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EVALUATING MANY VALUED MODUS PONENS

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This paper deals with many valued case of modus ponens. Cases with implicative and with clausal rules are studied. Many valued modus ponens via discrete connectives is studied with implicative rules as well as with clausal rules. Some properties of discrete modus ponens operator are given.

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1. INTRODUCTION

The aim of this paper is to design a sound and complete deduction in knowledge systems where uncertainty, vagueness and preference is modeled by many valued logic with arbitrary connectives (possibly obtained by an inductive procedure, see e. g. [5]).

In many systems, domain (background) knowledge is modeled using IF-THEN rules (Prolog/Datalog rule based systems). From the very beginning we face a problem. In two valued logic,

$$\boldsymbol{B} \longrightarrow \boldsymbol{H} \equiv \neg \boldsymbol{B} \vee \boldsymbol{H}$$

is a tautology. This need not be true in many valued logic. As far as our main concern is to make modeling as much realistic to real world data as possible, we do not make any restriction here. Instead we study both possibilities separately and compare them.

$$\frac{(\boldsymbol{B}, b), (\boldsymbol{B} \rightarrow \boldsymbol{H}, r)}{\boldsymbol{H}, f_{\rightarrow}(b, r)}, \qquad \frac{(\boldsymbol{B}, b), (\neg \boldsymbol{B} \vee \boldsymbol{H}, r)}{\boldsymbol{H}, g_{\vee\neg}(b, r)}.$$

We give a formula for the evaluation of $f_{\rightarrow}$, for evaluation of modus ponens with implicative rules, and of $g_{\vee\neg}$ for evaluation of modus ponens with clausal rules.

We build on works [8, 12]; in [12] there is an estimate of full resolution and in [8] there is an estimate of modus ponens for implicative rules.

We deal with multivalued (MV for short) logical connectives. Note that connectives in MV-logic with truth values range $[0, 1]$ are monotone extensions of the classical connectives. We recall notation and basic definitions used in the paper. We start with the basic logic connectives.

Definition 1.1. (see e. g. in Fodor and Roubens [2]) A function $N : [0, 1] \rightarrow [0, 1]$ is called a *fuzzy negator* if for each $a, b \in [0, 1]$ it satisfies the following conditions

- (i) $a < b \Rightarrow N(b) \leq N(a)$,
- (ii) $N(0) = 1, N(1) = 0$.

Remark 1.2. A *dual negator* $N^d : [0,1] \to [0,1]$ based on a negator N, is given by $N^d(x) = 1 - N(1-x)$. A fuzzy negator N is called *strict* if N is strictly decreasing and continuous for arbitrary $x, y \in [0,1]$. In classical logic we have that $(\boldsymbol{A}')' = \boldsymbol{A}$. In multivalued logic this equality is not satisfied for each negator. The negators with this equality are called *involutive negators.* The strict negator is *strong* if and only if it is involutive.

Some examples of strict and/or strong negators are included in the following example.

Example 1.3. The next functions are negators on $[0,1]$.

• $N_s(a) = 1 - a$	strong negator, standard negator;
• $N(a) = 1 - a^2$	strict, but not strong negator;
• $N(a) = \sqrt{1 - a^2}$	strong negator;
• $N_{G_1}(1) = 0, N_{G_1}(a) = 1$ if $a < 1$	non-continuous negator, the greatest fuzzy negator, dual Gödel negator;
• $N_{G_2}(0) = 1, N_{G_2}(a) = 0$ if $a > 0$	non-continuous negator, the least fuzzy negator, Gödel negator.

Remark 1.4. In this contribution we deal with *the standard negator* N_s which is a commonly used negator in applications.

Definition 1.5. ([7]) A non-decreasing mapping $C : [0,1]^2 \to [0,1]$ is called a *conjunctor* if

1. $C(x, y) = 0$ whenever $x = 0$ or $y = 0$, and
2. $C(1, 1) = 1$.

Remark 1.6. Note that the dual operator to a conjunctor C, defined by $D(x, y) = 1 - C(1-x, 1-y)$, is called a *disjunctor*. Equivalently, a disjunctor can be defined as a non-decreasing mapping $D : [0,1]^2 \to [0,1]$ such that $D(x, y) = 1$ whenever $x = 1$ or $y = 1$, and $D(0, 0) = 0$.

Commonly used conjunctors (disjunctors) in MV-logic are the triangular norms (conorms).

Definition 1.7. A *triangular norm* (*t-norm* for short) is a binary operation on the unit interval $[0,1]$, i.e., a function $T : [0,1]^2 \to [0,1]$ such that for all $x, y, z \in [0,1]$ the following four axioms are satisfied:

(T1) *Commutativity* $T(x, y) = T(y, x)$,

(T2) *Associativity* $T(x, T(y, z)) = T(T(x, y), z)$, (T3) *Monotonicity* $T(x,y) \le T(x,z)$ whenever $y \le z$,

(T4) *Boundary Condition* $T(x,1) = x$.

