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ON COPULAS THAT GENERALIZE SEMILINEAR COPULAS

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We study a wide class of copulas which generalizes well-known families of copulas, such as the semilinear copulas. We also study corresponding results for the case of quasi-copulas.

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1. INTRODUCTION

We first recall the concept of a copula. For a detailed study of its properties and applications, we refer to [14, 17].

Definition 1.1. A (bivariate) copula is a binary operation $C\colon [0,1]^2 \longrightarrow [0,1]$ which satisfies: (C1) $C(t,0) = C(0,t) = 0$ and $C(t,1) = C(1,t) = t$ for every $t \in [0,1]$, and (C2) C is 2-increasing, i. e., $C(u_2,v_2) - C(u_1,v_2) - C(u_2,v_1) + C(u_1,v_1) \geq 0$ for every $u_1, v_1, v_2, v_2 \in [0,1]$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$.

The importance of copulas in probability and statistics comes from Sklar's theorem [19, 20], which shows that the joint distribution of a pair of random variables and the corresponding marginal distributions F and G are linked by these functions. If F and G are continuous, then the copula is unique; otherwise, the copula is uniquely determined on $\operatorname{Range} F \times \operatorname{Range} G$ (see [4] for the description of these copulas).

Given a copula C, we define a set function μ_C as a finitely additive set function on a rectangle, i. e., if $R = [u_1,u_2] \times [v_1,v_2] \in [0,1]^2$, then $\mu_C(R) = C(u_2,v_2) - C(u_1,v_2) - C(u_2,v_1) + C(u_1,v_1)$. In the following, when we refer to "mass" on a set, we mean the value of μ_C for that set. We also note that μ_C can be also extended to a measure in the σ-algebra of the Borel sets.

Let Π denote the copula of independent random variables, i. e., $\Pi(u,v) = uv$ for every $(u,v) \in [0,1]^2$. We let M and W denote the respective Fréchet–Hoeffding upper and lower bound copulas, which, for any copula C, satisfy: $\max(u+v-1,0) = W(u,v) \leq C(u,v) \leq M(u,v) = \min(u,v)$ for every $(u,v) \in [0,1]^2$.

The *diagonal section* δ_C of a copula C is the function given by $\delta_C(t) = C(t,t)$ for every $t \in [0,1]$. We note that this definition can be also extended to other functions. On the other hand, a *diagonal* is a function $\delta\colon [0,1] \to \mathbb{R}$ which satisfies (i) $\delta(1) = 1$,

(ii) $\delta(t) \leq t$ for every $t \in [0,1]$, and (iii) $0 \leq \delta(t') - \delta(t) \leq 2(t'-t)$ for every $t, t' \in [0,1]$ such that $t \leq t'$. The diagonal section of any copula is a diagonal; and for any diagonal δ, there exist copulas whose diagonal section is δ [12, 18]. δ_C can be used to study the *tail dependence* of a random pair (X, Y) with associated copula C [15].

In this paper, we define and characterize a wide class of copulas which generalizes well-known families of copulas, such as the semilinear copulas [5, 8] and study the tail dependence property. We also characterize the members of the family to be quasi-copulas.

2. A WIDE CLASS OF COPULAS

2.1. Characterization and properties

Let D, E, F and G be functions defined on $[0,1]$ and C the function defined on $[0,1]^2$ by

$$C(x,y) = D(x \vee y)E(x \wedge y) + F(x \wedge y)G(x \vee y), \tag{1}$$

where $(x \vee y) = \max(x,y)$ and $(x \wedge y) = \min(x,y)$.

Our first step is to express the function given by (1), assuming it is a copula, in a more tractable manner.

Lemma 2.1. If C is a copula of type (1), then it can be written in the following way:

$$C(x,y) = A(x \vee y)Z(x \wedge y) + (x \wedge y)B(x \vee y), \tag{2}$$

where A, B and Z are three absolutely continuous functions defined on $[0,1]$ such that $A(1) = Z(0) = 0$ and $B(1) = 1$.

