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EIGENSPACE OF A THREE-DIMENSIONAL MAX-ŁUKASIEWICZ FUZZY MATRIX

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Eigenvectors of a fuzzy matrix correspond to stable states of a complex discrete-events system, characterized by a given transition matrix and fuzzy state vectors. Description of the eigenspace (set of all eigenvectors) for matrices in max-min or max-drast fuzzy algebra was presented in previous papers. In this paper the eigenspace of a three-dimensional fuzzy matrix in max-Łukasiewicz algebra is investigated. Necessary and sufficient conditions are shown under which the eigenspace restricted to increasing eigenvectors of a given matrix is non-empty, and the structure of the increasing eigenspace is described. Complete characterization of the general eigenspace structure for arbitrary three-dimensional fuzzy matrix, using simultaneous row and column permutations of the matrix, is presented in Sections 4 and 5, with numerical examples in Section 6.

Keywords: Łukasiewicz triangular norm, max-t fuzzy algebra, eigenproblem, monotone eigenvector

Classification: 93E12, 62A10

1. INTRODUCTION

The eigenproblem for a fuzzy matrix corresponds to finding a stable state (possibly: all stable states) of a complex discrete-events system, described by the given transition matrix and fuzzy state vectors. Hence, investigation of the eigenspace structure in fuzzy algebras is important for applications, see e. g. [3, 4, 13, 14]. The eigenproblem has been studied by many authors in the case of max-min fuzzy algebra, which is the basic one of fuzzy algebras. Many interesting results were found in describing the structure of the eigenspace, and algorithms were suggested for computing the maximal eigenvector of a given max-min matrix, see [1, 2, 5, 6]. The problem has also been studied in more general structures like semi-modules or distributive lattices [10, 11, 12, 15, 16, 17].

Complete structure of the eigenspace in max-min fuzzy algebra as a union of intervals of monotone eigenvectors was described in [7]. The approach from [7] can be transferred to other fuzzy algebras of type max-t with triangular norm t. This was done for the drastic t-norm in [8], where the structure of the eigenspace of a given max-drast fuzzy matrix was completely described.

In this paper we describe the structure of the eigenspace for matrices in max-Łukasiewicz algebra. As the eigenspace structure for matrices of higher dimensions is rather complex, we investigate the eigenproblem in full for three-dimensional fuzzy matrices. Necessary and sufficient conditions are proved under which the monotone eigenspace of a given matrix is non-empty, which means that the corresponding system has a stable state. Further, the structure of the monotone eigenspace is described and, using simultaneous row and column permutations of the matrix, complete characterization of the general eigenspace structure of a given three-dimensional fuzzy matrix is presented. In other words, all stable states of the corresponding system are described.

Results in the paper come from the research aimed on investigation of stable states of systems with fuzzy transition matrix in max-t fuzzy algebras with various triangular norm t. More complex cases of eigenspace structure for max-Łukasiewicz fuzzy matrices with higher dimension will be considered in future research.

2. EIGENVECTORS IN MAX-T ALGEBRA

Let us denote the real unit interval as $\mathcal{I} = \langle 0,\, 1\rangle$, let t be one of the triangular norms used in fuzzy theory. By max-t algebra we understand a triple $(\mathcal{I}, \oplus, \otimes)$ with $\oplus = \max$ and $\otimes = t$, binary operations on $\mathcal{I}$. For given natural n, we denote $N = \{1, 2, \ldots, n\}$. Further, the notation $\mathcal{I}(n, n)$ ($\mathcal{I}(n)$) denotes the set of all square matrices (all vectors) of a given dimension n over $\mathcal{I}$. Operations $\oplus$, $\otimes$ are extended to matrices and vectors by the usual definition.

The eigenproblem for a given matrix $A \in \mathcal{I}(n, n)$ in max-t algebra consists of finding an eigenvector $x \in \mathcal{I}(n)$ for which $A \otimes x = x$ holds true. Eigenspace of a matrix $A \in \mathcal{I}(n, n)$ is denoted by $\mathcal{F}(A) := \left\{\, x \in \mathcal{I}(n);\ A \otimes x = x \,\right\}$.

Investigation of the eigenspace structure can be simplified by permuting any vector $x \in \mathcal{I}(n)$ to an increasing form. For given permutations φ, $\psi \in P_n$ we denote by $A_{\varphi\psi}$ the matrix with rows permuted by φ and columns permuted by ψ, and we denote by x_φ the vector permuted by φ. It can be easily shown that the following theorem holds, see also [7].

