KYBERNETIKA — VOLUME *48* (2012), NUMBER 4, PAGES 589–599

EMPIRICAL ESTIMATOR OF THE REGULARITY INDEX OF A PROBABILITY MEASURE

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The index of regularity of a measure was introduced by Beirlant, Berlinet and Biau [1] to solve practical problems in nearest neighbour density estimation such as removing bias or selecting the number of neighbours. These authors proved the weak consistency of an estimator based on the nearest neighbour density estimator. In this paper, we study an empirical version of the regularity index and give sufficient conditions for its weak and strong convergence without assuming absolute continuity or other global properties of the underlying measure.

Keywords: regularity index, Lebesgue point, small ball probability

Classification: 62G05

1. INTRODUCTION

The subject of this paper is related to the general problem of the estimation of small ball probabilities. Beirlant et al. [1] introduced the notion of regularity index of a measure to specify the rate at which the ratio of ball measures converges at a Lebesgue point. Indeed, this index is the exponent appearing in the second order term of the expansion of the small ball probability. Then, they defined an estimator of this index based on the nearest neighbour density estimator and proved its weak consistency. This estimator was applied to solve practical problems in nearest neighbour density estimation such as removing bias or selecting the number of neighbours. More recently Berlinet and Servien [3] proved that this regularity index was the key parameter governing the limit distribution of nearest neighbour density estimators so that its estimation may be crucial in the derivation of confidence intervals. In the present paper, we study an empirical version of the regularity index and give sufficient conditions for its weak and strong convergence. Unlike Beirlant et al. [1] we do not assume absolute continuity of the underlying measure but only a pointwise property of small ball probabilities. Notation and former results are given in the next section. Section 3 gathers the definition of the estimator and its convergence properties. Section 4 is devoted to the proofs of the theorems and examples are given in Section 5.

2. NOTATION AND FORMER RESULTS

Let μ be a probability distribution and λ be the Lebesgue measure on $\mathbb{R}^d$ equipped with the Euclidean norm $||.||$. We denote by $B_\delta(x)$ the open ball with center x and radius δ. To evaluate the local behaviour of $\mu(B_\delta(x))$ in relation to $\lambda(B_\delta(x))$ one can consider the ratio of these two quantities. If, for fixed x, the following limit

$$\ell(x) = \lim_{\delta \to 0} \frac{\mu(B_\delta(x))}{\lambda(B_\delta(x))} \tag{1}$$

exists and is finite, then x is called a Lebesgue point of the measure μ (see Dudley [5] and Rudin [6]). This notion of Lebesgue point is essential to state elegant results with few restrictions on the functions to be estimated. In Berlinet and Levallois [2], examples where the density has a bad local behaviour at Lebesgue points are examined. To evaluate rates of convergence or investigate asymptotic normality of estimators, not only the convergence of the ratio of ball measures is required but also information on its higher order behaviour. In this context, Berlinet and Levallois [2] define a ρ-regularity point of the measure μ as any Lebesgue point x of μ satisfying

$$\left| \frac{\mu(B_\delta(x))}{\lambda(B_\delta(x))} - \ell(x) \right| \leq \rho(\delta), \tag{2}$$

where ρ is a measurable function such that $\lim_{\delta \downarrow 0} \rho(\delta) = 0$. To specify an exact rate of convergence of the ratio of ball measures, Beirlant et al. [1] assumed that a more precise relation than (2) holds at the Lebesgue point x; namely

$$\frac{\mu(B_\delta(x))}{\lambda(B_\delta(x))} = \ell(x) + C_x \delta^{\alpha_x} + o(\delta^{\alpha_x}) \text{ as } \delta \downarrow 0, \tag{3}$$

where C_x is a non-zero constant and α_x is a positive real number. It is easy to show that Equation (3) implies ρ-regularity at the point x with $\rho(\delta) = D_x \delta^{\alpha_x}$ and $D_x > C_x$. The constants C_x and α_x are unique (provided they exist). Examples are provided in Section 5 with an absolute continuous measure and a measure with discrete part. The index α_x is a regularity index that controls the degree of smoothness of the symmetric derivative of μ with respect to λ. The larger the value of α_x, the smoother the derivative of μ is at the point x. Beirlant et al.[1] showed the interest of estimating the regularity index to solve practical problems in nearest neighbour density estimation, such as removing bias or selecting the number of neighbours. More recently Berlinet and Servien [3] analyzed the effect of the value of α_x on limit distributions of nearest neighbour density estimators. They gave a necessary and sufficient condition involving α_x and the number of neighbours to have a limit distribution for the estimator.