Example 1.8. The next operators T_M, T_P, T_L are the basic continuous t-norms:

(i) Minimum t-norm $T_M(x,y) = \min(x,y)$.

(ii) Product t-norm $T_P(x,y) = x.y$.

(iii) Łukasiewicz t-norm $T_L(x,y) = \max(0, x+y-1)$.

The dual operator to the t-norm T is the triangular conorm (t-conorm) $S : [0,1]^2 \to [0,1]$, which is given by

$$S(x,y) = 1 - T(1-x, 1-y).$$

Remark 1.9. Note that the corresponding t-conorms to the basic continuous t-norms T_M, T_P, T_L are denoted by S_M, S_P and S_L.

In this contribution we often use t-seminorms C and t-semiconorms D as the truth functions for conjunctions and disjunctions.

Definition 1.10. (Schweizer and Sklar [13])

(i) A *t-seminorm* C is a conjunctor that satisfied the boundary condition

$$C(1,x) = C(x,1) = x \text{ for all } x \in [0,1].$$

(ii) A *t-semiconorm* D is a disjunctor that satisfied the boundary condition

$$D(0,x) = D(x,0) = x \text{ for all } x \in [0,1].$$

In the literature, one can find several different definitions of fuzzy implicators. In this paper we will use the following one, which is equivalent to the definition introduced by Fodor and Roubens in [2].

Definition 1.11. A function $I : [0,1]^2 \to [0,1]$ is called a *fuzzy implicator* if it satisfies the following conditions:

(I1) I is non-increasing in its first variable,

(I2) I is non-decreasing in its second variable,

(I3) $I(1,0) = 0$, $I(0,0) = I(1,1) = 1$.

2. THE AGGREGATION DEFICITS AND FULL RESOLUTION TRUTH FUNCTION

In Pavelka's language of evaluated expressions, we would like to achieve the following: from $(\boldsymbol{C} \vee_D \boldsymbol{A}, x)$ and $(\boldsymbol{B} \vee_D \neg \boldsymbol{A}, y)$ to infer $(\boldsymbol{C} \vee_D \boldsymbol{B}, f_{\vee_D}(x,y))$ where $f_{\vee_D}(x,y)$ should be the best promise, we can give the truth function of disjunction $\vee_D$ and x and y.

First, we introduce a new operator, called an *aggregation deficit R_D, which is based on a disjunctor D.* We recall its definition and important properties; their proofs can be found in [12]. The motivation is following. Assume the truth value $TV(\boldsymbol{A}) = a$. We would like to know conditions on truth values $TV(\boldsymbol{B}) = b$ and $TV(\boldsymbol{C}) = c$ such that they aggregate together with a or $1-a$ to have $D(c,a) \geq x$ and $D(b, 1-a) \geq y$. In order to obtain this aggregation deficit, R_D is defined by the next inequalities:

$$x \leq D(c,a) \qquad \text{and} \qquad y \leq D(b, 1-a).$$

$$c \geq R_D(a,x) \qquad \text{and} \qquad b \geq R_D(1-a, y).$$

This leads naturally to the following definition.

Definition 2.1. (Smutná-Hliněná and Vojtáš [12]) Let D be a disjunctor. The aggregation deficit is defined by

$$R_D(x,y) = \inf\{z \in [0,1]; D(z,x) \geq y\}.$$

Example 2.2. (Smutná-Hliněná and Vojtáš [12]) For the basic t-conorms S_M, S_P and S_L we obtain the following aggregation deficits:

$$R_{S_M}(x,y) = \begin{cases} 0 & \text{if } x \geq y, \\ y & \text{otherwise,} \end{cases} \qquad R_{S_P}(x,y) = \begin{cases} 0 & \text{if } x \geq y, \\ \frac{y-x}{1-x} & \text{otherwise,} \end{cases}$$

$$R_{S_L}(x,y) = \begin{cases} 0 & \text{if } x \geq y, \\ y-x & \text{otherwise.} \end{cases}$$

Remark 2.3. Note that one easily verifies the hybrid monotonicity of the aggregation deficit R_D. Let D_1 and D_2 be the disjunctors such that $\forall x, y \in [0,1]; D_1(x,y) \leq D_2(x,y)$.

Then $R_{D_1}(x,y) \geq R_{D_2}(x,y)$ for every x, y. This follows from the fact that the aggregation deficit R_D is non-increasing in its first argument.

