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A CONSTRUCTION OF LARGE GRAPHS OF DIAMETER TWO AND GIVEN DEGREE FROM ABELIAN LIFTS OF DIPOLES

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For any $d \geq 11$ we construct graphs of degree d, diameter 2, and order $\frac{8}{25}d^2 + O(d)$, obtained as lifts of dipoles with voltages in cyclic groups. For Cayley Abelian graphs of diameter two a slightly better result of $\frac{9}{25}d^2 + O(d)$ has been known [3] but it applies only to special values of degrees d depending on prime powers.

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1. INTRODUCTION

Two types of restrictions that appear frequently in the design of large interconnection networks are limitations on the number of links emanating from a node and on the length of the shortest path between a pair of nodes. If networks are modeled by undirected graphs, the two requirements lead to design of large graphs of a given maximum degree and a given diameter. The search for *largest* such graphs is known as the *degree-diameter problem.* Since we will be interested only in the case of diameter 2, we just mention that by the Moore bound [5] the largest order (i. e., number of vertices) of a graph of diameter 2 and maximum degree d is d^2+1 and that graphs of such an order exist only for degrees $d = 2, 3, 7$ and possibly 57.

In the past decades a number of techniques for constructing large graphs of a given degree and diameter have been developed. A fruitful method appears to be lifting graphs of a small order to comparatively large graphs by means of voltage assignments in finite groups; if the groups are Abelian one speaks about *Abelian lifts.* To avoid repetitiousness we refer to the basics of the method of lifting to [5] and references therein. In particular, Abelian lifts of dipoles (graphs of order 2) gave rise to the largest vertex-transitive and almost vertex-transitive graphs of diameter 2 and a given degree $d = (3q \pm 1)/2$, q an odd prime power, whose order is $\frac{8}{9}d^2 + O(d)$, cf. [4, 8]. This led to interest in largest possible Abelian lifts of graphs of order 1 (equivalently, Cayley graphs of Abelian groups) and 2. From [7] it follows that the largest order of a graph of diameter 2 and degree d obtained as an Abelian lift of a dipole is $\leq 0.932d^2 + O(d)$. In the other direction, constructions of [3] furnish Cayley graphs of degree d and diameter 2 on Abelian groups of order $\frac{1}{3}(d+1)^2$ if $d = 3q-1$ and $\frac{3}{8}(d^2-4)$ if $d = 4q-2$, where in both cases q is an odd prime power. Moreover, in [3] the authors gave a construction of a Cayley graph of diameter 2 and degree $d = 5p-3$, where p is a prime congruent to 2 mod 3, on a cyclic group of order $\frac{9}{25}d^2 + O(d)$.

In this note we offer a construction of graphs of degree d, diameter 2, and order $\frac{8}{25}d^2 + O(d)$, obtained as lifts of dipoles with voltages in cyclic groups. This is slightly worse than the aforementioned result of [3] but has the advantage that the construction works for *general* degrees $d \geq 11$.

2. RESULTS

Our graphs will be always finite but may have loops and parallel (that is, multiple) edges. By $D_{r,s}$ we denote a dipole, that is, a graph consisting of exactly two vertices joined by r parallel edges and having s loops at each vertex. Such a dipole is a regular graph of degree $d = r + 2s$; with unspecified r and s we just speak about a dipole D of degree d.

We are now ready to present and prove our results.

Theorem 2.1. For any $d \geq 11$ there exists a graph of order $\frac{8}{25}d^2 + O(d)$, degree d, and diameter 2, arising as a lift of a dipole with voltages in a cyclic group.

Proof. Because of the nature of the statement it is sufficient to prove it for all sufficiently large d and we will do so for all $d \geq 11$. We begin with degrees $d \equiv 1 \bmod 10$, that is, we let $d = 10\ell + 1$ where $\ell \geq 1$. For $r = 8\ell + 1$ and $s = \ell$, consider the dipole $D = D_{r,s}$ as introduced before, of degree $d = r + 2s = 10\ell + 1$ and with vertices u and v. Further, let $G = \mathbb{Z}_n$ be the cyclic group of order $n = 16\ell^2 + 8\ell = \frac{4}{25}d^2 + O(d)$. On the dipole D we introduce a voltage assignment α in G as follows. Letting $k = 4\ell + 1$, the $r = 2k - 1$ darts from u to v will be mapped bijectively by α onto the set $A = \{0, -1, -2, \ldots, -k+1, k, 2k, \ldots, (k-1)k\}$, and the set of all the $2\ell = (k-1)/2$ loops at both u and v are mapped bijectively by α onto the set $B = \{1, 2, 3, \ldots, (k-1)/2\}$. The lift D^α has $2n = 2(k^2 - 1) = \frac{8}{25}d^2 + O(d)$ vertices and has degree d.