The link with the small ball probability is clear since Equation (3) is equivalent to the expansion

$$P\left(\|X - x\| \leq \delta\right) = V_d \delta^d \left(\ell(x) + C_x \delta^{\alpha_x} + o(\delta^{\alpha_x})\right)$$

where X has probability distribution μ and $V_d = \pi^{d/2}/\Gamma(1 + d/2)$ denotes the volume of the unit ball in $\mathbb{R}^d$. In other words, the second order term in the expansion of the small ball probability of radius δ at x is equal, up to a multiplicative constant, to $\delta^{d+\alpha_x}$.

Hence it appears that to estimate α_x one needs some information on the behaviour of $\mu(B_\delta(x))$ as a function of δ. This is why the following theorem, proved by Beirlant et al. [1] will be useful in the sequel.

Theorem 2.1. Suppose that $x \in \mathbb{R}^d$ is a Lebesgue point of μ with regularity index α_x. Then, for any $\tau > 1$,

$$\lim_{\delta \to 0} \frac{\varphi_{\tau^2\delta}(x) - \varphi_{\tau\delta}(x)}{\varphi_{\tau\delta}(x) - \varphi_\delta(x)} = \tau^{\alpha_x}$$

where we denote, for $\delta > 0$,

$$\varphi_\delta(x) = \frac{\mu(B_\delta(x))}{\lambda(B_\delta(x))}.$$

Now let $X_1, \dots, X_n$ denote n independent random variables with distribution μ on $\mathbb{R}^d$, μ being unknown. Using the k_n-nearest neighbour density estimator

$$f_{k_n}(x) = \frac{k_n}{nV_d \left\|X_{(k_n)}(x) - x\right\|^d}$$

where $X_{(k_n)}(x)$ is the k_n^{th}-nearest neighbour of x and V_d is the volume of the unit ball in $\mathbb{R}^d$, Beirlant et al. [1] introduced an estimator $\bar{\alpha}_{n,x}$ of the regularity index inspired by the above theorem by setting, for $\tau > 1$,

$$\bar{\alpha}_{n,x} = \frac{d}{\ln \tau} \ln \frac{f_{\lfloor \tau^2 k_n \rfloor}(x) - f_{\lfloor \tau k_n \rfloor}(x)}{f_{\lfloor \tau k_n \rfloor}(x) - f_{\lfloor k_n \rfloor}(x)}, \tag{4}$$

if $[f_{\lfloor \tau^2 k_n \rfloor}(x) - f_{\lfloor \tau k_n \rfloor}(x)]/[f_{\lfloor \tau k_n \rfloor}(x) - f_{\lfloor k_n \rfloor}(x)] > 1$ and $\bar{\alpha}_{n,x} = 0$ otherwise, and proved the weak consistency of $\bar{\alpha}_{n,x}$.

In the paper by Beirlant et al. [1] most results are stated under the assumption of absolute continuity of the measure μ with respect to Lebesgue measure. This is required for instance to get a beta distribution for the random variable $\mu\left(B_{\left[X_{(k_n)}(x)-x\right]}(x)\right)$.

Our goal in the present paper is to define an empirical estimator inspired by the same theorem. For this, we simply replace in the expression of $\varphi_\delta(x)$ the unknown quantity $\mu(B_\delta(x))$ by its empirical counterpart. We prove the weak and strong consistency of the resulting estimator under the sole assumption that Equation (3) holds true. The present paper stays at a theoretical level, giving conditions on the deterministic sequence (δ_n) to get consistency. This is a first step. Further work should lead to an automatic choice of this sequence from the observed data.

