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ON AN ALGORITHM FOR TESTING T4 SOLVABILITY OF MAX-PLUS INTERVAL SYSTEMS

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In this paper, we shall deal with the solvability of interval systems of linear equations in max-plus algebra. Max-plus algebra is an algebraic structure in which classical addition and multiplication are replaced by $\oplus$ and $\otimes$, where $a \oplus b = \max\{a, b\}$, $a \otimes b = a + b$.

The notation $\boldsymbol{A} \otimes x = \boldsymbol{b}$ represents an interval system of linear equations, where $\boldsymbol{A} = [\underline{A}, \overline{A}]$ and $\boldsymbol{b} = [\underline{b}, \overline{b}]$ are given interval matrix and interval vector, respectively. We can define several types of solvability of interval systems. In this paper, we define the T4 solvability and give an algorithm for checking the T4 solvability.

Keywords: max-plus algebra, interval system, T4 vector, T4 solvability

Classification: 15A06, 65G30

1. INTRODUCTION

In many systems which we deal in engineering or physics with, the state varies continuously through the time. A simple example is an electrical circuit, where the voltage at a particular time may be described as a function of the continuous variable t representing time: $V = V(t)$. A change in voltage depending on the time is expressed by differential equations. By contrast, many others systems, especially those which occur in a digital signal processing or industrial are often more conveniently expressed in terms of events. We may speak about discrete events systems in which the individual components move from event to event rather than varying continuously through the time. Among interesting real-life applications let us mention, e. g., a large scale model of Dutch railway network or synchronizing traffic lights in Delft [11]. Behavior of such systems can be described by systems of linear equations in the form $A \otimes x = b$, where the $\otimes$-matrix product is computed formally in the same way as the classical matrix product, just instead of addition we use the operation $\oplus = \max$ and instead of multiplication we use the operation $\otimes = +$.

However, when the matrix and vector entries are estimated incorrectly, the obtained theoretical results may become useless in practice, due to imprecise results. A possible method of restoring solvability is to replace the matrix A and vector b by a matrix interval and a vector interval. Then we talk about an interval system of linear equations. The theory of interval computations and in particular of interval systems in the classical algebra is already quite developed, see e. g. the monograph [6] or [13, 14]. An interesting approach to interval computations was published in [5, 12]. In max-plus and max-min algebra, interval systems of linear equations have been studied by K. Cechlárová and R. A. Cuninghame-Green in [2, 3]. They dealt with the weak, strong and tolerance solvability. In [7, 8], we studied the weak tolerance, weak control, control, universal and weak universal solvability in max-plus and max-min algebra. The T1, T2 and T3 solvability have been studied in [9] only for the max-min case, because in max-plus algebra they are trivial, so they do not have a practical importance. In this paper, we define the T4 solvability and give necessary and sufficient condition in max-plus algebra.

There is also motivation coming from applications for the use of interval systems. One of possible applications follows from the following example, taken from [11], but slightly modified and generalized.

Example 1.1. There are two railway stations S_1 and S_2 in a metropolitan area, which are interconnected by a railway system consisting of two inner circles and two outer circles (see Figure 1). The number $a_{ij},\ i, j \in \{1, 2\}$ indicates the transit time from station S_j to station S_i including the time necessary for passengers to change over.

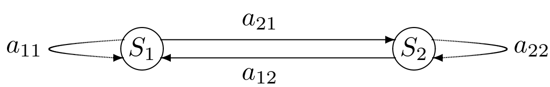

Fig. 1.

Suppose that there are four trains (two at each station) and two of them in the same station S_i leave simultaneously at the time x_i. The time at which both trains are already in station S_i is equal to $\max\{a_{i1}+x_1, a_{i2}+x_2\}$. Suppose that there are two schools near the two stations that begin their daily programme at the times b_1, b_2. It is required to find the departure times x_i which allow the students to catch the beginning of classes, i. e.,

$$\max_j (a_{ij} + x_j) = b_i \tag{1}$$

for $i = 1, 2$. Using the symbols $\oplus$ and $\otimes$ for operations of maximum and addition, respectively, we can rewrite (1) to the matrix form

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}. \tag{2}$$

The above described model system can be easily rewritten in general case of n stations. If there is no traffic from station S_j to station S_i we set $a_{ij} = -\infty$.

2. PRELIMINARIES

By *max-plus algebra* we understand a triple $(B, \oplus, \otimes)$, where

$$B = \mathbb{R} \cup \{\varepsilon\},\ a \oplus b = \max\{a, b\},\ a \otimes b = a + b,$$

and $\varepsilon = -\infty$. Denote by M and N the index sets $\{1, 2, \ldots, m\}$ and $\{1, 2, \ldots, n\}$, respectively. The set of all $m \times n$ *matrices* over B is denoted by $B(m, n)$ and the set of

all column *n-vectors* over B by $B(n)$.
Operations $\oplus$ and $\otimes$ are extended to matrices and vectors in the same way as in classical algebra. We shall consider the *ordering* $\leq$ on the sets $B(m,n)$ and $B(n)$ defined as follows:

- for $A, C \in B(m,n): \ A \leq C$ if $a_{ij} \leq c_{ij}$ for all $i \in M, \ j \in N$,
- for $x, y \in B(n): \ x \leq y$ if $x_j \leq y_j$ for all $j \in N$.

We shall use the *monotonicity of* $\otimes$ which means that for each $A, C \in B(m,n)$ and for each $x, y \in B(n)$ the implication

$$\text{if } A \leq C \text{ and } x \leq y \text{ then } A \otimes x \leq C \otimes y$$

holds true.
In max-plus algebra we can write the system of equations (2) in the form

$$A \otimes x = b \tag{3}$$

which represents a *max-plus system of linear equations*.
To give a necessary and sufficient condition for solvability of (3), we add some conditions. We shall suppose that

i) $b_i > \varepsilon$ for all $i \in M$,

ii) A contains no column with full ε-s.