We proceed by showing that the lift D^α has diameter 2. It suffices to show that for any $g \in \mathbb{Z}_n$ there exists a walk W in D of length at most two starting and ending at any of the two vertices u, v of D and such that $\alpha(W) = g$. First we examine the $u \to v$ walks. If $g = kt \in A$ for some t such that $0 \leq t \leq k-1$, then W consists of the dart from u to v carrying the voltage $kt \in A$. For $g = ik + j$, where $i \in \{0, 1, 2, \ldots, k-1\}$ and $j \in B \cup -B$, we can take W of length 2 composed of the dart from u to v with voltage ik and a suitable loop at u or at v carrying the voltage j. Considering $u \to u$ walks, for $g \in A \cup -A$ the walk W consists of the dart from u to v with voltage g followed by the v to u dart with voltage 0. If $g = ik + h$, where $i, h \in \{1, 2, \ldots k-1\}$, then we choose W consisting of the $u \to v$ dart with voltage ik and the $v \to u$ dart with voltage h. The cases of $v \to v$ and $v \to u$ walks can be dealt with in a similar way. This implies that the lift D^α has diameter two.

We have thus proved the statement for all $d \geq 11$ such that $d \equiv 1 \bmod 10$. For the remaining $d = 10\ell + 1 + \delta$, where $\ell \geq 1$ and $1 \leq \delta \leq 9$ we modify the dipole D by inserting extra $\lfloor \delta/2 \rfloor$ loops at both u and v that carry arbitrary distinct voltages in the set $\{2\ell+1, \ldots, 2\ell+\lfloor \delta/2 \rfloor\} \subset Z_n$; if δ is odd we also insert an extra dart from u to v carrying the voltage $1 \in Z_n$. By the above argument, the lift will have diameter 2, degree d, and order $\frac{8}{25}d^2 + O(d)$. □

The natural question of possible vertex-transitivity of the graphs constructed above is answered in the negative by our next result.

Theorem 2.2. The graphs constructed in the proof of Theorem 2.1 are not vertex-transitive if $d \geq 21$.

P r o o f. We keep to the notation introduced in the proof of Theorem 2.1. Let $F_u = \{u_i;\ i \in Z_n\}$ and $F_v = \{v_i;\ i \in Z_n\}$ be the fibres above u and v, respectively, in the covering $D^\alpha \to D$ induced by the voltage assignment α in Z_n. Since k is relatively prime to $n = k^2 - 1$, the element $k \in Z_n$ has order n. Let $k_0 = k(k-1)/2$ and $k_1 = k(k+1)/2$ be elements of Z_n. If $k \geq 9$, which is the case if $d \geq 21$, the dart of D from u to v that carries the voltage k_0 is contained in no walk of length 3 of zero voltage, and the same is true for the dart from u to v of voltage k_1. (The condition $k \geq 9$ is needed because of the additional loops in the construction for $d \not\equiv 1 \bmod 10$.) It follows that no edge of the form $u_i v_{i+m}$ for $m \in \{k_0, k_1\}$ in the lift D^α lies in a triangle for any $i \in Z_n$. But as $k_1 - k_0 = k$, the cycle C of the form

$$u_0 \to v_{k_1} \to u_k \to v_{k+k_1} \ldots \to u_{jk} \to v_{jk+k_1} \to u_{(j+1)k} \to v_{(j+1)k+k_1} \to \cdots$$

is a Hamilton cycle of D^α consisting of edges belonging to no triangle. Note also that every edge of D^α with both ends in F_u lies in a triangle, with a similar conclusion for any edge with both ends in F_v.

Suppose now that D^α was a vertex-transitive graph and let f be an automorphism that takes a vertex from F_u onto a vertex from F_v. Since $f(C)$ is a Hamilton cycle again, with edges contained in no triangles, it follows that f must interchange the sets F_u and F_v. In other words, the fibres F_u and F_v form a block system for the automorphism group of D^α. By the construction of D^α it is obvious that any edge of D^α that is a lift of a loop lies in a triangle containing vertices from both fibres, and such an edge lies in a largest number of such triangles if and only if the edge is a lift of the loop carrying the voltage 1. But such edges are either all in F_u or all in F_v. Consequently, no automorphism f as above exists, and we conclude that D^α is not a vertex-transitive graph. □

Let us remark that there is a lot of flexibility regarding the voltage assignment α in the proof of Theorem 2.1. It might be possible that a better choice of a voltage assignment could give vertex-transitive graphs but we have not been able to identify such assignments for general degrees d, and not even for small d by computer [1].

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