3. THE EMPIRICAL ESTIMATOR AND ITS CONVERGENCE

Let $(X_i)_{i \geq 1}$ be a sequence of independent real d-dimensional random vectors with distribution μ. The empirical measure μ_n associated with $X_1, \dots, X_n$ is defined by

$$\mu_n(A) = \frac{1}{n} \sum_{i=1}^{n} I_{(X_i \in A)}, \qquad A \subseteq \mathbb{R}^d,$$

where

$$I_{(X_i \in A)} = \begin{cases} 1 \text{ if } X_i \in A \\ 0 \text{ otherwise} \end{cases}$$

and the associated empirical estimator of $\varphi_\delta(x)$ by

$$\varphi_{n,\delta}(x) = \frac{\mu_n(B_\delta(x))}{\lambda(B_\delta(x))}.$$

The following theorems state the weak and strong consistency of the empirical estimator defined by

$$\widehat{\alpha}_{n,x} = \frac{1}{\ln \tau} \ln \frac{\varphi_{n,\tau^2\delta_n}(x) - \varphi_{n,\tau\delta_n}(x)}{\varphi_{n,\tau\delta_n}(x) - \varphi_{n,\delta_n}(x)} \tag{5}$$

if $\left[\varphi_{n,\tau^2\delta_n}(x) - \varphi_{n,\tau\delta_n}(x)\right] / \left[\varphi_{n,\tau\delta_n}(x) - \varphi_{n,\delta_n}(x)\right] > 1$ and $\widehat{\alpha}_{n,x} = 0$ otherwise, (δ_n) being a sequence of positive numbers which will be assumed to tend to zero.

Theorem 3.1. (Weak consistency) Suppose that $x \in \mathbb{R}^d$ is a Lebesgue point of μ with regularity index α_x. Then, under the conditions

$$\lim_{n\to\infty} \delta_n = 0 \quad \text{and} \quad \lim_{n\to\infty} n\delta_n^{d+2\alpha_x} = \infty$$

the empirical estimator $\widehat{\alpha}_{n,x}$ converges to α_x in probability.

As is usually the case almost sure consistency is obtained under stronger conditions on the sequence (δ_n).

Theorem 3.2. (Strong consistency) Suppose that $x \in \mathbb{R}^d$ is a Lebesgue point of μ with regularity index α_x. Then, under the conditions

$$\lim_{n\to\infty} \delta_n = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{n\delta_n^{2(d+\alpha_x)}}{\ln n} = \infty$$

the empirical estimator $\widehat{\alpha}_{n,x}$ converges to α_x almost surely.

4. PROOFS

The weak (respectively strong) consistency of $\widehat{\alpha}_{n,x}$ is equivalent to the weak (resp. strong) consistency, for any $\tau > 0$, of the ratio

$$R_n(\delta_n) = \frac{\varphi_{n,\tau^2\delta_n}(x) - \varphi_{n,\tau\delta_n}(x)}{\varphi_{n,\tau\delta_n}(x) - \varphi_{n,\delta_n}(x)}$$

to τ^{α_x}. Let us fix $\tau > 0$ and set

$$S_n(\delta_n) = \frac{\varphi_{n,\tau\delta_n}(x) - \varphi_{n,\delta_n}(x)}{\varphi_{\tau\delta_n}(x) - \varphi_{\delta_n}(x)}.$$

We can write

$$\frac{\varphi_{n,\tau^2\delta_n}(x)-\varphi_{n,\tau\delta_n}(x)}{\varphi_{n,\tau\delta_n}(x)-\varphi_{n,\delta_n}(x)} = \frac{\varphi_{\tau^2\delta_n}(x)-\varphi_{\tau\delta_n}(x)}{\varphi_{\tau\delta_n}(x)-\varphi_{\delta_n}(x)}\,\frac{\varphi_{n,\tau^2\delta_n}(x)-\varphi_{n,\tau\delta_n}(x)}{\varphi_{\tau^2\delta_n}(x)-\varphi_{\tau\delta_n}(x)} \times \left(\frac{\varphi_{n,\tau\delta_n}(x)-\varphi_{n,\delta_n}(x)}{\varphi_{\tau\delta_n}(x)-\varphi_{\delta_n}(x)}\right)^{-1}$$

or equivalently

$$R_n(\delta_n) = \frac{\varphi_{\tau^2\delta_n}(x)-\varphi_{\tau\delta_n}(x)}{\varphi_{\tau\delta_n}(x)-\varphi_{\delta_n}(x)}\;\frac{S_n(\tau\delta_n)}{S_n(\delta_n)}.$$

Let us first look at the variance of $S_n(\delta_n)$. For this let us write

$$S_n(\delta_n) = 1 + \frac{A_n(\tau\delta_n)-A_n(\delta_n)}{\Delta_n}$$

where

$$A_n(\delta_n) = \varphi_{n,\delta_n}(x) - \varphi_{\delta_n}(x) \qquad \text{and} \qquad \Delta_n = \varphi_{\tau\delta_n}(x) - \varphi_{\delta_n}(x).$$

The following lemma gives the asymptotic variance of $A_n(\delta_n)$, the asymptotic covariance of $(A_n(\tau\delta_n), A_n(\delta_n))$ and the asymptotic variance of $S_n(\delta_n)$.