Proof. First assume that $D(1) = 0$. Then, for $x \leq y$, $x = C(x,1) = F(x)G(1)$ for every $x \in [0,1]$, and hence we have $G(1) \neq 0$ and $F(x) = \frac{x}{G(1)}$, which implies that

$$C(x,y) = D(x \vee y)E(x \wedge y) + (x \wedge y)\frac{G(y \vee x)}{G(1)}.$$

Taking $A(x) = D(x)$, $Z(x) = E(x)$ and $B(x) = \frac{G(x)}{G(1)}$, we obtain (2). Now, if $D(1) \neq 0$, we have $x = C(x,1) = D(1)E(x) + F(x)G(1)$, i. e., $E(x) = \frac{x - F(x)G(1)}{D(1)}$, which implies that

$$C(x,y) = F(x \wedge y)\left(G(x \vee y) - \frac{D(x \vee y)G(1)}{D(1)}\right) + (x \wedge y)\frac{D(x \vee y)}{D(1)}. \tag{3}$$

Taking $Z(x) = F(x)$, $B(x) = \frac{D(x)}{D(1)}$ and $A(x) = G(x) - B(x)G(1)$ in (3), we obtain (2).

Since $A(y)Z(x) + xB(y)$ is absolutely continuous when $x \leq y$, then Z is absolutely continuous when $x \leq y$, and tending y to 1, then Z is absolutely continuous in $[0,1]$. If Z is not a linear function, we can express A and B as a linear combination of two absolutely continuous functions, therefore, both A and B are absolutely continuous. If Z is linear of the form $Z(x) = ax$ for a constant a, then

$$C(x,y) = (aA(x \vee y) + B(x \vee y) - 1)(x \wedge y) + (x \wedge y)$$

and, in this case, we take $A^*(x) = aA(x) + B(x) - 1$, $B^*(x) = 1$, $Z^*(x) = x$, and thus A^*, B^* and Z^* are absolutely continuous. $\square$

The following result characterizes the function given by (1) to be a copula.

Theorem 2.2. The function C given in (1) is a copula if and only if it holds:

(i) C can be expressed in the form (2), with $A(1) = Z(0) = 0$ and $B(1) = 1$,

(ii) $A'(y)Z'(x) + B'(y) \geq 0$ for $x \leq y$ in a set of measure $1/2$,

(iii) $A'(x)Z(x) - A(x)Z'(x) - B(x) + xB'(x) \leq 0$ in a set of measure 1.

Proof. First assume (1) is a copula. Then, Lemma 2.1, and the fact that $0 = C(0,y) = A(y)Z(0)$ for every $y \in [0,1]$, give us item (i). Let x_1, x_2, y_1, y_2 be points in $[0,1]$ such that $x_1 < \min(x_2, y_1) < y_2$. Since C is 2-increasing, we have $[A(y_2) - A(y_1)][Z(x_2) - Z(x_1)] + (x_2 - x_1)[B(y_2) - B(y_1)] \geq 0$, or equivalently,

$$\frac{[A(y_2) - A(y_1)][Z(x_2) - Z(x_1)]}{(y_2 - y_1)(x_2 - x_1)} + \frac{B(y_2) - B(y_1)}{y_2 - y_1} \geq 0. \tag{4}$$

If we tend $x_2 \to x_1$ and $y_2 \to y_1$, we obtain (ii). Since A, B and Z are derivable in a set of measure 1, we have

$$\frac{\partial^2}{\partial x \partial y} C(x,y) = A'(x \vee y)Z'(x \wedge y) + B'(x \vee y)$$

except in a set of measure 0. Assuming $b > a$, since the sum of the masses of the rectangles $R_1 = [b,1] \times [a,b]$ and $R_2 = [0,a] \times [a,b]$ must be less or equal to $b - a$, then we have

$$\begin{aligned} \mu_C(R_1) + \mu_C(R_2) &= b - a - A(b)Z(b) - bB(b) + 2A(b)Z(a) \\ &\quad +2aB(b) - A(a)Z(a) - aB(a) \\ &\leq b - a, \end{aligned}$$

i.e.,

$$-A(b)Z(b) - bB(b) + 2A(b)Z(a) + 2aB(b) - A(a)Z(a) - aB(a) \leq 0,$$

or equivalently,

$$Z(a)[A(b) - A(a)] - A(b)[Z(b) - Z(a)] - (b - a)B(b) + a[B(b) - B(a)] \leq 0.$$

Dividing this last expression by $b - a$ and tending $a \to b$, we obtain that the derivative $A'(x)Z(x) - A(x)Z'(x) - B(x) + xB'(x)$ (if it exists) must be less or equal to zero. But in a set of measure 1 the existence is guaranteed, since the functions are absolutely continuous, whence (iii) follows.