Let $D : [0,1]^2 \to [0,1]$ be a t-semiconorm. Then $R_D(x,y) \leq y$ for $(x,y) \in [0,1]^2$. If $x \geq y$, then $R_D(x,y) = 0$. It means, that for any aggregation deficit R_D it holds that $R_D \leq R_{S_M}$. More, if the partial mappings of disjunctor D are infimum-morphism ($\inf_{a \in M} D(x,a) = D(x, \inf_{a \in M} a)$, where M is subset of interval $[0,1]$) then $x \geq y$ if and only if $R_D(x,y) = 0$. It follows from boundary condition and monotonicity of t-semiconorm D. Consider an aggregation deficit R_D, then the partial mapping $R_D(.,1)$ is negator on $[0,1]$. The aggregation deficit R_S of t-conorm S coincides with residual coimplicator J_S, which was introduced by Bernard De Baets in [1] for different purpose.

For the formulation of a result on sound and complete full resolution, Smutná - Hliněná and Vojtáš in [12] investigated the *resolution truth function* $f_{R_D} : [0,1]^2 \to [0,1]$, which is defined by

$$f_{R_D}(x,y) = \inf_{a \in [0,1]} \{D(R_D(a,x), R_D(1-a,y))\}.$$

Example 2.4. (Smutná-Hliněná and Vojtáš [12]) For the aggregation deficits R_{S_M}, R_{S_P} and R_{S_L}, which are corresponded with the basic t-conorms, we obtain the following functions:

$$f_{R_{S_M}}(x,y) = \begin{cases} 0 & \text{if } x+y \leq 1, \\ \min(x,y) & \text{otherwise,} \end{cases} \qquad f_{R_{S_P}}(x,y) = \begin{cases} 0 & \text{if } x+y \leq 1, \\ \frac{x+y-1}{\max(x,y)} & \text{otherwise,} \end{cases}$$

$$f_{R_{S_L}}(x,y) = \begin{cases} 0 & \text{if } x+y \leq 1, \\ x+y-1 & \text{otherwise.} \end{cases}$$

Theorem 2.5. (Smutná-Hliněná and Vojtáš [12]) Assume the truth evaluation of proposition variables is a model of $(\boldsymbol{C} \vee_D \boldsymbol{A}, x)$ and $(\boldsymbol{B} \vee_D \neg \boldsymbol{A}, y)$. Then

$$TV(\boldsymbol{C} \vee_D \boldsymbol{B}) \geq f_{R_D}(x,y).$$

3. MODUS PONENS FOR CLAUSE BASED RULES

For implicative rules, the following estimation of modus ponens is in [3] and [4]

$$\frac{(\boldsymbol{B}, b), (\boldsymbol{B} \to \boldsymbol{H}, r)}{\boldsymbol{H}, f_{\to}(b,r)}.$$

We know that the implication $(\boldsymbol{B} \to \boldsymbol{H})$ is true to degree r (at least). Therefore $\boldsymbol{H}$ must be true to some degree h such that $I(b,h) \geq r$. We need to find the least value h with this property in order to guarantee that $TV(\boldsymbol{H}) \geq h$. Let I be the truth function of implication $\to$, then truth function $f_{\to}$ is residual conjunctor of implicator I (note mnemonic body-head-rule notation of variables)

$$f_{\to}(b,r) = C_I(b,r) = \inf\{h \in [0,1]; I(b,h) \geq r\}.$$

To be consistent with body-head-rule notation of [8], we will use it also here for clausal rules.

Example 3.1. The following are the logical operators of material implicator which are corresponding to basic t-conorms: maximum S_M, probabilistic sum S_P, and Łukasiewicz t-conorm S_L and standard negator N_s.

$$I_{S_M}(b,h) = \max(1-b,h), \quad I_{S_P}(b,h) = 1-b+b \cdot h,$$

$$I_{S_L}(b,h) = \min(1-b+h, 1).$$

Note, that $I_S(b,h) = S(N(b),h)$, where N is a negator and S is a t-conorm. For an arbitrary disjunctor D and the standard negator N_s we get $I_D(b,h) = D(1-b,h)$.

First idea to mimic implicative rules, is to take residua to material implicators. The residual conjunctors of previous implicators are:

$$C_{I_{S_M}}(b,r) = \begin{cases} 0 & \text{if } b+r \le 1, \\ r & \text{otherwise}, \end{cases} \qquad C_{I_{S_P}}(b,r) = \begin{cases} 0 & \text{if } b+r \le 1, \\ \frac{b+r-1}{b} & \text{otherwise}, \end{cases}$$

$$C_{I_{S_L}}(b,r) = \max(0, b+r-1).$$

Note that all residua to material implicator in previous example are zero in the triangle $b + r \le 1$.

Another possibility is to calculate the lower bound on the truth value of $\boldsymbol{H}$ using aggregation deficit.