Lemma 4.1. Suppose that $x \in \mathbb{R}^d$ is a Lebesgue point of μ with regularity index α_x. Then, under the condition

$$\lim_{n\to\infty} \delta_n = 0$$

we have

$$\lim_{n\longrightarrow\infty} n\ \delta_n^d\ E\left[(A_n(\delta_n))^2\right] = \frac{\ell(x)}{V_d},$$

$$\lim_{n\longrightarrow\infty} n\ \delta_n^d\ E\left[A_n(\tau\delta_n)A_n(\delta_n)\right] = \frac{\ell(x)}{\tau^d V_d}$$

and

$$\lim_{n\longrightarrow\infty} n\ \delta_n^{d+2\alpha_x} E\left[(S_n(\delta_n)-1)^2\right] = \frac{\ell(x)\left(\tau^d-1\right)}{\tau^d V_d\ C_x^2(\tau^{\alpha_x}-1)^2}.$$

Proof of Lemma 4.1. First note that Equation (3) implies that

$$\Delta_n = \varphi_{\tau\delta_n}(x) - \varphi_{\delta_n}(x) = C_x\delta_n^{\alpha_x}(\tau^{\alpha_x}-1) + o(\delta_n^{\alpha_x})$$

and

$$\lim_{n\longrightarrow\infty} \frac{\delta_n^{2\alpha_x}}{\Delta_n^2} = \frac{1}{C_x^2(\tau^{\alpha_x}-1)^2}.$$

Now, using the fact that the random variable $n\mu_n(B_{\delta_n}(x))$ has the binomial distribution $\mathcal{B}(n, \mu(B_{\delta_n}(x)))$ we get

$$\begin{aligned} n\ \delta_n^d\ E\left[(A_n(\delta_n))^2\right] &= \frac{\delta_n^d\ \mu(B_{\delta_n}(x))(1-\mu(B_{\delta_n}(x)))}{\left[\lambda(B_{\delta_n}(x))\right]^2} \\ &= \left[1-\mu(B_{\delta_n}(x))\right]\frac{\mu(B_{\delta_n}(x))}{\lambda(B_{\delta_n}(x))}\frac{1}{V_d} \end{aligned}$$

which gives the asymptotic variance of $A_n(\delta_n)$.

As $B_{\delta_n}(x) \subset B_{\tau\delta_n}(x)$ the covariance

$$E\left([\mu_n(B_{\tau\delta_n}(x)) - \mu(B_{\tau\delta_n}(x))]\,[\mu_n(B_{\delta_n}(x)) - \mu(B_{\delta_n}(x))]\right)$$

is equal to

$$\frac{1}{n}\,(1 - \mu(B_{\tau\delta_n}(x)))\;\mu(B_{\delta_n}(x))$$

and therefore

$$\begin{aligned} n\;\delta_n^d\;E\left[A_n(\tau\delta_n)A_n(\delta_n)\right] &= \frac{\delta_n^d\;(1-\mu(B_{\tau\delta_n}(x)))\;\mu(B_{\delta_n}(x))}{\lambda(B_{\tau\delta_n}(x))\;\lambda(B_{\delta_n}(x))} \\ &= \left[1 - \mu(B_{\tau\delta_n}(x))\right]\frac{\mu(B_{\delta_n}(x))}{\lambda(B_{\delta_n}(x))}\frac{1}{\tau^d V_d} \end{aligned}$$

which gives the asymptotic covariance of $(A_n(\tau\delta_n), A_n(\delta_n))$.
Changing δ_n into $\tau\delta_n$ as argument of $A_n(.)$ gives

$$\lim_{n\longrightarrow\infty} n\;(\tau\delta_n)^d\;E\left[(A_n(\tau\delta_n))^2\right] = \frac{\ell(x)}{V_d}.$$