To justify the first assumption, we show how to get rid of ε-s. Namely, denote by M_0 the set $M_0 = \{i \in M; \ b_i = \varepsilon\}$. Then any solution x of (3) has $x_j = \varepsilon$ for each $j \in N_0$, where $N_0 = \{j \in N; \ a_{ij} \neq \varepsilon$ for some $i \in M_0\}$. Therefore it is possible to omit the equations with indices from M_0 and columns of A with indices from N_0 and the solutions of the original and reduced systems correspond to each other by setting $x_j = \varepsilon$ for $j \in N_0$ in the former.
To justify the second assumption denote by $N_1 = \{j \in N; \ a_{ij} = \varepsilon$ for each $i \in M\}$. Then x_j can be arbitrary for each $j \in N_1$. Therefore it is possible to omit the columns of A with indices from N_1 and the solutions of the original and reduced systems correspond to each other by setting $x_j = x, \ x$ for all $j \in N_1$, where x is an arbitrary element from B.
By now, we can define a principal solution of system (3) as follows:

$$x^*_j(A,b) = \min_{i \in M}\{b_i - a_{ij}\} \tag{4}$$

for each $j \in N$.

The following assertions describe the importance of the principal solution for the solvability of (3).

Lemma 2.1. (Cuninghame-Green [4], Zimmermann [15]) Let $A \in B(m,n)$ and $b \in B(m)$ be given.

i) If $A \otimes x = b$ for $x \in B(n)$, then $x \leq x^*(A,b)$.

ii) $A \otimes x^*(A,b) \leq b$.

Theorem 2.2. (Cuninghame-Green [4]) Let $A \in B(m,n)$ and $b \in B(m)$ be given. Then the system $A \otimes x = b$ is solvable if and only if $x^*(A,b)$ is its solution.

Lemma 2.3. (Myšková [7]) Let $A \in B(m,n),\ b,d \in B(m)$ be such that $b \le d$. Then $x^*(A,b) \le x^*(A,d)$.

Lemma 2.4. (Myšková [7]) Let $b \in B(m),\ C,D \in B(m,n)$ be such that $D \le C$. Then $x^*(C,b) \le x^*(D,b)$.

3. INTERVAL SYSTEMS

In practice, the travelling times in Example 1.1 may depend on outside conditions, so the values a_{ij} are from an interval of possible values, i.e., $a_{ij} \in [\underline{a}_{ij}, \overline{a}_{ij}]$ for each $i \in N,\ j \in N$. Also we shall require the arrival times to be not precise values but they are rather from given intervals, i.e., $b_i \in [\underline{b}_i, \overline{b}_i]$ for each $i \in N$.

Similarly to [2, 7, 8, 12] we define an *interval matrix* $\boldsymbol{A}$ and *interval vector* $\boldsymbol{b}$ as follows:

$$\boldsymbol{A} = [\underline{A}, \overline{A}] = \left\{ A \in B(m,n);\ \underline{A} \le A \le \overline{A} \right\}$$

and

$$\boldsymbol{b} = [\underline{b}, \overline{b}] = \left\{ b \in B(m);\ \underline{b} \le b \le \overline{b} \right\},$$

where $\underline{A}, \overline{A} \in B(m,n)$, $\underline{A} \le \overline{A}$ and $\underline{b}, \overline{b} \in B(m)$, $\underline{b} \le \overline{b}$.

Denote by

$$\boldsymbol{A} \otimes x = \boldsymbol{b} \tag{5}$$

the set of all of max-plus systems of linear equations of the form (3) such that $A \in \boldsymbol{A},\ b \in \boldsymbol{b}$. We shall call (5) a *max-plus interval system of linear equations.* A system of the form (3) is called a *subsystem* of (5) if $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$. We say, that interval system (5) has the *constant matrix* if $\underline{A} = \overline{A}$ and has the *constant right-hand side*, if $\underline{b} = \overline{b}$.

To use the arguments from the previous section we shall suppose for interval system (5) that

- $\underline{b}_i \neq \varepsilon$ for each $i \in M$,
- for each $j \in N$ there exists $i \in M$ such that $\underline{a}_{ij} \neq \varepsilon$.

We can define several conditions which the given interval system is required to fulfill. According to them we shall define several solvability concepts. Table 1 contains the list of all up to now studied types of the solvability of (5) in max-plus algebra. There are omitted solvability concepts which lead to trivial conditions.

Solvability concept	**Definition**
Weak solvability [2]	$(\exists x \in B(n))(\exists A \in \boldsymbol{A})(\exists b \in \boldsymbol{b}): A \otimes x = b$
Strong solvability [3]	$(\forall A \in \boldsymbol{A})(\forall b \in \boldsymbol{b})(\exists x \in B(n)): A \otimes x = b$
Tolerance solvability [2]	$(\exists x \in B(n))(\forall A \in \boldsymbol{A})(\exists b \in \boldsymbol{b}): A \otimes x = b$
Weak tolerance solvability [7]	$(\forall A \in \boldsymbol{A})(\exists x \in B(n))(\exists b \in \boldsymbol{b}): A \otimes x = b$
Control solvability [8]	$(\exists x \in B(n))(\forall b \in \boldsymbol{b})(\exists A \in \boldsymbol{A}): A \otimes x = b$
Weak control solvability [8]	$(\forall b \in \boldsymbol{b})(\exists x \in B(n))(\exists A \in \boldsymbol{A}): A \otimes x = b$
Universal solvability [7]	$(\exists x \in B(n))(\forall b \in \boldsymbol{b})(\forall A \in \boldsymbol{A}): A \otimes x = b$
Weak universal solvability [8]	$(\forall b \in \boldsymbol{b})(\exists x \in B(n))(\forall A \in \boldsymbol{A}): A \otimes x = b$

Table 1.

4. T4 SOLVABILITY

In this section, we define the notions of T4 vector and T4 solvability of interval system (5). We present a procedure for checking T4 solvability.

Definition 4.1.

i) A vector $b \in \boldsymbol{b}$ is called a *T4 vector* of interval system (5) if there exists $x \in B(n)$ such that $A \otimes x = b$ for each $A \in \boldsymbol{A}$.

ii) Interval system (5) is *T4 solvable* if there exists $b \in \boldsymbol{b}$ such that b is a T4 vector of (5).

To give a necessary and sufficient condition for T4 solvability, we use a notion of a *universal solution*, which has been studied by K. Cechlárová [2].
Recall, that a vector $x \in B(n)$ is a universal solution of interval system (5) if $A \otimes x = b$ for each $A \in \boldsymbol{A}$ and for each $b \in \boldsymbol{b}$.