Example 3.2. To have a sound clause based modus ponens, we make following observation. Let $D : [0,1]^2 \to [0,1]$ be a commutative disjunctor. If for all $b, r \in [0,1]$

$$(\boldsymbol{B}, b) \text{ and } (\neg \boldsymbol{B} \vee_D \boldsymbol{H}, r) \text{ should imply } (\boldsymbol{H}, g_D(b,r)),$$

then using Theorem 2.2

$$r \le D(1-b,h) \implies r \le D(h, 1-b) \implies h \ge R_D(1-b,r).$$

Hence the best possible estimate for h is

$$g_D(b,r) = \inf_{b' \ge b} R_D(1-b',r).$$

Since the aggregation deficit R_D is non-increasing in the first argument, hence $\inf_{b' \ge b} R_D(1-b',r) = R_D(1-b,r)$, it means that

$$g_D(b,r) = R_D(1-b,r).$$

Remark 3.3. Note that the truth value of $\boldsymbol{H}$ depends on the truth functions of disjunction and negation. Therefore, on a very formal level, one would write $g_{\vee_D \neg_N}$. To make the notation shorter we omit the symbols of disjunction and negation, since it they do not bear any additional information. Because we deal only with the standard negator N_s in this article, symbol N is omitted as well. We thus use g_D.

For commutative disjunctors we get:

Theorem 3.4.

1. Let $D_1 \le D_2$, then $g_{D_1} \ge g_{D_2}$.
2. Let D be a t-semiconorm, then $g_D \le g_{S_M}$.
3. Function g_D is non-decreasing in both arguments.
4. Let $D : [0,1]^2 \to [0,1]$ be a commutative $t-$semiconorm. For function g_D we get $g_D(1,1) = 1, g_D(0,x) = g_D(x,0) = 0$. It means, the function g_D is the conjunctor.

P r o o f. The parts 1.−2. directly follow from Remark 2.3. The part 3. is implied from Remark 2.3 and from equality $g_D(b,r) = R_D(1-b,r)$. In the last part we deal with a commutative $t-$semiconorm D. For $t-$semiconorm we have $D(x,0) = x$, therefore we get:

$$g_D(1,1) = R_D(0,1) = \inf\{z \in [0,1]; D(z,0) \geq 1\} = 1.$$

Since $D(x,1) = 1$ we have:

$$g_D(0,x) = R_D(1,x) = \inf\{z \in [0,1]; D(z,1) \geq x\} = 0.$$

Since $D(x,y) \geq 0$ we get:

$$g_D(x,0) = R_D(1-x,0) = \inf\{z \in [0,1]; D(z,1-x) \geq 0\} = 0.$$

Since function g_D is non-decreasing in both arguments (part 3.), g_D is a conjunctor. □

Remark 3.5. If a commutative $t-$semiconorm D possesses the properties

$$D(x,y) = 1 \text{ for all } x,y \in [0,1], \text{ such that } x+y = 1$$

$$D(x,y) < 1 \text{ for all } x,y \in [0,1], \text{ such that } x+y < 1$$

then g_D is a $t-$seminorm. These properties guarantee that the boundary condition $g_D(x,1) = x$ is satisfied for all $x \in [0,1]$. The second boundary condition, $g_D(1,x) = x$, is satisfied for arbitrary commutative $t-$semiconorm D. Note that, for example, $t-$conorm S_L possesses these properties.

Estimation for clause rules and implicative rules are in some cases identical:

Theorem 3.6. Let $g_D : [0,1]^2 \to [0,1]$ be truth function based on R_D, where D is a commutative disjunctor and $C_I : [0,1]^2 \to [0,1]$ be a truth function based on I, where $I(b,h) = D(h,1-b)$. Then

$$C_I(b,r) = g_D(b,r)$$

for all $b,r \in [0,1]$.

P r o o f. Let D be a disjunctor and $I(b,h) = D(h,1-b)$. Equality $R_D(1-b,r) = C_I(b,r)$ follows directly from definitions of R_D and C_I. According to Example 3.2, $g_D(b,r) = R_D(1-b,r)$, and therefore also $g_D(b,r) = C_I(b,r)$. □

4. DISCRETE MANY VALUED MODUS PONENS

Assume users will evaluate preference on attributes X and Y with fuzzy or linguistic values x and y. In this part we will estimate modus ponens via discrete connectives. It is known ([15] and `http://en.wikipedia.org/wiki/Likert_scale`), that people are not able to sort according to quality to more than 7 ± 2 categories. In accordance with this fact we use coefficients k,l as follows:

$$k \in \{5,6,7,8,9\} \quad \text{and} \quad l \in \{5,6,7,8,9\}.$$

And for m (the number of roundings) we take $m = k * l$, which provides us with good ordering of results. The meaning of these coefficients will be come obvious in the next definition of a discrete conjunctor:

Definition 4.1. Let $C : [0,1]^2 \to [0,1]$ be a conjunctor, $k \in \{5,6,7,8,9\}, l \in \{5,6,7,8,9\}$ and $m = k * l$. Mapping $C^m_{k,l} : [0,1]^2 \to [0,1]$ which is defined as follows

$$C^m_{k,l}(x,y) = \frac{\left\lceil m \cdot C\left(\frac{\lceil k \cdot x \rceil}{k}, \frac{\lceil l \cdot y \rceil}{l}\right)\right\rceil}{m}$$

is called a *discrete conjunctor.*

Obviously this mapping is a conjunctor. However it is not a t-seminorm. Commutative or associative conjunctor C may lead to $C^m_{k,l}$ without these properties. Note, that if a conjunctor C is commutative, then the discrete conjunctor $C^m_{k,k}$ is commutative, too. Dual mapping to the discrete conjunctor is given by a similar equality.

Theorem 4.2. Let $C : [0,1]^2 \to [0,1]$ and $D : [0,1]^2 \to [0,1]$ be the dual conjunctor and disjunctor which are continuous, $k \in \{5,6,7,8,9\}, l \in \{5,6,7,8,9\}$ and $m = k * l$. Then the dual discrete disjunctor to $C^m_{k,l}$ is the mapping $D^m_{k,l} : [0,1]^2 \to [0,1]$ such that

$$D^m_{k,l}(x,y) = \frac{\left\lfloor m \cdot D\left(\frac{\lfloor k \cdot x \rfloor}{k}, \frac{\lfloor l \cdot y \rfloor}{l}\right)\right\rfloor}{m}. \tag{1}$$

P r o o f. The dual disjunctors to conjunctors C and $C^m_{k,l}$ are given by $D(x,y) = 1 - C(1-x, 1-y)$ and $D^m_{k,l}(x,y) = 1 - C^m_{k,l}(1-x, 1-y)$, respectively. For any $k \in \mathbb{N}$ and $t \in [0,1]$ it holds that $\lceil k - k \cdot t \rceil = k - \lfloor k \cdot t \rfloor$ and $k - \lceil k \cdot t \rceil = \lfloor k - k \cdot t \rfloor$. Using these two facts, the rest of the proof is straightforward:

$$D^m_{k,l}(x,y) = 1 - \frac{\left\lceil m \cdot C\left(\frac{\lceil k - k \cdot x \rceil}{k}, \frac{\lceil l - l \cdot y \rceil}{l}\right)\right\rceil}{m}$$

$$= \frac{\left\lfloor m - m \cdot C\left(1 - \frac{\lfloor k \cdot x \rfloor}{k}, 1 - \frac{\lfloor l \cdot y \rfloor}{l}\right)\right\rfloor}{m} = \frac{\left\lfloor m \cdot D\left(\frac{\lfloor k \cdot x \rfloor}{k}, \frac{\lfloor l \cdot y \rfloor}{l}\right)\right\rfloor}{m}.$$

□

For an illustration we introduce the following example:

Example 4.3. Let C be a product t-norm T_P. We, for example, calculate the value $C^{25}_{5,5}(\frac{1}{3}, \frac{2}{3})$:

$$C^{25}_{5,5}(\frac{1}{3}, \frac{2}{3}) = \frac{\left\lceil 25 \cdot C\left(\frac{\lceil 5 \cdot \frac{1}{3} \rceil}{5}, \frac{\lceil 5 \cdot \frac{2}{3} \rceil}{5}\right)\right\rceil}{25} = \frac{\left\lceil 25 \cdot C\left(\frac{2}{5}, \frac{4}{5}\right)\right\rceil}{25} = \frac{\left\lceil 25 \cdot \frac{8}{25} \right\rceil}{25} = \frac{8}{25}.$$

Conjunctor $C^{25}_{5,5}(x,y)$ and its dual disjunctor $D^{25}_{5,5}$ are given in Tables 1 and 2.