Gathering the above results one gets

$$\begin{aligned} \lim_{n\longrightarrow\infty} n\;\delta_n^{d+2\alpha_x} E\left[(S_n(\delta_n)-1)^2\right] &= \lim_{n\longrightarrow\infty} n\;\delta_n^{d+2\alpha_x} E\left[\left(\frac{A_n(\tau\delta_n) - A_n(\delta_n)}{\Delta_n}\right)^2\right] \\ &= \frac{\ell(x)\left(\tau^d - 1\right)}{\tau^d V_d\;C_x^2(\tau^{\alpha_x}-1)^2}. \end{aligned}$$

This ends the proof of the lemma. □

Remark. Note that under the assumptions

$$\lim_{n\to\infty} \delta_n = 0 \quad \text{and} \quad \ell(x) > 0$$

the condition

$$\lim_{n\to\infty} n\delta_n^{d+2\alpha_x} = \infty$$

is not only sufficient but also necessary for the L_2 convergence of $(S_n(\delta_n))$.

Proof of Theorem 3.1. Under the conditions of Theorem 3.1, $(S_n(\delta_n))$ and $(S_n(\tau\delta_n))$ converge to the constant 1 in the L_2 sense and therefore also in probability. Thus, their ratio tends to 1 in probability. By Theorem 2.1 $(R_n(\delta_n))$ tends to τ^{α_x} in probability. This ends the proof. □

P r o o f o f T h e o r e m 3.2. As already said the conclusion of Theorem 3.2, is equivalent to the following property: For any $\tau > 1$,

$$\lim_{n\to\infty} \frac{\varphi_{n,\tau^2\delta_n}(x) - \varphi_{n,\tau\delta_n}(x)}{\varphi_{n,\tau\delta_n}(x) - \varphi_{n,\delta_n}(x)} = \tau^{\alpha_x} \qquad \text{almost surely.}$$

From Hoeffding's inequality (see [4]) for a binomial distribution we have

$$\forall t > 0, \qquad \mathbb{P}\big(|\mu_n(B_{\delta_n}(x)) - \mu(B_{\delta_n}(x))| \geq t\big) \leq 2\exp\left(-2nt^2\right).$$

Taking

$$\varepsilon > 0 \qquad \text{and} \qquad t = \varepsilon\lambda(B_{\delta_n}(x))\,|\Delta_n|\,,$$

we get,

$$\forall \varepsilon > 0, \qquad \mathbb{P}\left(\left|\frac{A_n(\delta_n)}{\Delta_n}\right| \geq \varepsilon\right) \leq 2\exp\left(-2n\left[\varepsilon\lambda(B_{\delta_n}(x))\Delta_n\right]^2\right).$$

By Borel–Cantelli lemma, we have the convergence

$$\frac{A_n(\delta_n)}{\Delta_n} \longrightarrow 0 \qquad \text{almost completely}$$

if

$$\forall \varepsilon > 0, \qquad \sum_{n=1}^{\infty} \exp\left(-2n\left[\varepsilon\lambda(B_{\delta_n}(x))\Delta_n\right]^2\right) < \infty. \tag{6}$$

Now, set

$$\gamma_n = \frac{V_d^2\ \Delta_n^2}{\delta_n^{2\alpha_x}}.$$

As we have from Equation (3)

$$\Delta_n^2 = \delta_n^{2\alpha_x}(C_x^2\ (\tau^{\alpha_x} - 1)^2 + o(1))$$

we have

$$\gamma_n = V_d^2\ (C_x^2\ (\tau^{\alpha_x} - 1)^2 + o(1))$$

and the summand in Condition (6) writes

$$\exp\left[-n\delta_n^{2(d+\alpha_x)}\gamma_n\ \varepsilon^2\right] = \exp\left[-\frac{n\delta_n^{2(d+\alpha_x)}}{\ln n}\gamma_n\ \varepsilon^2 \ln n\right] = \frac{1}{n^{u_n}}$$

with (γ_n) tending to

$$\gamma = V_d^2\ C_x^2\ (\tau^{\alpha_x} - 1)^2 > 0$$

as n tends to infinity and

$$u_n = \frac{n\delta_n^{2(d+\alpha_x)}}{\ln n}\gamma_n\ \varepsilon^2.$$

The condition imposed on the sequence $(n\delta_n^{2(d+\alpha_x)}/\ln n)$ implies that for any $\varepsilon > 0$, the sequence (u_n) tends to infinity and therefore Condition (6) is satisfied. Thus