Theorem 4.2. (Myšková [7]) Interval system (5) with the constant right-hand side $b = \underline{b} = \overline{b}$ has a universal solution if and only if

$$\underline{A} \otimes x^*(\overline{A}, b) = b \tag{6}$$

and in this case $x^*(\overline{A}, b)$ is the maximum universal solution.

Lemma 4.3. A vector $b \in \boldsymbol{b}$ is a T4 vector of interval system (5) if and only if it satisfies equality (6).

Proof. A vector $b \in \boldsymbol{b}$ is a T4 vector if and only if interval system (5) with the constant right-hand side $\underline{b} = \overline{b} = b$ has a universal solution, which is according to Theorem 4.2 equivalent to (6). □

For any fixed $b \in \boldsymbol{b}$ we denote

$$M_j(\boldsymbol{A}) = \{i \in M : \underline{a}_{ij} = \overline{a}_{ij} \neq \varepsilon\}, \quad P_j(\boldsymbol{A}, b) = \{i \in M : x_j^*(\overline{A}, b) = b_i - \overline{a}_{ij}\}$$
$$\text{and } L_j(\boldsymbol{A}, b) = M_j(\boldsymbol{A}) \cap P_j(\boldsymbol{A}, b)$$

for each $j \in N$.

Lemma 4.4. A vector $b \in \boldsymbol{b}$ is a T4 vector of interval system (5) if and only if $\cup_{j \in N} L_j(\boldsymbol{A}, b) = M$.

Proof. Let $b \in \boldsymbol{b}$ be such that $\cup_{j\in N} L_j(\boldsymbol{A}, b) = M$. Then for each $k \in M$ there exists $r \in N$ such that $k \in L_r(\boldsymbol{A}, b)$, i.e., $\underline{a}_{kr} = \overline{a}_{kr}$ and $x^*_r(\overline{A}, b) = b_k - \overline{a}_{kr}$. Then

$$b_k \geq [\overline{A} \otimes x^*(\overline{A}, b)]_k \geq [\underline{A} \otimes x^*(\overline{A}, b)]_k = \max_{j\in N}\{\underline{a}_{kj} + x^*_j(\overline{A}, b)\}$$

$$\geq \underline{a}_{kr} + x^*_r(\overline{A}, b) = \underline{a}_{kr} + b_k - \overline{a}_{kr} = b_k$$

holds true for each $k \in M$. We have $\underline{A} \otimes x^*(\overline{A}, b) = b$, so the vector b is according to Lemma 4.3 a T4 vector of (5).

For the converse implication suppose that for some $b \in \boldsymbol{b}$ there exists $k \in M$ such that $k \notin \cup_{j\in N} L_j(\boldsymbol{A}, b)$ which means that for each $j \in N$ one of the following cases has occurred:

$$\underline{a}_{kj} < \overline{a}_{kj} \quad \text{or} \quad x^*_j(\overline{A}, b) < b_k - \overline{a}_{kj}.$$

In the first case we get $\underline{a}_{kj} + x^*_j(\overline{A}, b) = \underline{a}_{kj} + \min_{i\in M}\{b_i - \overline{a}_{ij}\} \leq \underline{a}_{kj} + b_k - \overline{a}_{kj} < b_k$.
In the second case $\underline{a}_{kj} + x^*_j(\overline{A}, b) < \underline{a}_{kj} + b_k - \overline{a}_{kj} \leq b_k$.
Then $[\underline{A} \otimes x^*(\overline{A}, b)]_k = \max_{j\in N}\{\underline{a}_{kj} + x^*_j(\overline{A}, b)\} < b_k$ which implies that the vector b is not a T4 vector of (5). □

Corollary 4.5. If $\cup_{j\in N} M_j(\boldsymbol{A}) \neq M$, then interval system (5) is not T4 solvable.

In the following we shall suppose that $\cup_{j\in N} M_j(\boldsymbol{A}) = M$.

For each $i \in M$ denote by J_i the set $J_i = \{j \in N : \underline{a}_{ij} = \overline{a}_{ij} \neq \varepsilon\}$. Let $p = (p_1, p_2, \ldots, p_m) \in J_1 \times J_2 \times \cdots \times J_m$ be arbitrary but fixed. Denote by

$$\boldsymbol{A}^{(p)} \otimes x = \boldsymbol{b} \tag{7}$$

an interval system with $\boldsymbol{A}^{(p)} = [\underline{A}^{(p)}, \overline{A}^{(p)}]$ where $\overline{A}^{(p)} = \overline{A}$ and $\underline{A}^{(p)} = (\underline{a}^{(p)}_{ij})$ is defined as follows:

$$\underline{a}^{(p)}_{ij} = \begin{cases} \underline{a}_{ij} & \text{for } (i,j) = (i, p_i), i \in M, \\ \varepsilon & \text{else.} \end{cases} \tag{8}$$

By (8), for a given $p \in J_1 \times J_2 \times \cdots \times J_m$ the matrix $\underline{A}^{(p)}$ has in each row exactly one element equal to the corresponding element in $\underline{A}$ and the others are equal to ε.

Theorem 4.6. Interval system (5) is T4 solvable if and only if there exists an m-tuple $p \in J_1 \times J_2 \times \cdots \times J_m$ such that the interval system $\boldsymbol{A}^{(p)} \otimes x = \boldsymbol{b}$ is T4 solvable.

Proof. If interval system (5) is T4 solvable then there exists a vector b such that b is a T4 vector of (5). By Lemma 4.4, for each $k \in M$ there exists $r_k \in N$ such that $k \in L_{r_k}(\boldsymbol{A}, b)$, i.e., $\underline{a}_{kr_k} = \overline{a}_{kr_k}$ and $x^*_{r_k}(\overline{A}, \overline{b}) = b_k - \overline{a}_{kr_k}$. Then $r_k \in J_k$ for each $k \in M$. Let us set $r = (r_1, r_2, \ldots, r_m)$. For each $k \in M$ we have

$$[\underline{A}^{(r)} \otimes x^*(\overline{A}^{(r)}, b)]_k = [\underline{A}^{(r)} \otimes x^*(\overline{A}, b)]_k = \underline{a}_{kr_k} \otimes x^*_{r_k}(\overline{A}, b) = \overline{a}_{kr_k} + b_k - \overline{a}_{kr_k} = b_k.$$

Since $\underline{A}^{(r)} \otimes x^*(\overline{A}^{(r)}, b) = b$ the interval system $\boldsymbol{A}^{(r)} \otimes x = \boldsymbol{b}$ is T4 solvable with the vector b as a T4 vector.