We can see that the conjunctor $C^{25}_{5,5}$ in the example is left-continuous. Since the functions $\lceil x \rceil$ and $\lfloor x \rfloor$ are left- and right-continuous, respectively, we are able to generalise this fact:

$y \setminus x$	0	$]0, \frac{1}{5}]$	$]\frac{1}{5}, \frac{2}{5}]$	$]\frac{2}{5}, \frac{3}{5}]$	$]\frac{3}{5}, \frac{4}{5}]$	$]\frac{4}{5}, 1]$
0	0	0	0	0	0	0
$]0, \frac{1}{5}]$	0	$\frac{1}{25}$	$\frac{2}{25}$	$\frac{3}{25}$	$\frac{4}{25}$	$\frac{1}{5}$
$]\frac{1}{5}, \frac{2}{5}]$	0	$\frac{2}{25}$	$\frac{4}{25}$	$\frac{6}{25}$	$\frac{8}{25}$	$\frac{2}{5}$
$]\frac{2}{5}, \frac{3}{5}]$	0	$\frac{3}{25}$	$\frac{6}{25}$	$\frac{9}{25}$	$\frac{12}{25}$	$\frac{3}{5}$
$]\frac{3}{5}, \frac{4}{5}]$	0	$\frac{4}{25}$	$\frac{8}{25}$	$\frac{12}{25}$	$\frac{16}{25}$	$\frac{4}{5}$
$]\frac{4}{5}, 1]$	0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	1

Tab. 1. Conjunctor $(T_P)^{25}_{5,5}$

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$y \setminus x$	$[0, \frac{1}{5}[$	$[\frac{1}{5}, \frac{2}{5}[$	$[\frac{2}{5}, \frac{3}{5}[$	$[\frac{3}{5}, \frac{4}{5}[$	$[\frac{4}{5}, 1[$	1
$[0, \frac{1}{5}[$	0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	1
$[\frac{1}{5}, \frac{2}{5}[$	$\frac{1}{5}$	$\frac{9}{25}$	$\frac{13}{25}$	$\frac{17}{25}$	$\frac{21}{25}$	1
$[\frac{2}{5}, \frac{3}{5}[$	$\frac{2}{5}$	$\frac{13}{25}$	$\frac{16}{25}$	$\frac{19}{25}$	$\frac{22}{25}$	1
$[\frac{3}{5}, \frac{4}{5}[$	$\frac{3}{5}$	$\frac{17}{25}$	$\frac{19}{25}$	$\frac{21}{25}$	$\frac{23}{25}$	1
$[\frac{4}{5}, 1[$	$\frac{4}{5}$	$\frac{21}{25}$	$\frac{22}{25}$	$\frac{23}{25}$	$\frac{24}{25}$	1
1	1	1	1	1	1	1

Tab. 2. Disjunctor $(S_P)^{25}_{5,5}$.

Theorem 4.4. Let $C : [0,1]^2 \to [0,1]$ be a continuous conjunctor. Then the discrete conjunctor $C^m_{k,l}$ is left-continuous and the discrete disjunctor $D^m_{k,l}$ is right-continuous.

Remark 4.5. Let $C : [0,1]^2 \to [0,1]$ be a conjunctor and $D : [0,1]^2 \to [0,1]$ be a disjunctor. Then the following inequalities hold:

- $C \leq C^m_{k,l}$,
- $D \geq D^m_{k,l}$.

The first fact follows from inequality $x \leq \frac{\lceil k \cdot x \rceil}{k}$ and monotonicity of a conjunctor. The second one follows from inequality $x \geq \frac{\lfloor k \cdot x \rfloor}{k}$ and monotonicity of a disjunctor.

Formula similar to equation (1) holds also for the aggregation deficit R_D and its discrete counterpart. The discrete aggregation deficit is denoted by R^*_D. By definition, the aggregation deficit R^*_D is given by the formula

$$R^*_D(x,y) = \inf\{z \in [0,1]; \left\lfloor m \cdot D\left(\frac{\lfloor k \cdot z \rfloor}{k}, \frac{\lfloor l \cdot x \rfloor}{l}\right)\right\rfloor \geq m \cdot y\}.$$

Theorem 4.6. Let $D : [0,1]^2 \to [0,1]$ be a continuous disjunctor and $D^m_{k,l}$ be a discrete disjunctor. Let $R_D : [0,1]^2 \to [0,1]$ and $R^*_D : [0,1]^2 \to [0,1]$ be the aggregation deficits given by D and $D^m_{k,l}$ respectively. Then the following equality holds:

$$R^*_D(x,y) = \frac{\left\lceil k \cdot R_D\left(\frac{\lfloor l\cdot x\rfloor}{l}, \frac{\lceil m\cdot y\rceil}{m}\right)\right\rceil}{k}.$$

P r o o f. From definition we have that

$$R_D\left(\frac{\lfloor l\cdot x\rfloor}{l}, \frac{\lceil m\cdot y\rceil}{m}\right) = \inf\left\{z \in [0,1]; D\left(z, \frac{\lfloor l\cdot x\rfloor}{l}\right) \geq \frac{\lceil m\cdot y\rceil}{m}\right\},$$

$$R^*_D(x,y) = \inf\left\{z \in [0,1]; D\left(\frac{\lfloor k\cdot z\rfloor}{k}, \frac{\lfloor l\cdot x\rfloor}{l}\right) \geq \frac{\lceil m\cdot y\rceil}{m}\right\}.$$

(The second formula is equivalent to the definition of R^*_D.) Take $n \in \mathbb{N}$, such that $D\left(\frac{n}{k}, \frac{\lfloor l\cdot x\rfloor}{l}\right) \geq \frac{\lceil m\cdot y\rceil}{m}$ and n is the smallest number with this property. Such n always exists and $0 \leq n \leq k$. Now we need to distinguish between two cases: $n = 0$ and $n > 0$.