$(A_n(\delta_n)/\Delta_n)$ converges to 0 almost completely. In the same way one proves that $(A_n(\tau\delta_n)/\Delta_n)$ converges to 0 almost completely. It follows that

$$S_n(\delta_n) = 1 + \frac{A_n(\tau\delta_n) - A_n(\delta_n)}{\Delta_n}$$

and $(S_n(\tau\delta_n))$ converge to 1 almost completely. Finally, using Theorem 2.1, we get the conclusion that $\widehat{\alpha}_{n,x}$ converges to α_x almost surely. □

5. EXAMPLES

5.1. An example with an absolutely continuous measure

First consider the measure μ, absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, with density

$$f(x) = 1 - \frac{\sqrt{2}}{3} + \sqrt{|x|}\ \mathbf{1}_{(-1/2,1/2)}(x).$$

The distribution function F of μ is given by

$$F(x) = \begin{cases} 0 & \text{if} \quad x \leq -1/2, \\ (1/2) + (1-\sqrt{2}/3)\ x + (2/3)\, x\sqrt{|x|} & \text{if} \quad -1/2 \leq x \leq 1/2, \\ 1 & \text{if} \quad x \geq 1/2. \end{cases}$$

For $x \in (0, 1/2)$ and $\delta > 0$, δ small enough, one has

$$F(x+\delta) - F(x-\delta) = 2\delta(1-\sqrt{2}/3) + (2/3)\ \left[(x+\delta)^{(3/2)} - (x-\delta)^{(3/2)}\right].$$

For $x \in (-1/2, 0)$ and $\delta > 0$, δ small enough, one has

$$\begin{aligned} F(x+\delta) - F(x-\delta) &= 2\delta(1-\sqrt{2}/3) + (2/3)\ \left[-(-x-\delta)^{(3/2)} + (-x+\delta)^{(3/2)}\right] \\ &= 2\delta(1-\sqrt{2}/3) + (2/3)\ \left[(-x+\delta)^{(3/2)} - (-x-\delta)^{(3/2)}\right]. \end{aligned}$$

Now, for $0 < |u| < 1$,

$$(1+u)^{(3/2)} = 1 + \frac{3}{2}u + \frac{3}{8}u^2 - \frac{1}{16}u^3 + \frac{3}{128}u^4 + o(u^4),$$

hence

$$(1+u)^{(3/2)} - (1-u)^{(3/2)} = 3u - \frac{1}{8}u^3 + o(u^4)$$

and, for $x \neq 0$,

$$\begin{aligned} (x+\delta)^{(3/2)} - (x-\delta)^{(3/2)} &= x^{(3/2)} \left[\left(1+\frac{\delta}{x}\right)^{(3/2)} - \left(1-\frac{\delta}{x}\right)^{(3/2)}\right] \\ &= x^{(3/2)} \left[3\frac{\delta}{x} - \frac{1}{8}\left(\frac{\delta}{x}\right)^3 + o(\delta^4)\right]. \end{aligned}$$

If $x > 0$,

$$\frac{F(x+\delta)-F(x-\delta)}{2\delta} = (1-\sqrt{2}/3) + x^{(3/2)}\left[\frac{1}{x} - \frac{1}{24}\frac{1}{x^3}\delta^2\right] + o(\delta^3)$$

$$\frac{F(x+\delta)-F(x-\delta)}{2\delta} = (1-\sqrt{2}/3) + \sqrt{x} - \frac{1}{24}\frac{1}{x^{(3/2)}}\delta^2 + o(\delta^3).$$

If $x < 0$,

$$\frac{F(x+\delta)-F(x-\delta)}{2\delta} = (1-\sqrt{2}/3) + (-x)^{(3/2)}\left[\frac{1}{-x} - \frac{1}{24}\frac{1}{(-x)^3}\delta^2\right] + o(\delta^3)$$

$$\frac{F(x+\delta)-F(x-\delta)}{2\delta} = (1-\sqrt{2}/3) + \sqrt{-x} - \frac{1}{24}\frac{1}{(-x)^{(3/2)}}\delta^2 + o(\delta^3).$$

Thus, for $x \neq 0$,

$$\frac{F(x+\delta)-F(x-\delta)}{2\delta} - f(x) = -\frac{1}{24}\frac{1}{|x|^{(3/2)}}\delta^2 + o(\delta^3).$$

This implies Equation (3) with

$$\alpha_x = 2 \qquad \text{and} \qquad C_x = -\frac{1}{24}\frac{1}{|x|^{(3/2)}}.$$