Conversely, the boundary conditions are immediately satisfied. To check the 2-increasing property, we first consider two cases:

(I) If the rectangle $R=[x_1,x_2]\times[y_1,y_2]$ has no intersection with the main diagonal, since item (ii) is satisfied, we obtain (4) by using the same idea as above backwards, and thus $\mu_C(R)=[A(y_2)-A(y_1)][Z(x_2)-Z(x_1)]+(x_2-x_1)[B(y_2)-B(y_1)]\geq 0$.

(II) If two vertices of a rectangle R are on the main diagonal, i.e., $R=[a,b]^2$, then we have

$$\begin{aligned}\mu_C(R) &= bB(b)+aB(a)-2aB(b)+A(b)Z(b)+A(a)Z(a)-2A(b)Z(a)\\ &= \int_a^b [B(b)-aB'(x)+Z'(x)A(b)-A'(x)Z(a)]\,\mathrm{d}x\\ &= \int_a^b [B(x)-xB'(x)+Z'(x)A(x)-A'(x)Z(x)]\,\mathrm{d}x\\ &\quad+\int_a^b\!\!\int_a^x [B'(x)+A'(x)Z'(y)]\,\mathrm{d}y\mathrm{d}x+\int_a^b\!\!\int_x^b [B'(y)+A'(y)Z'(x)]\,\mathrm{d}y\mathrm{d}x.\end{aligned}$$

Thus, by hypothesis, we obtain $\mu_C(R)\geq 0$.

Finally, note that any rectangle can be expressed as the finite union of rectangles of the forms given in cases (I) and (II); hence C is 2-increasing, and this completes the proof. □

Any copula C can be written as the sum of an *absolutely continuous* component A_C and a singular component S_C. When $C=A_C$ (respectively, $C=S_C$), then it is said that C is *absolutely continuous* (respectively, *singular*). The following result characterizes the function C given by (2) to be absolutely continuous or have a singular component.

Theorem 2.3. The function C given by (2) is an absolutely continuous copula if and only if

$$A'(x)Z(x)-A(x)Z'(x)+xB'(x)-B(x)=0 \tag{5}$$

for $x\in[0,1]$. Moreover, if C has a singular component, then it must be concentrated on the main diagonal, and has density function

$$-A'(x)Z(x)+A(x)Z'(x)-xB'(x)+B(x).$$

Proof. Consider a rectangle $R=[x_1,x_2]\times[y_1,y_2]\in[0,1]^2$ such that, without loss of generality, $x_1<\min(x_2,y_1)<y_2$. Then, we have

$$\begin{aligned}\int_{y_1}^{y_2}\!\!\int_{x_1}^{x_2}\frac{\partial^2}{\partial x\partial y}C(x,y)\,\mathrm{d}x\mathrm{d}y &= \int_{y_1}^{y_2}\!\!\int_{x_1}^{x_2}[A'(y)Z'(x)+B'(y)]\,\mathrm{d}x\mathrm{d}y\\ &= [A(y_2)-A(y_1)][Z(x_2)-Z(x_1)]\\ &\quad+(x_2-x_1)[(B(y_2)-B(y_1)]\\ &= \mu_C(R).\end{aligned}$$

Thus, if the singular component exists, then it must be concentrated on the main diagonal. Therefore, C is absolutely continuous if and only if

$$\int_0^a\!\!\int_0^a\frac{\partial^2}{\partial x\partial y}C(x,y)\,\mathrm{d}x\mathrm{d}y=C(a,a).$$

Since

$$\begin{aligned}\int_0^a\int_0^a \frac{\partial^2}{\partial x\partial y}C(x,y)\,\mathrm{d}x\mathrm{d}y &= \int_0^a\int_0^x [A'(x)Z'(y)+B'(x)]\,\mathrm{d}y\mathrm{d}x \\ &\quad+\int_0^a\int_x^a [A'(y)Z'(x)+B'(y)]\,\mathrm{d}y\mathrm{d}x \\ &= \int_0^a [A'(x)Z(x)-A(x)Z'(x)+xB'(x)-B(x)]\,\mathrm{d}x \\ &\quad+A(a)Z(a)+aB(a)\end{aligned}$$

and $C(a,a)=A(a)Z(a)+aB(a)$, we have

$$\int_0^a [A'(x)Z(x)-A(x)Z'(x)+xB'(x)-B(x)]\,\mathrm{d}x=0.$$

Since A, B and Z are absolutely continuous, we obtain (5). If C has a singular component (on the main diagonal), then