- If $n > 0$ then $D\left(\frac{n}{k}, \frac{\lfloor l\cdot x\rfloor}{l}\right) \geq \frac{\lceil m\cdot y\rceil}{m} > D\left(\frac{n-1}{k}, \frac{\lfloor l\cdot x\rfloor}{l}\right)$, and therefore we have that $R^*_D(x,y) = \frac{n}{k}$. It is obvious that $R_D(x,y) \leq R^*_D(x,y) = \frac{n}{k}$.

  Since D is continuous, $\frac{n-1}{k} < \inf\{z \in [0,1]; D\left(z, \frac{\lfloor l\cdot x\rfloor}{l}\right) \geq \frac{\lceil m\cdot y\rceil}{m}\}$. (In the other case we get that $D\left(\frac{n-1}{k}, \frac{\lfloor l\cdot x\rfloor}{l}\right) \geq \frac{\lceil m\cdot y\rceil}{m}$. That is not possible since $\frac{n}{k}$ is the smallest k-fraction with mentioned property.)

  Summarizing previous two facts we have $\frac{n-1}{k} < R_D\left(\frac{\lfloor l\cdot x\rfloor}{l}, \frac{\lceil m\cdot y\rceil}{m}\right) \leq \frac{n}{k}$. Therefore, $\frac{\left\lceil k\cdot R_D\left(\frac{\lfloor l\cdot x\rfloor}{l}, \frac{\lceil m\cdot y\rceil}{m}\right)\right\rceil}{k} = \frac{n}{k} = R^*_D(x,y)$.

- Let $n = 0$. This implies $D\left(0, \frac{\lfloor l\cdot x\rfloor}{l}\right) \geq \frac{\lceil m\cdot y\rceil}{m}$, which means that $R^*_D(x,y) = 0$. It also holds that $D(0,x) \geq y$, because $D(0,x) \geq D\left(0, \frac{\lfloor l\cdot x\rfloor}{l}\right) \geq \frac{\lceil m\cdot y\rceil}{m} \geq y$. It means that $R_D(x,y) = 0$, and therefore in case $n = 0$ we again get the equality $R^*_D(x,y) = \frac{\left\lceil k\cdot R_D\left(\frac{\lfloor l\cdot x\rfloor}{l}, \frac{\lceil m\cdot y\rceil}{m}\right)\right\rceil}{k}$.

□

Corollary 4.7. Let $g_D : [0,1]^2 \to [0,1]$ and $g^*_D : [0,1]^2 \to [0,1]$ be the estimaties of modus ponens with commutative disjunctors D and $D^m_{k,k}$ respectively. Then the following equality holds:

$$g^*_D(b,r) = \frac{\left\lceil k \cdot g_D\left(\frac{\lceil k\cdot b\rceil}{k}, \frac{\lceil m\cdot r\rceil}{m}\right)\right\rceil}{k}.$$

Since $f_{\rightarrow}(b,r) = C_{I_D}(b,r)$, it may seem that one can obtain discrete operator $f^*_{\rightarrow}$ simply from conjunctor C_{I_D} using Definition 4.1 However, this is not a correct procedure – residual conjunctor to I^*_D is different. The following fact is proved in a similar manner as Theorem 4.6

Theorem 4.8. Let $D : [0,1]^2 \rightarrow [0,1]$ be a continuous disjunctor. Let $I^*_D : [0,1]^2 \rightarrow [0,1]$ be a material implicator given by discrete disjunctor $D^m_{k,l}$. Then the discrete residual conjunctor to I^*_D is given by

$$C_{I^*_D}(b,r) = \frac{\left\lceil k \cdot C_{I_D}\left(\frac{\lceil l \cdot b \rceil}{l}, \frac{\lceil m \cdot r \rceil}{m}\right)\right\rceil}{k}.$$

The last example shows estimation of modus ponens with the disjunctor $(S_P)^{25}_{5,5}$ derived from probabilistic sum.

Example 4.9. Let $C_{I^*_D}$ be a residual conjunctor obtained from the disjunctor $(S_P)^{25}_{5,5}$. $C_{I^*_D}$ is given by Table 3.