Theorem 2.1. Let $A \in \mathcal{I}(n, n)$, $x \in \mathcal{I}(n)$ and $\varphi \in P_n$. Then $x \in \mathcal{F}(A)$ if and only if $x_\varphi \in \mathcal{F}(A_{\varphi\varphi})$.

We define the *increasing* eigenspace of a matrix $A \in \mathcal{I}(n, n)$ as

$$\mathcal{F}^{\leq}(A) := \left\{\, x \in \mathcal{I}(n);\ A \otimes x = x,\ (\forall\, i, j)[\, i \leq j \ \Rightarrow\ x_i \leq x_j \,] \,\right\},$$

and the *strictly increasing* eigenspace as

$$\mathcal{F}^{<}(A) := \left\{\, x \in \mathcal{I}(n);\ A \otimes x = x,\ (\forall\, i, j)[\, i < j \ \Rightarrow\ x_i < x_j \,] \,\right\}.$$

Similar notation $\mathcal{I}^{\leq}(n)$ and $\mathcal{I}^{<}(n)$ will be used without the condition $A \otimes x = x$. We shall also use the notation $\mathcal{I}^{\leq}_{\varphi}(n) := \left\{\, x \in \mathcal{I}(n);\ x_\varphi \in \mathcal{I}^{\leq}(n) \,\right\}$, $\mathcal{I}^{<}_{\varphi}(n) := \left\{\, x \in \mathcal{I}(n);\ x_\varphi \in \mathcal{I}^{<}(n) \,\right\}$, $\mathcal{F}^{\leq}_{\varphi}(A) := \left\{\, x \in \mathcal{F}(A);\ x_\varphi \in \mathcal{I}^{\leq}(n) \,\right\}$ and $\mathcal{F}^{<}_{\varphi}(A) := \left\{\, x \in \mathcal{F}(A);\ x_\varphi \in \mathcal{I}^{<}(n) \,\right\}$.

In this notation, the assertions of Theorem 2.1 can be expressed as $\mathcal{F}^{\leq}_{\varphi}(A) = \left\{\, x \in \mathcal{I}(n);\ x_\varphi \in \mathcal{F}^{\leq}(A_{\varphi\varphi}) \,\right\}$ and $\mathcal{F}^{<}_{\varphi}(A) = \left\{\, x \in \mathcal{I}(n);\ x_\varphi \in \mathcal{F}^{<}(A_{\varphi\varphi}) \,\right\}$.

It has been proved in [7] that if the binary operation $\otimes$ coincides with the minimum operation, then the strictly increasing eigenspace $\mathcal{F}^{<}(A)$ can be described as an interval of strictly increasing eigenvectors, where the bounds $m^\star(A)$, $M^\star(A) \in \mathcal{I}(n)$ of the interval are defined as follows

$$m^{(j)}(A) := \max_{k>j} a_{jk} \qquad M^{(j)}(A) := \max_{k\geq j} a_{jk}$$
$$m^\star_i(A) := \max_{j\leq i} m^{(j)}(A) \qquad M^\star_i(A) := \min_{j\geq i} M^{(j)}(A).$$

Theorem 2.2. [7] Let $A \in \mathcal{I}(n,n)$ and let $x \in \mathcal{I}(n)$ be a strictly increasing vector in max-min algebra (max-t algebra with the t-norm equal to minimum operation). Then $x \in \mathcal{F}(A)$ if and only if $m^\star(A) \leq x \leq M^\star(A)$. In formal notation,

$$\mathcal{F}^{<}(A) = \langle m^\star(A), M^\star(A)\rangle \cap \mathcal{I}^{<}(n)\,.$$

Remark 2.3. In [7], Theorem 2.1, Theorem 2.2 are presented in a slightly different formulation, namely for a max-min algebra $(\mathcal{B}, \oplus, \otimes)$ with an arbitrary bounded linearly ordered set $\mathcal{B}$ and with the operations $\oplus = \max$, $\otimes = \min$. Moreover, analogous description is given for the non-strictly increasing eigenspace $\mathcal{F}^{\leq}(A)$, and for constant eigenvectors. Hence, the structure of $\mathcal{F}(A)$ in a max-min algebra $\mathcal{B}$ is completely described for any matrix $A \in \mathcal{B}(n,n)$.