$$S_C(x,x)=C(x,x)-\int_0^x\int_0^x \frac{\partial^2}{\partial u\partial v}C(u,v)\,\mathrm{d}u\mathrm{d}v,$$

whence the result easily follows. □

2.2. Tail dependence

If (X,Y) is a pair of continuous random variables with associated copula copula C, and $x_t, y_t \in \mathbb{R}$ are respective $100t$-th percentiles for every $t\in(0,1)$, then $\delta_C(t)=\mathbb{P}[X\le x_t, Y\le y_t]$ for every $t\in(0,1)$. The *upper* and *lower tail dependence parameters* λ_U and λ_L, which are defined as $\lambda_U=\lim_{t\to 1^-}\mathbb{P}[Y>y_t|X>x_t]$ and $\lambda_L=\lim_{t\to 0^+}\mathbb{P}[Y\le y_t|X\le x_t]$ (if the limits exist) can be computed as follows: $\lambda_U(C)=2-\delta_C'(1^-)$ and $\lambda_L(C)=\delta_C'(0^+)$ [15, 17]. Since copulas are used to build models for dependence between risks in financial and actuarial risk management, specially dependence between extreme events, tail dependence has been shown to be useful to describe this dependence (see, for instance, [9, 10]).

Now we compute the upper and lower tail coefficients for any copula C defined by (2) — the proof is immediate, and we omit it.

Theorem 2.4. Let C be a copula given by (2). Then we have $\lambda_U(C)=1-A'(1^-)Z(1)-B'(1^-)$ and $\lambda_L(C)=A(0)Z'(0^+)+B(0)$, as long as the derivatives exist.

2.3. Examples

We provide several examples of copulas given by (1) (some of them are generalizations of well-known families of copulas). The first example shows the reason for the title of this work.

Example 2.5. Suppose $Z(x) = x$, $B(x) = 1$, and $H(x) = A(x) + 1$ for all $x \in [0, 1]$. Then, the function C in (2) is given by

$$C(x, y) = H(x \vee y)(x \wedge y).$$

Thus, C is copula if and only if $H(x)$ is nondecreasing and $\frac{H(x)}{x}$ is nonincreasing for $x \in [0, 1]$. These copulas (called lower semilinear) are defined and characterized in [5, 8], where the function H is given by $H(x) = \frac{\delta_C(x)}{x}$. Note also that $\lambda_U(C) = 1 - H'(1^-)$ and $\lambda_L(C) = H(0)$.

As a particular case, if we consider the function $H(x) = x^{1-\alpha}$, with $\alpha \in [0, 1]$, in (2), then we obtain the well-known *Cuadras–Augé* family of copulas [3], which is given by

$$C(x, y) = (x \vee y)^{1-\alpha}(x \wedge y). \tag{6}$$

Example 2.6. Assume $B(x) = x$ for all $x \in [0, 1]$. Then the function C in (1) is given by

$$C(x, y) = A(x \vee y)Z(x \wedge y) + xy,$$

with A and Z absolutely continuous functions. Thus, C is a copula if and only if the following conditions hold: (i) $A(1) = Z(0) = 0$, (ii) $A'(y)Z'(x) \geq -1$ such that $x \leq y$ in a set of measure 1/2, and (iii) $A'(x)Z(x) - A(x)Z'(x) \leq 0$ in a set of measure 1. These copulas have been studied in [7].

As a particular case, if we consider the functions $A(x) = 1 - x$ and $Z(x) = \beta x$ for $x \in [0, 1]$, with $\beta \in [0, 1]$, in (2), we obtain a member of the well-known *Fréchet* family of copulas [11, 17]

$$C(x, y) = \beta(x \wedge y) + (1 - \beta)xy,$$

i. e., $C(x, y) = \beta M(x, y) + (1 - \beta)\Pi(x, y)$.

Example 2.7. Consider the functions $A(x) = x(1-x)$, $Z(x) = x^2$ and $B(x) = x^2$ for all $x \in [0, 1]$. It is easy to check that these functions fulfill the hypotheses in Theorem 2.2. Thus, the function given by

$$C(x, y) = (x \vee y)[1 - (x \vee y)](x \wedge y)^2 + (x \wedge y)(x \vee y)^2$$

is a copula. Moreover, the equality (5) is satisfied, whence C is absolutely continuous.