Observe that $C_{I^*_D}(b,1) = 0$ if $b = 0$ and $C_{I^*_D}(b,1) = 1$ otherwise. This fact holds for any conjunctor $C_{I^*_D}$ obtained using disjunctor D without non-trivial zero divisors. It is generalized in the following theorem:

Theorem 4.10. Let $D^m_{k,l}$ be a discrete disjunctor without non-trivial zero divisors, then $C_{I^*_D}(0,1) = 0$ and $C_{I^*_D}(b,1) = 1$ for all $b > 0$.

Proof. Let $D^m_{k,l}$ be a disjunctor without non-trivial zero divisors, i. e.

$$x < 1, y < 1 \Leftrightarrow D(x,y) < 1.$$

Since $I^*_D(x,y) = D^m_{k,l}(y, 1-x)$, we have $I^*_D(x,y) = 1 \Leftrightarrow x = 0 \vee y = 1$. From definition of C_I we have

$$C_{I^*_D}(b,1) = \inf\{h \in [0,1]; I^*_D(b,h) = 1\}.$$

The set at the right side is either $[0,1]$ (if $b = 0$), or $\{1\}$. Infima of these sets are 0 and 1 respectively, therefore the proof is complete. □

This paper has presented some investigations connected with two generalizations of classical modus ponens rule to fuzzy logic and discrete case of this generalization. The contribution of first part is in presentation of two formulas for evaluation of modus ponens with implicative rules and with clausal rules. We plan to compare these two approaches in future. In the second part we have studied many valued modus ponens via discrete connectives and its properties.

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$r \setminus b$	0	$]0, \frac{1}{5}]$	$]\frac{1}{5}, \frac{2}{5}]$	$]\frac{2}{5}, \frac{3}{5}]$	$]\frac{3}{5}, \frac{4}{5}]$	$]\frac{4}{5}, 1]$
0	0	0	0	0	0	0
$]0, \frac{1}{25}]$	0	0	0	0	0	$\frac{1}{5}$
$]\frac{1}{25}, \frac{2}{25}]$	0	0	0	0	0	$\frac{1}{5}$
$]\frac{2}{25}, \frac{3}{25}]$	0	0	0	0	0	$\frac{1}{5}$
$]\frac{3}{25}, \frac{4}{25}]$	0	0	0	0	0	$\frac{1}{5}$
$]\frac{4}{25}, \frac{5}{25}]$	0	0	0	0	0	$\frac{1}{5}$
$]\frac{5}{25}, \frac{6}{25}]$	0	0	0	0	$\frac{1}{5}$	$\frac{2}{5}$
$]\frac{6}{25}, \frac{7}{25}]$	0	0	0	0	$\frac{1}{5}$	$\frac{2}{5}$
$]\frac{7}{25}, \frac{8}{25}]$	0	0	0	0	$\frac{1}{5}$	$\frac{2}{5}$
$]\frac{8}{25}, \frac{9}{25}]$	0	0	0	0	$\frac{1}{5}$	$\frac{2}{5}$
$]\frac{9}{25}, \frac{10}{25}]$	0	0	0	0	$\frac{2}{5}$	$\frac{2}{5}$
$]\frac{10}{25}, \frac{11}{25}]$	0	0	0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$
$]\frac{11}{25}, \frac{12}{25}]$	0	0	0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$
$]\frac{12}{25}, \frac{13}{25}]$	0	0	0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$
$]\frac{13}{25}, \frac{14}{25}]$	0	0	0	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{3}{5}$
$]\frac{14}{25}, \frac{15}{25}]$	0	0	0	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{3}{5}$
$]\frac{15}{25}, \frac{16}{25}]$	0	0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$
$]\frac{16}{25}, \frac{17}{25}]$	0	0	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{3}{5}$	$\frac{4}{5}$
$]\frac{17}{25}, \frac{18}{25}]$	0	0	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{4}{5}$
$]\frac{18}{25}, \frac{19}{25}]$	0	0	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{4}{5}$
$]\frac{19}{25}, \frac{20}{25}]$	0	0	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{4}{5}$	$\frac{4}{5}$
$]\frac{20}{25}, \frac{21}{25}]$	0	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{4}{5}$	1
$]\frac{21}{25}, \frac{22}{25}]$	0	$\frac{2}{5}$	$\frac{4}{5}$	$\frac{4}{5}$	1	1
$]\frac{22}{25}, \frac{23}{25}]$	0	$\frac{3}{5}$	$\frac{4}{5}$	1	1	1
$]\frac{23}{25}, \frac{24}{25}]$	0	$\frac{4}{5}$	1	1	1	1
$]\frac{24}{25}, 1]$	0	1	1	1	1	1

Tab. 3. Estimation of modus ponens with material implicator I_D^*

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