For the converse implication suppose that there exists an m-tuple $p \in J_1 \times J_2 \times \ldots \times J_m$ such that the interval system $\boldsymbol{A}^{(p)} \otimes x = \boldsymbol{b}$ is T4 solvable, i.e., there exists $b \in \boldsymbol{b}$ such that $\underline{A}^{(p)} \otimes x^*(\overline{A}^{(p}, b) = b$. Then

$$b \geq \underline{A} \otimes x^*(\overline{A}, b) \geq \underline{A}^{(p)} \otimes x^*(\overline{A}^{(p)}, b) = b$$

which implies $\underline{A} \otimes x^*(\overline{A}, b) = b$, so interval system (5) is T4 solvable and the vector b is a T4 vector of (5). □

From the previous theorem it follows that for checking solvability of interval system (5) it suffice to deal with T4 solvability of systems of the form (7).

Definition 4.7. The *T4 sequence* of interval system (7) is a sequence $\{b^{(k)}\}_{k=0}^{\infty}$ defined as follows:

$$b^{(k)} = \begin{cases} \overline{b} & \text{for } k = 0, \\ \underline{A}^{(p)} \otimes x^*(\overline{A}^{(p)}, b^{(k-1)}) & \text{for } k \geq 1. \end{cases} \tag{9}$$

Lemma 4.8. Let $\{b^{(k)}\}_{k=0}^{\infty}$ be the T4 sequence of interval system (7) and $l \in \mathbb{N}_0$ be arbitrary. The following assertions hold true:

i) The sequence $\{b^{(k)}\}_{k=0}^{\infty}$ is non-increasing.

ii) If $b^{(l+1)} = b^{(l)}$ then for each $k \in \mathbb{N}$ the equality $b^{(l+k)} = b^{(l)}$ holds.

Proof.

i) By monotonicity of $\otimes$ we have

$$b^{(k+1)} = \underline{A}^{(p)} \otimes x^*(\overline{A}^{(p)}, b^{(k)}) \leq \overline{A}^{(p)} \otimes x^*(\overline{A}^{(p)}, b^{(k)}) \leq b^{(k)}.$$

ii) Suppose that $b^{(l+1)} = b^{(l)}$. By mathematical induction on k we prove that for each $k \in \mathbb{N}$ the equality $b^{(l+k)} = b^{(l)}$ holds.

1. For $k = 1$ the equality $b^{(l+1)} = b^{(l)}$ follows from the assumption.
2. We prove that if $b^{(l+k)} = b^{(l)}$ then $b^{(l+k+1)} = b^{(l)}$. We have

$$b^{(l+k+1)} = \underline{A}^{(p)} \otimes x^*(\overline{A}^{(p)}, b^{(l+k)}) = \underline{A}^{(p)} \otimes x^*(\overline{A}^{(p)}, b^{(l)}) = b^{(l+1)} = b^{(l)}.$$

□

Theorem 4.9. Let $b \in \boldsymbol{b}$ be a T4 vector of interval system (7). Then for each $k \in \mathbb{N}_0$ the inequality $b \leq b^{(k)}$ is satisfied.

Proof. By mathematical induction on k

1. For $k = 0$ the inequality $b \leq \overline{b} = b^{(0)}$ trivially holds.
2. We prove that if $b \leq b^{(k)}$ then $b \leq b^{(k+1)}$.
For the sake of a contradiction suppose that $b \leq b^{(k)}$ and $b \nleq b^{(k+1)}$, i.e., there exists $r \in M$ such that $b_r > b_r^{(k+1)}$. We get
$$b_r > [\underline{A}^{(p)} \otimes x^*(\overline{A}^{(p)}, b^{(k)})]_r \geq [\underline{A}^{(p)} \otimes x^*(\overline{A}^{(p)}, b)]_r$$
which implies $\underline{A}^{(p)} \otimes x^*(\overline{A}^{(p)}, b) \neq b$, a contradiction.

□

Remark 4.10. We can define in the same way the T4 sequence for an arbitrary interval system and the assertions of Lemma 4.8 and Theorem 4.9 hold true.

For a fixed $p \in J_1 \times J_2 \times \cdots \times J_m$ denote by
$$\boldsymbol{C} \otimes x = \boldsymbol{b} \tag{10}$$
the interval system with $\boldsymbol{C} = [\underline{C}, \overline{C}]$, where $\boldsymbol{C}$ is obtained from $\boldsymbol{A}^{(p)}$ by deleting all columns $\boldsymbol{A}_j^{(p)}$ (where $\boldsymbol{A}_j^{(p)}$ denotes the jth column of $\boldsymbol{A}^{(p)}$) such that $j \in L$, where $L = \{j \in N : a_{ij}^{(p)} = \varepsilon \text{ for each } i \in M\}$.

Lemma 4.11. Let $\{b^{(k)}\}_{k=0}^{\infty}$ and $\{c^{(k)}\}_{k=0}^{\infty}$ be the T4 sequences of interval systems (7) and (10), respectively. Then $b^{(k)} = c^{(k)}$ for each $k \in \mathbb{N}_0$.