Note that

$$\lim_{x \longrightarrow 0} C_x = -\infty.$$

At the point $x = 0$ one uses the fact that for $\delta \in (0,1)$ one has

$$\frac{F(\delta)-F(-\delta)}{2\delta} - f(0) = \frac{2}{3}\sqrt{\delta}$$

to conclude that Equation (3) holds with

$$\alpha_0 = \frac{1}{2} \qquad \text{and} \qquad C_0 = \frac{2}{3}.$$

5.2. An example with a measure having a discrete part

Now, consider the discrete probability measure ν supported by the sequence

$$x_i = \frac{1}{i}, \qquad i \in \mathbb{N}^*,$$

with masses

$$\nu(\{x_i\}) = \frac{1}{i(i+1)}.$$

For any $\delta \in (0,1)$, there exists a unique positive integer $k(\delta)$ such that

$$\frac{1}{k(\delta)+1} \leq \delta < \frac{1}{k(\delta)} \tag{7}$$

and we have

$$\nu\left[B(0,\delta)\right] = \sum_{i>k(\delta)} \frac{1}{i(i+1)} = \frac{1}{k(\delta)+1}$$

and

$$\frac{\nu\left[B(0,\delta)\right]}{\lambda\left[B(0,\delta)\right]} = \frac{1}{2\delta\left(k(\delta)+1\right)}.$$

From the definition of $k(\delta)$ it follows that

$$\lim_{\delta \longrightarrow 0^+} k(\delta) = \infty \qquad \text{and} \qquad \lim_{\delta \longrightarrow 0^+} \delta k(\delta) = 1^-$$

thus

$$\lim_{\delta \longrightarrow 0^+} \frac{\nu\left[B(0,\delta)\right]}{\lambda\left[B(0,\delta)\right]} = \frac{1}{2}.$$

Now,

$$\frac{\nu\left[B(0,\delta)\right]}{\lambda\left[B(0,\delta)\right]} - \frac{1}{2} = \frac{1-\delta\left(k(\delta)+1\right)}{2\delta\left(k(\delta)+1\right)}$$

and, from (7), we have

$$-\frac{1}{k(\delta)} < 1-\delta\left(k(\delta)+1\right) \leq 0$$

and therefore

$$-\frac{\delta^{1-\alpha}}{\delta k(\delta)} < \frac{1-\delta\left(k(\delta)+1\right)}{\delta^\alpha} \leq 0, \qquad \alpha \in (0,1).$$

Finally, gathering the above results, one gets

$$\forall \alpha \in (0,1), \qquad \frac{\nu\left[B(0,\delta)\right]}{\lambda\left[B(0,\delta)\right]} = \frac{1}{2} + o(\delta^\alpha).$$

The probability measure $\eta = (\mu+\nu)/2$ satisfies

$$\frac{\eta\left[B(0,\delta)\right]}{\lambda\left[B(0,\delta)\right]} = \frac{3}{4} - \frac{\sqrt{2}}{6} + \frac{1}{3}\delta^{1/2} + o(\delta^{1/2}).$$

So, the measure η, which is clearly not absolutely continuous, satisfies Equation (3) at the point $x = 0$ with

$$l(0) = \frac{9-2\sqrt{2}}{12}, \qquad C_0 = \frac{1}{3} \qquad \text{and} \qquad \alpha_0 = \frac{1}{2}.$$

ACKNOWLEDGEMENT

We wish to thank the referees for their careful reading of the paper and helpful comments.

(Received January 24, 2011)

REFERENCES

[1] J. Beirlant, A. Berlinet, and G. Biau: Higher order estimation at Lebesgue points. Ann. Inst. Statist. Math. *60* (2008), 651–677.

[2] A. Berlinet and S. Levallois: Higher order analysis at Lebesgue points. In: G. G. Roussas Festschrift – Asymptotics in Statistics and Probability (M. L. Puri, ed.), 2000, pp. 17–32.

[3] A. Berlinet and R. Servien: Necessary and sufficient condition for the existence of a limit distribution of the nearest neighbour density estimator. J. Nonparametr. Statist. *23* (2011), 633–643.

[4] L. Devroye and G. Lugosi: Combinatorial Methods in Density Estimation. Springer, New York 2001.

[5] R. M. Dudley: Real Analysis and Probability. Chapman and Hall, New York 1989.

[6] W. Rudin: Real and Complex Analysis. McGraw-Hill, New York 1987.

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