Example 2.8. Consider the functions $A(x) = 1 - x$, $Z(x) = x^2$ and $B(x) = x^2$ for all $x \in [0, 1]$. It is easy to check that these functions fulfill the hypotheses in Theorem 2.2. Thus, the function given by

$$C(x, y) = [1 - (x \vee y)](x \wedge y)^2 + (x \wedge y)(x \vee y)^2$$

is a copula. Moreover, the equality (5) is not satisfied, whence C has a singular component along the main diagonal and such that the segment that joins the point $(0, 0)$ to (x, x) has a mass equals to $x^2\left(1 - \frac{2x}{3}\right)$.

2.4. Quasi-copulas

The notion of (bivariate) quasi-copula was introduced in [1] in order to show that a certain class of operations on univariate distribution functions is not derivable from corresponding operations on random variables defined on the same probability space. A *quasi-copula* is a function $Q\colon [0,1]^2 \to [0,1]$ that satisfies (C1) in Definition 1.1, but instead of (C2), the weaker conditions [13]: (Q1) Q is increasing in each variable; and (Q2) Q is *1-Lipschitz*, i.e, for all $u_1, v_1, u_2, v_2 \in [0,1]$ it holds that $|Q(u_1,v_1) - Q(u_2,v_2)| \le |u_1 - u_2| + |v_1 - v_2|$.

While every copula is a quasi-copula, there exist *proper* quasi-copulas, i. e. quasi-copulas that are not copulas. However, quasi-copulas are also bounded by the copulas W and M.

Nowadays, quasi-copulas are used in aggregation processes because they ensure that the aggregation is stable, in the sense that small error inputs correspond to small error outputs. For an overview, we refer to [2].

The following result provides under which conditions the function C defined by (2) is a quasi-copula.

Theorem 2.9. Let C be the function given by (2). Then, C is a quasi-copula if and only if it satisfies the following conditions:

(i) A, B and Z are absolutely continuous,

(ii) $A(1) = Z(0) = 0$ and $B(1) = 1$,

(iii) $0 \le A'(x)Z(y) + yB'(x) \le 1$ for $y \le x$ in a set of measure $1/2$,

(iv) $0 \le A(x)Z'(y) + B(x) \le 1$ for $y \le x$ in a set of measure $1/2$.

Proof. First suppose C is a quasi-copula. Conditions (i) and (ii) are obtained in a similar manner to those in Theorem 2.2. Since C is nondecreasing and 1-Lipschitz (with respect to x), this implies that

$$0 \le [A(x_2) - A(x_1)]Z(y) + y[B(x_2) - B(x_1)] \le x_2 - x_1 \tag{7}$$

when $y \le x_1 < x_2$. We divide (7) by $x_2 - x_1$. From the fact that A, B and Z are absolutely continuous, and tending x_2 to x_1, we have condition (iii). We also obtain condition (iv) in a similar way (with respect to y).

Conversely, the boundary conditions (C1) are satisfied via condition (ii). On the other hand, since

$$\int_{x_1}^{x_2} [A'(t)Z(y) + yB'(t)]\,\mathrm{d}t = [A(x_2) - A(x_1)]Z(y) + y[B(x_2) - B(x_1)],$$

the inequalities in (iii) imply that C is increasing and 1-Lipschitz with respect to x. Similarly, by using condition (iv), we obtain that C is increasing and 1-Lipschitz with respect to y, and hence C is a quasi-copula. $\square$

Example 2.10. Consider the function A given by

$$A(x) = \begin{cases} x, & \text{if } 0 \leq x \leq 1/3 \\ 1-2x, & \text{if } 1/3 < x \leq 2/3 \\ x-1, & \text{otherwise,} \end{cases}$$

$Z(x) = -A(x)$ and $B(x) = x$ for every $x \in [0,1]$. It is easy to check that the functions A, Z and B satisfy the hypotheses in Theorem 2.9, and the function C, defined in (2) and given by $C(x,y) = xy - A(x)A(y)$ (note that C is symmetric), is a proper quasi-copula [6].

3. CONCLUSION AND FURTHER WORK

In this paper we have constructed a wide class of copulas generalizing well-known families such as the semilinear copulas. In [16], the authors introduce new types of semilinear copulas based on diagonal and opposite diagonal functions. The generalization of the so-called orbital semilinear copulas (in a similar manner to that done in this work) is subject of further research.

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