_\psi^{[1]}$, $\mathscr{A}_\psi^{[2]}$ are ordered in the following way:

$$A_{\vartheta_1} \le A_{\vartheta_2} \text{ on } \mathbb{I} \;\Rightarrow\; H_{A_{\vartheta_1}} \ge H_{A_{\vartheta_2}}.$$

In particular we have $H_{A_\vartheta} \ge H^\perp$, where $H^\perp(u_1,u_2) = u_1u_2$. This yields positive quadratic dependence for H_{A_ϑ}. Archimax copulas $H_{g,A}$ can represent more complex types of dependence. The families $\mathscr{A}_F^{[2]}$ and $\mathscr{A}_\psi^{[2]}$ induce families of archimax copulas. This follows from the fact that to every $v \in \boldsymbol{v}$ (under a certain condition) corresponds a pair of function parameters (g_v, A_v), where g_v is an additive generator (of an archimedean copula H_{g_v}). To justify this, we quote a fact from Wysocki [10]. Let $\boldsymbol{V}$ be the family of all continuous functions $v : \mathbb{I} \to [-1,0]$ satisfying the following conditions:

(B_1) $-1 \le v(0) \le 0$, $v(1) = 0$ and $v(t) < 0$ for all $t \in (0,1)$,

(B_2) $v(t) \ge t - 1$ for all $t \in \mathbb{I}$,

(B_3) v is differentiable on $(0,1)$,

(B_4) $v^{(1)}(t) \le 1$ for all $t \in (0,1)$ and $\lim_{t\to 1_-} v^{(1)}(t) = 1$.

The elements of $\boldsymbol{V}$ are called *vector generators* (of archimedean copulas). Define

$$\alpha^v(t) = \frac{1}{t-1} - \frac{1}{v(t)}, \quad t \in (0,1).$$

Lemma 2. If $v \in \boldsymbol{V}$ has a second order left derivative at $t = 1$, then the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{y}{v(t)} \tag{14}$$

with initial condition $\lim_{t\to 0_+} y^{(1)}(t) = 1$ has a unique solution given by

$$y(t) = g(t) = (1-t)\exp\left[\int_t^1 \alpha^v(s)\,\mathrm{d}s\right] \quad \text{for all } t \in \mathbb{I}. \tag{15}$$

The solution (15) is a twice differentiable additive generator.

From (15) and the definition of α^v we obtain

Corollary 3. Every element $v \in \boldsymbol{V}$ is of the form

$$v(t) = \frac{g(t)}{g^{(1)}(t)}.$$

Obviously, we have the inclusion $\boldsymbol{v} \subset \boldsymbol{V}$. For $v \in \boldsymbol{v}$ the solution (15) is three times differentiable. If ψ has a right derivative at $t = -\frac{1}{2}$, then the limit $\lim_{t\to 0_+} v_\vartheta^{(2)}(t)$ exists for $v_\vartheta(t) = 2\vartheta \int_{-1/2}^{t-1/2} \psi(s)\,\mathrm{d}s + t^2 - t$.

In turn, if there exists an interval $(-\infty, a)$, $a < 0$, on which the function $x \mapsto F^{(1)}(x)/F(x)$ is bounded and monotone, then the limit $\lim_{t\to 0_+} v_\vartheta^{(2)}(t)$ exists for $v_\vartheta(t) = 2\int_0^t F[\vartheta F^{-1}(s)]\,\mathrm{d}s - t$.

By Theorem 4 and Lemma 2 we introduce the family of archimax copulas $H_{g,A}$ for $g = g_{v_\vartheta}$ and $A = A_\vartheta$, where $v_\vartheta(t) = 2\vartheta \int_{-1/2}^{t-1/2} \psi(s)\,\mathrm{d}s + t^2 - t$, $A_\vartheta(u) = \exp[v_\vartheta(u)]$, which we denote $\mathcal{H}(\psi)$. Similarly $\mathcal{H}(F)$ is the family of archimax copulas for $g = g_{v_\vartheta}$, $v_\vartheta(t) = 2\int_0^t F[\vartheta F^{-1}(s)]\,\mathrm{d}s - t$, $A_\vartheta(u) = \exp[v_\vartheta(u)]$.

The dependence measures obtained by Hürlimann [5], Kendall's tau and Spearman's rho, for elements of $\mathcal{H}(\psi)$ and $\mathcal{H}(F)$, can only be found numerically. It suggests itself that dependence measures constructed specifically for archimax copulas may be more "sensitive" and easier to compute. Information about such copulas is represented by the function parameters g and A. Consequently, a dependence measure for an archimax copula may measure the "closeness" of the pair (g, A) (resp. (v, α)) to $(g^\perp, A^+)$ (resp. $(v^\perp, \alpha^\perp)$), where $g^\perp(t) = -\ln t$, $v^\perp(t) = t\ln t$ and $\alpha^\perp \equiv 0$. Such measures may be, for example, functions based on suitably scaled metrics on $\mathscr{G} \times \mathscr{A}$ (resp. $\boldsymbol{V} \times \boldsymbol{\alpha}_\mathrm{s}$). These issues will be discussed in another publication.

In terms of generators of dependence functions, one can give necessary and sufficient conditions for convergence of dependence functions. For example, if a sequence (α_k) of generators of dependence functions converges uniformly to a function $\neq 0$ on $(0,1)\backslash\{\frac{1}{2}\}$, then the corresponding sequence of dependence functions $(A_k) \subset \mathscr{A}_\mathrm{s}^-$ converges uniformly to a C^1 dependence function. Omitting condition (a_4) in the definition of the family $\boldsymbol{\alpha}_\mathrm{s}$ (which requires modifying (a_1)–(a_3) in the obvious way), we obtain a family $\boldsymbol{\alpha}$. It turns out that in terms of vector generators and generators of dependence functions, one can characterize archimedean copulas and study convergence of sequences of such copulas (see Wysocki [10]).

(Received October 17, 2011)

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