Proof. We prove the equality $b^{(k)} = c^{(k)}$ by mathematical induction on k:

1. For $k = 0$ we have $b^{(0)} = c^{(0)} = \overline{b}$.
2. Suppose that $b^{(k)} = c^{(k)}$. Then for each $i \in M$ we get
$$b_i^{(k+1)} = [\underline{A}^{(p)} \otimes x^*(\overline{A}^{(p)}, b^{(k)})]_i = \bigoplus_{j \in N} \underline{a}_{ij}^{(p)} \otimes x_j^*(\overline{A}^{(p)}, b^{(k)}) = \bigoplus_{j \in N} \underline{a}_{ij}^{(p)} \otimes x_j^*(\overline{A}^{(p)}, c^{(k)})$$
$$= \bigoplus_{j \in N-L} \underline{a}_{ij}^{(p)} \otimes x_j^*(\overline{A}^{(p)}, c^{(k)}) = \bigoplus_{j \in N-L} \underline{c}_{ij} \otimes x_j^*(\overline{C}, c^{(k)}) = c_i^{(k+1)}$$
because of $x_j^*(\overline{C}, c^{(k)}) = x_j^*(\overline{A}^{(p)}, c^{(k)})$ for each $j \in N - L$.

□

Now we shall deal with the properties of the T4 sequence of interval system (10). The size of interval matrix $\boldsymbol{C}$ is $m \times n^*$, where $n^* = n - |L|$. Let us define the *equality matrix* E of the size $m \times n^*$ as follows:
$$e_{ij} = \begin{cases} 1 & \text{if } \underline{c}_{ij} = \overline{c}_{ij} \neq \varepsilon, \\ 0 & \text{otherwise,} \end{cases} \tag{11}$$

for each $i \in M,\ j \in N^* = \{1, 2, \dots, n^*\}$. It is easy to see that there is exactly one unit in each row and at least one unit in each column in E. Without lost of generality we can suppose that

$$e_{ij} = \begin{cases} 1 & \text{for } j \in N^*,\ i \in \{i_{j-1}+1, \dots, i_j\}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\{i_k\}_{k=0}^{n^*}$ is an increasing sequence of indices with $i_0 = 0,\ i_{n^*} = m$.

For clarity, we use the notations $x^{(k)}$ instead of $x^*(\overline{C}, c^{(k)})$ and $\bar{c}_{i,j}$ instead of $\bar{c}_{ij}$. From the equality $c^{(k+1)} = \underline{C} \otimes x^{(k)}$ we get

$$c^{(k+1)} = (\bar{c}_{1,1} + x_1^{(k)}, \bar{c}_{2,1} + x_1^{(k)}, \dots \bar{c}_{i_1,1} + x_1^{(k)}, \bar{c}_{i_1+1,2} + x_2^{(k)}, \dots \bar{c}_{i_2,2} + x_2^{(k)}, \dots,$$
$$\bar{c}_{i_{s-1}+1,s} + x_s^{(k)}, \dots, \bar{c}_{i_s,s} + x_s^{(k)}, \dots, \bar{c}_{i_{m-1}+1,n^*} + x_{n^*}^{(k)}, \dots, \bar{c}_{m,n^*} + x_{n^*}^{(k)})^T.$$

By (4) we get

$$x_j^{(k+1)} = \min\{(x_1^{(k)} + \min_{r \in \{1,2,\dots,i_1\}} \{\bar{c}_{r,1} - \bar{c}_{r,j}\}), (x_2^{(k)} + \min_{r \in \{i_1+1,\dots,i_2\}} \{\bar{c}_{r,2} - \bar{c}_{r,j}\}), \dots,$$
$$(x_s^{(k)} + \min_{r \in \{i_{s-1}+1,\dots,i_s\}} \{\bar{c}_{r,s} - \bar{c}_{r,j}\}), \dots, (x_{n^*}^{(k)} + \min_{r \in \{i_{n^*-1}+1,\dots,m\}} \{\bar{c}_{r,n^*} - \bar{c}_{r,j}\})\}$$
$$\min_{s \in N^*} \{= x_s^{(k)} + \min_{r \in \{i_{s-1}+1,\dots,i_s\}} \{\bar{c}_{r,s} - \bar{c}_{r,j}\}\}$$
$$= -\max_{s \in N^*} \{-x_s^{(k)} + \max_{r \in \{i_{s-1}+1,\dots,i_s\}} \{\bar{c}_{r,j} - \bar{c}_{r,s}\}\},$$

or equivalently

$$-x_j^{(k+1)} = \max_{s \in N^*} \{-x_s^{(k)} + \max_{r \in \{i_{s-1}+1,\dots,i_s\}} \{\bar{c}_{r,j} - \bar{c}_{r,s}\}\} \tag{12}$$

for each $j \in N^*$. Let us define the *difference matrix* $D = (d_{js})$ of the size $n^* \times n^*$ as follows:

$$d_{js} = \max_{r \in \{i_{s-1}+1,\dots,i_s\}} \{\bar{c}_{r,j} - \bar{c}_{r,s}\}. \tag{13}$$

for each $j, s \in N^*$. Denote by $\hat{x}^{(k)}$ the opposite vector to the vector $x^{(k)}$, i. e., $\hat{x}^{(k)} = -x^{(k)}$. By now, we can write equality (12) in the form

$$\hat{x}^{(k+1)} = D \otimes \hat{x}^{(k)},$$

which implies

$$\hat{x}^{(k)} = D^k \otimes \hat{x}^{(0)} \tag{14}$$

for each $k \in \mathbb{N}$, where D^k denotes the kth power of D. Therefore the properties of the sequence $\{\hat{x}^{(k)}\}_{k=1}^{\infty}$ follow from the properties of the sequence $\{D^k\}_{k=1}^{\infty}$.

We shall use known properties of the sequence of powers of matrices, studied in [10]. Now, we introduce necessary notions and recall known results.

Definition 4.12. Let $A \in B(n, n)$. The *digraph of matrix* A, in notation $\mathcal{G}(A)$, is a weighted digraph (N, E, v), with the vertex set $N = \{1, 2, \dots, n\}$, the arc set $E = \{(i, j);\ i, j \in N, a_{ij} \neq \varepsilon\}$ and a weight function $v:\ E \to B$ defined by $v(i, j) = a_{ij}$ for every $(i, j) \in E$.

For any $i, j \in N$, we denote by $W_{\mathcal{G}(A)}(i,j)$ the set of all walks in $\mathcal{G}(A)$, beginning in vertex i and ending in vertex j. If $w = (w_0, w_1, \ldots, w_r) \in W_{\mathcal{G}(A)}(i,j)$ is a walk in $\mathcal{G}(A)$ of length r, then the weight of w is defined by the sum $v(w) = \sum_{1\leq k\leq r} v(w_{k-1}, w_k)$. If c is a cycle of positive length $|c|$, then $\overline{c} = w(c)/|c|$ denotes the cycle mean of c. The maximal cycle mean is denoted by $\lambda(A)$. We say, that c is a *critical cycle*, if $\overline{c} = \lambda(A)$.