In fuzzy sets theory, various triangular norms are used. The most frequent of them are

Gödel norm $\quad G(x,y) = \min(x,y)$

product norm $\quad \mathrm{prod}(x,y) = (x \cdot y)$

drastic norm

$$\mathrm{drast}(x,y) = \begin{cases} \min(x,y) & \text{if } \max(x,y) = 1 \\ 0 & \text{if } \max(x,y) < 1 \end{cases}$$

Łukasiewicz norm $\quad Ł(x,y) = \max(x+y-1, 0).$

The max-t fuzzy algebra with the Gödel norm is a special case of the max-min algebra, and by Remark 2.3 the eigenspace for this case is described in [7]. The eigenspace for the max-drast fuzzy algebra and max-prod fuzzy algebra with the drastic norm and product norm was discussed in [8] and [9] respectively.

3. EIGENVECTORS IN MAX-ŁUKASIEWICZ ALGEBRA

In this section, the above considerations will be transferred to the case of max-Łukasiewicz algebra, which is a special case of max-t fuzzy algebra with Łukasiewicz triangular norm. In the rest of this paper we shall work with the max-Ł fuzzy algebra $(\mathcal{I}, \oplus, \otimes_l)$ with binary operation $\oplus = \max$ and $\otimes_l = Ł$. Hence, for every $x, y \in \mathcal{I}(n)$ and for every $i \in N$ we have

$$(x \oplus y)_i = \max(x_i, y_i)$$
$$(x \otimes_l y)_i = Ł(x_i, y_i) = \begin{cases} x_i + y_i - 1 & \text{if } \min(x_i + y_i - 1, 0) = 0 \\ 0 & \text{otherwise.} \end{cases}$$

The following proposition contains several logical consequences of the definition of Łukasiewicz triangular norm which will be used in development of the further theory.

Proposition 3.1. Let $a, b, c \in [0,1]$. Then

(i) $a \otimes_ł b = b$ if and only if $a = 1$ or $b = 0$

(ii) $a \otimes_ł c = b$ if and only if $a = 1 + b - c$ or ($a \leq 1 - c$ and $b = 0$)

(iii) $a \otimes_ł c \leq b$ if and only if $a \leq 1 + b - c$

(iv) $a \otimes_ł c > b$ if and only if $a > 1 + b - c$

(v) If $c < b$ then $a \otimes_ł c < b$.

Proof. (i) Let $a \otimes_ł b = b$. We have either $0 \leq a + b - 1$ or $a + b - 1 \leq 0$. That is, by definition of Łukasiewicz triangular norm, either $a = 1$ or $b = 0$. For the converse implication, first consider that $a = 1$. Then $a + b - 1 = 1 + b - 1 = b \geq 0$ implies that $a \otimes_ł b = b$. If $b = 0$, then $a + b - 1 = a - 1 \leq 0$. Hence $a \otimes_ł b = b$ follows again from the definition.

(ii) Let $a \otimes_ł c = b$. By definition, either $0 \leq a + c - 1 = b$ or $a + c - 1 \leq 0 (= b)$. That is, either $a = 1 + b - c$ or ($a \leq 1 - c$ and $b = 0$). Conversely, let $a = 1 + b - c$ or ($a \leq 1 - c$ and $b = 0$). Then either $a + c - 1 = b \geq 0$ or ($a + c - 1 \leq 0$ and $b = 0$). It follows directly from the definition that in either case $a \otimes_ł c = b$.

(iii) By definition $a \otimes_ł c \leq b$ implies that, either $a + c - 1 \leq 0 (\leq b)$ or $0 \leq a + c - 1 \leq b$. It follows directly that in either case $a \leq 1 + b - c$. Conversely, $a \leq 1 + b - c$ implies $a + b - 1 \leq b$. Also, $0 \leq b$. Then by definition $a \otimes_ł c \leq b$.

(iv) The proof is analogous to (iii).