Lemma 4.13. Let D be the matrix defined by (13). Then $\lambda(D) \geq 0$.

Proof. By (13), we have $d_{ii} = 0$ for each $i \in N^*$. This follows that $\mathcal{G}(D)$ contains a loop of weight zero at each vertex, i.e. there exist cycles in $\mathcal{G}(D)$ with the cycle mean equal to zero. By definition of $\lambda(D)$, we have $\lambda(D) \geq 0$. □

We introduce the notion of an *almost linear periodicity*.

Definition 4.14. We say that a sequence $a^* = \{a^{(r)}; r \in N\}$ in B is *almost linear periodic*, if there are $q \in B,\ q \neq \varepsilon,\ p \in \mathbb{N}$ and $d \in \mathbb{N}$ such that for every $r > d$

$$a^{(r+p)} = a^{(r)} + p \times q \tag{15}$$

where $+$ and $\times$ are classical addition and multiplication, respectively. The element $q = \mathrm{lfac}(a^*)$ is called the *linear factor* of a^* and the smallest numbers $d = \mathrm{ldef}(A),\ p = \mathrm{per}(a^*)$ with the above properties are called the *linear defect* and *linear period* of a^*, respectively.

Theorem 4.15. [10] Let $A \in B(n,n)$ be a matrix with digraph $\mathcal{G}(A) = (N, E, v)$, maximum cycle mean $\lambda(A)$ and critical cycle c. Then the sequences $a^*_{ij} = \{a^r_{ij}; r \in \mathbb{N}\}$ and $a^*_{ji} = \{a^r_{ji}; r \in \mathbb{N}\}$ for each $i \in c$ and each $j \in N$ are almost linear periodic with linear period $p = |c|$, linear defect $d \leq p \cdot (n-1)$ and linear factor $q = \lambda(A)$.

Lemma 4.16. Let $\{b^{(k)}\}_{k=0}^{\infty}$ be a T4 sequence of interval system (7). The following conditions are satisfied

i) If $\lambda(D) > 0$ then there exists $r \in M$ such that the sequence $\{b^{(k)}_r\}_{k=0}^{\infty}$ is almost linear periodic with linear factor $q = -\lambda(D)$.

ii) If $\lambda(D) = 0$ then $b^{(n^*+2)} = b^{(n^*+1)}$.

Proof.

i) If $\lambda(D) > 0$, then there is a cycle c of the length at least two with $\overline{c} = \lambda(D)$ in $\mathcal{G}(D)$ (all cycles of length one have the cycle mean equal to zero). Let $s \in N,\ s \in c$. By Theorem 4.15 the sequence $d^*_{sj} = \{d^r_{sj}; r \in \mathbb{N}\}$ is almost linear periodic for each $j \in N^*$ with linear period $p = |c|$, linear defect $d \leq p \cdot (n^* - 1)$ and linear factor $\lambda(D)$. This follows that for each $k \in \mathbb{N},\ k > d$ we have $d^{k+p}_{sj} = d^k_{sj} + p \times \lambda(D)$. By (14) we get

$$\hat{x}^{(k+p)}_s = [D^{k+p} \otimes \hat{x}^{(0)}]_s = \max_{j\in N^*}\{d^{k+p}_{sj} + \hat{x}^{(0)}_j\} = \max_{j\in N^*}\{d^k_{sj} + p \times \lambda(D) + \hat{x}^{(0)}_j\}$$

$$= p \times \lambda(D) + \max_{j \in N^*}\{d_{sj}^k + \hat{x}_j^{(0)}\} = p \times \lambda(D) + \hat{x}_s^{(k)}.$$

Then $x_s^{(k+p)} = x_s^{(k)} + p \times (-\lambda(D))$ and consequently for each $k > d$ the equality $c_r^{(k+p+1)} = c_r^{(k+1)} + p \times (-\lambda(D))$ holds true for each $r \in \{i_{s-1}+1, i_s\}$, so the sequence $\{c_r^{(k)}\}_{k=0}^{\infty}$ is almost linear periodic with $q = -\lambda(D)$. According to Lemma 4.11 the sequence $\{b_r^{(k)}\}_{k=0}^{\infty}$ is almost linear periodic with $q = -\lambda(D)$, too.

ii) If $\lambda(D) = 0$ then there is a loop of weight zero at each node of $\mathcal{G}(D)$. So each node lies on a critical cycle of length one. By Theorem 4.15 the sequences $\{d_{ij}^k\}_{k=1}^{\infty}$ are almost linear periodic for each $i, j \in N^*$ with $p = |c| = 1$, $q = \lambda(D) = 0$ and $d \le n^* - 1$. Consequently the equality $D^{k+1} = D^k$ holds at least for each $k \ge n^*$. It implies $\hat{x}^{(n^*+1)} = \hat{x}^{(n^*)}$ and consequently $x^{(n^*+1)} = x^{(n^*)}$, $c^{(n^*+2)} = c^{(n^*+1)}$ which is equivalent to $b^{(n^*+2)} = b^{(n^*+1)}$.

□

Corollary 4.17. If $\lambda(D) > 0$ then $b^{(l+1)} \ne b^{(l)}$ for each $l \in \mathbb{N}_0$.

Proof. According to Lemma 4.16i) we have $b_r^{(l+p)} = b_r^{(l)} + p \times (-\lambda(D))$ for each $l > d$ which means that $b^{(l+p)} \ne b^{(l)}$. Using Lemma 4.8ii) we get $b^{(l+1)} \ne b^{(l)}$ for each $l \in \mathbb{N}_0$. □

Theorem 4.18. Interval system (7) is T4 solvable if and only if there exists $l \in \mathbb{N}_0$ such that $b^{(l+1)} = b^{(l)}$ and $b^{(l)} \in \boldsymbol{b}$.

Proof. If $b^{(l+1)} = b^{(l)} \in \boldsymbol{b}$ for some $l \in \mathbb{N}_0$ then $\underline{A}^{(p)} \otimes x^*(\overline{A}^{(p)}, b^{(l)}) = b^{(l)} \in \boldsymbol{b}$ which means that the vector $b^{(l)}$ is a T4 vector of (7), so interval system (7) is T4 solvable.

For the converse implication suppose that interval system (7) is T4 solvable and there does not exist any $l \in \mathbb{N}_0$ such that $b^{(l+1)} = b^{(l)} \in \boldsymbol{b}$. From Lemma 4.8 it follows that we shall distinguish two cases.

Case 1. For each $l \in \mathbb{N}_0$ the inequality $b^{(l+1)} \ne b^{(l)}$ holds.
Suppose that for each $l \in \mathbb{N}_0$ the inequality $b^{(l+1)} \ne b^{(l)}$ is satisfied and there exists a vector $b \in \boldsymbol{b}$ such that b is T4 vector of (7). According to Lemma 4.16ii) we have $\lambda(D) > 0$ and by Lemma 4.16i) there exists $r \in M$ such that the sequence $\{b_r^{(k)}\}_{k=0}^{\infty}$ is almost linear periodic with linear factor $q = -\lambda(D)$. It follows that there exists $l^* \in \mathbb{N}_0$ such that $b_r^{(l^*)} < \underline{b}_r$. Then $b_r \le b_r^{(l^*)} < \underline{b}$, i. e., $b \notin \boldsymbol{b}$, a contradiction.

Case 2. There exists $l \in \mathbb{N}_0$ such that $b^{(l+1)} = b^{(l)}$ and $b^{(l)} \notin \boldsymbol{b}$.
Suppose that there exists $l \in \mathbb{N}_0$ such that $b^{(l+1)} = b^{(l)} \notin \boldsymbol{b}$ and there exists a vector $b \in \boldsymbol{b}$ such that b is the T4-vector of (7). The definition of the sequence $\{b^{(k)}\}_{k=0}^{\infty}$ and Lemma 4.8i) imply that $\underline{b} \not\le b^{(l)}$. This means that there exists an index $r \in M$ such that $b_r^{(l)} < \underline{b}_r$. By Theorem 4.9 we get $b_r \le b_r^{(l)} < \underline{b}_r$ and consequently $b \notin \boldsymbol{b}$, a contradiction. □

Theorem 4.19. Interval system (7) is T4 solvable if and only if

$$b^{(n^*+2)} = b^{(n^*+1)} \in \boldsymbol{b}.$$

Proof. Suppose that there exists a vector $b \in \boldsymbol{b}$ such that b is T4 vector of (7). By Theorem 4.18 there exists $l \in \mathbb{N}_0$ such that $b^{(l+1)} = b^{(l)}$. According to Lemma 4.16 we have $\lambda(D) = 0$ which implies $b^{(n^*+2)} = b^{(n^*+1)}$. By Theorem 4.9 we get $\underline{b} \le b \le b^{(n^*+1)}$, so $b^{(n^*+1)} \in \boldsymbol{b}$ (the inequality $b^{(n^*+1)} \le \overline{b}$ follows from (9).

For the converse implication suppose that $b^{(n^*+2)} = b^{(n^*+1)} \in \boldsymbol{b}$. By Theorem 4.18 interval system (7) is T4 solvable. □

From Theorem 4.19 it follows that it suffices to compute $n^* + 2$ members of the sequence $\{b^{(k)}\}_{k=1}^{\infty}$ for checking the T4 solvability of (7). As for different $p \in J_1 \times J_2 \times \ldots \times J_m$ the number n^* is changing we use the inequality $n^* + 2 \le n + 2$ and for each $p \in M$ we restrict the number of members of the sequence $\{b^{(k)}\}_{k=1}^{\infty}$ by $n + 2$.

The previous assertions enable us to give the following algorithm for checking T4 solvability.

Algorithm T4.
Input: $\boldsymbol{A}, \boldsymbol{b}$
Output: 'yes' in variable $t4$ if the given interval system is T4 solvable, and 'no' in $t4$ otherwise

begin
Step 1. Find the sets $J_1, J_2, \ldots, J_m$; $P := J_1 \times J_2 \times \cdots \times J_m$; $t4$:='no'
Step 2. If $P = \emptyset$ or $t4$ ='yes' then go to end
Step 3. Choose arbitrary $p \in P$
Step 4. $b^{(0)} := \overline{b},\ k := 0$
Step 5. If $k = n + 2$ then go to *Step 9* else $b^{(k+1)} := \underline{A}^{(p)} \otimes x^*(\overline{A}^{(p)}, b^{(k)})$
Step 6. If $\underline{b} \not\le b^{(k+1)}$ then go to *Step 9*
Step 7. If $b^{(k+1)} = b^{(k)}$ then $t4$:='yes'; go to *Step 9*
Step 8. $k := k + 1$; go to *Step 5*
Step 9. $P := P - \{p\}$; go to *Step 2*
end

Now, we shall deal with the computational complexity of Algorithm T4. The most time-consuming is Step 5 which requires $O(m \cdot n)$ operations. The number of repetitions of the loop 5–8 is bounded by $n + 2$, so the loop 5–8 requires $O(m \cdot n^2)$ operations. The complexity of whole algorithm is given by the number of repetitions of the loop 2–9 which is given by the cardinality of the set P which is bounded by n^m so Algorithm T4 is not polynomial.

Remark 4.20. Using Algorithm T4 for the model system described in Example 1.1 we can find the vector of arrival times $b^* \in \boldsymbol{b}$ which can be achieved by suitable choice of the vector x of the departure times (for example the vector $x^*(\overline{A}, b^*)$) independently of the transit times between the stations, if such a vector of arrival times b exists.

Example 4.21. Let us take

$$\boldsymbol{A} = \begin{pmatrix} [18,19] & [20,20] & [14,15] & [18,19] & [10,10] \\ [10,10] & [7,8] & [8,9] & [9,10] & [7,12] \\ [10,11] & [10,11] & [9,10] & [12,12] & [8,11] \\ [4,5] & [16,17] & [18,18] & [16,16] & [1,10] \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} [17,25] \\ [8,17] \\ [9,16] \\ [11,15] \end{pmatrix}.$$

We have $J_1 = \{2,5\}$, $J_2 = \{1\}$, $J_3 = \{4\}$, $J_4 = \{3,4\}$, so $P = \{(5,1,4,3), (2,1,4,3), (5,1,4,4), (2,1,4,4)\}$.
For $p^{(1)} = (5,1,4,3)$ we have

$$\boldsymbol{A}^{(p^{(1)})} = \begin{pmatrix} [\varepsilon,19] & [\varepsilon,20] & [\varepsilon,15] & [\varepsilon,19] & [10,10] \\ [10,10] & [\varepsilon,8] & [\varepsilon,9] & [\varepsilon,10] & [\varepsilon,12] \\ [\varepsilon,11] & [\varepsilon,11] & [\varepsilon,10] & [12,12] & [\varepsilon,8] \\ [\varepsilon,5] & [\varepsilon,17] & [18,18] & [\varepsilon,16] & [\varepsilon,1] \end{pmatrix}.$$

The T4 sequence of the interval system $\boldsymbol{A}^{(p^{(1)})} \otimes x = \boldsymbol{b}$ is

$$\{b^{(k)}\}_{k=0}^{\infty} = \left\{ \overset{b^{(0)}}{\begin{pmatrix} 25 \\ 17 \\ 16 \\ 15 \end{pmatrix}}, \overset{b^{(1)}}{\begin{pmatrix} 15 \\ 15 \\ 11 \\ 15 \end{pmatrix}}, \dots \right\}.$$

Since $\underline{b} \not\leq b^{(1)}$, the interval system $\boldsymbol{A}^{(p^{(1)})} \otimes x = \boldsymbol{b}$ is not T4 solvable.
For $p^{(2)} = (2,1,4,3)$ we have

$$\boldsymbol{A}^{(p^{(2)})} = \begin{pmatrix} [\varepsilon,19] & [20,20] & [\varepsilon,15] & [\varepsilon,19] & [\varepsilon,10] \\ [10,10] & [\varepsilon,8] & [\varepsilon,9] & [\varepsilon,10] & [\varepsilon,12] \\ [\varepsilon,11] & [\varepsilon,11] & [\varepsilon,10] & [12,12] & [\varepsilon,8] \\ [\varepsilon,5] & [\varepsilon,17] & [18,18] & [\varepsilon,16] & [\varepsilon,1] \end{pmatrix}.$$

The T4 sequence of the interval system $\boldsymbol{A}^{(p^{(2)})} \otimes x = \boldsymbol{b}$ is

$$\{b^{(k)}\}_{k=0}^{3} = \left\{ \overset{b^{(0)}}{\begin{pmatrix} 25 \\ 17 \\ 16 \\ 15 \end{pmatrix}}, \overset{b^{(1)}}{\begin{pmatrix} 18 \\ 15 \\ 11 \\ 15 \end{pmatrix}}, \overset{b^{(2)}}{\begin{pmatrix} 18 \\ 9 \\ 11 \\ 15 \end{pmatrix}} \right\}, \quad b^{(k)} = b^{(2)} \text{ for } k \geq 3,$$

so interval system $\boldsymbol{A}^{(p^{(2)})} \otimes x = \boldsymbol{b}$ is T4 solvable. By Theorem 4.6 the given interval system $\boldsymbol{A} \otimes x = \boldsymbol{b}$ is T4 solvable.

The following example describes the extreme case, due to the complexity of the Algorithm T4.

Example 4.22. Let us take

$$\boldsymbol{A} = \begin{pmatrix} [14,14] & [12,15] & [36,44] \\ [10,19] & [19,19] & [48,48] \\ [1,3] & [42,42] & [46,46] \end{pmatrix}, \ \boldsymbol{b} = \begin{pmatrix} [21.23] \\ [25,50] \\ [28,31] \end{pmatrix}.$$

We have $J_1 = \{1\}$, $J_2 = \{2,3\}$, $J_3 = \{2,3\}$, so $P = \{(1,2,2),\ (1,2,3),\ (1,3,2),\ (1,3,3)\}$.

For $p^{(1)} = (1,2,2)$ we get $b^{(1)} = (23,8,31)^T \not\geq \underline{b}$, so interval system $\boldsymbol{A}^{(p^{(1)})} \otimes x = \boldsymbol{b}$ is not T4 solvable.

For $p^{(2)} = (1,2,3)$ we get $b^{(1)} = (23,8,25)^T \not\geq \underline{b}$, so interval system $\boldsymbol{A}^{(p^{(2)})} \otimes x = \boldsymbol{b}$ is not T4 solvable.

For $p^{(3)} = (1,3,2)$ we get the T4 sequence

$$\{b^{(k)}\}_{k=0}^{5} = \left\{ \overset{b^{(0)}}{\begin{pmatrix} 23 \\ 50 \\ 31 \end{pmatrix}}, \overset{b^{(1)}}{\begin{pmatrix} 23 \\ 27 \\ 31 \end{pmatrix}}, \overset{b^{(2)}}{\begin{pmatrix} 22 \\ 27 \\ 31 \end{pmatrix}}, \overset{b^{(3)}}{\begin{pmatrix} 22 \\ 26 \\ 31 \end{pmatrix}}, \overset{b^{(4)}}{\begin{pmatrix} 21 \\ 26 \\ 31 \end{pmatrix}}, \overset{b^{(5)}}{\begin{pmatrix} 21 \\ 25 \\ 31 \end{pmatrix}} \right\}.$$

Since we have $n^* = 3$ and $b^{(5)} \neq b^{(4)}$, by Theorem 4.19 interval system $\boldsymbol{A}^{(p^{(3)})} \otimes x = \boldsymbol{b}$ is not T4 solvable.

For $p^{(4)} = (1,3,3)$ we get $b^{(1)} = (23,27,25)^T \not\geq \underline{b}$, so interval system $\boldsymbol{A}^{(p^{(2)})} \otimes x = \boldsymbol{b}$ is not T4 solvable.

Since there is no T4 solvable interval system $\boldsymbol{A}^{(p)} \otimes x = \boldsymbol{b}$, according to Theorem 4.6 interval system $\boldsymbol{A} \otimes x = \boldsymbol{b}$ is not T4 solvable.

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