(v) Let $0 \leq c < b$. That is $b > 0$. As $a \leq 1$ then $a + c < 1 + b$ implies that $a + c - 1 < b$. Hence, $a \otimes_ł c < b$ follows from the definition. □

Proposition 3.2. Let $A \in \mathcal{I}(n,n)$, $x \in \mathcal{I}(n)$. Then $x \in \mathcal{F}(A)$ if and only if for every $i \in N$ the following hold

$$a_{ij} \otimes_ł x_j \leq x_i \quad \text{for every} \quad j \in N, \tag{1}$$

$$a_{ij} \otimes_ł x_j = x_i \quad \text{for some} \quad j \in N. \tag{2}$$

Proof. By definition, $x \in \mathcal{F}(A)$ is equivalent with the condition $\max_{j \in N} a_{ij} \otimes_ł x_j \leq x_i$ for every $i \in N$, which is equivalent to (1) and (2). □

Proposition 3.3. Let $A \in \mathcal{I}(n,n)$, $x \in \mathcal{I}^<(n)$. Then $x \in \mathcal{F}^<(A)$ if and only if for every $i \in N$ the following hold

$$a_{ij} \otimes_ł x_j \leq x_i \quad \text{for every} \quad j \in N, \quad j \geq i, \tag{3}$$

$$a_{ij} \otimes_ł x_j = x_i \quad \text{for some} \quad j \in N, \quad j \geq i. \tag{4}$$

Proof. By Proposition 3.1(v), $a_{ij} \otimes_l x_j < x_i$ for every $j < i$, $x_j < x_i$. Hence the terms with $j < i$ in (1) and (2) of Proposition 3.2 can be left out. □

Theorem 3.4. Let $A \in \mathcal{I}(n,n)$ and $x \in \mathcal{I}^<(n)$. Then $x \in \mathcal{F}^<(A)$ if and only if for every $i \in N$ the following hold

(i) $a_{ij} \leq 1 + x_i - x_j$ for every $j \in N,\, j \geq i$,

(ii) if $i = 1$, then $x_1 = 0$ or $a_{1j} = 1 + x_1 - x_j$ for some $j \in N$,

(iii) if $i > 1$, then $a_{ij} = 1 + x_i - x_j$ for some $j \in N,\, j \geq i$.

Proof. Suppose that $x \in \mathcal{F}^<(A)$, that is $A \otimes_l x = x$. Then $a_{ij} \otimes_l x_j \leq x_i$, for every $j \in N, j \geq i$ by Proposition 3.1(iii) gives $a_{ij} \leq 1 + x_i - x_j$. If $i = 1$ then $a_{1j} \otimes_l x_j = x_1$, for some $j \in N$. Then by Proposition 3.1(ii), we have $x_1 = 0$ or $a_{1j} = 1 + x_1 - x_j$, for some $j \in N$. To prove (iii), consider $a_{ij} \otimes_l x_j = x_i$ for some $j \in N$, $j \geq i > 1$. By definition we have, either $a_{ij} + x_j - 1 \leq 0$ or $0 \leq a_{ij} + x_j - 1$. The case $a_{ij} \otimes_l x_j = 0 = x_i$ is not possible, as for $i > 1$, $x_i > 0$. Then we must have $0 \leq a_{ij} + x_j - 1 = x_i$, that is $a_{ij} = 1 + x_i - x_j$.

Conversely, suppose that conditions (i), (ii), (iii) hold true. We show that $x \in \mathcal{F}^<(A)$, that is $A \otimes_l x = x$. In other words, $\max_{j\in N} a_{ij} \otimes_l x_j = x_i$ for every $i \in N$. Let $i \in N$ be fixed. By (i) and Proposition 3.1(iii), $a_{ij} \otimes_l x_j \leq x_i$, for every $j \in N, j \geq i$. If $i = 1$ then by (ii) and Proposition 3.1(ii), $a_{1j} = 1 + x_1 - x_j$, for some $j \in N$. If $i > 1$ then by (iii) $a_{ij} = 1 + x_i - x_j$, for some $j \in N$, $j \geq i > 1$, we have $a_{ij} + x_j - 1 = x_i > 0$, because $i > 1$. This implies $\max(0, a_{ij} + x_j - 1) = x_i$, that is $a_{ij} \otimes_l x_j = x_i$. Hence $\max_{j\in N} a_{ij} \otimes_l x_j = x_i$ for every $i \in N$, that is $A \otimes_l x = x$ or $x \in \mathcal{F}^<(A)$. □

The following theorem describes necessary conditions under which a given square matrix can have a strictly increasing eigenvector.

Theorem 3.5. Let $A \in \mathcal{I}(n,n)$. If $\mathcal{F}^<(A) \neq \emptyset$, then the following conditions are satisfied

(i) $a_{ij} < 1$ for all $i,j \in N, i < j$,

(ii) $a_{nn} = 1$.

Proof. Let $\mathcal{F}^<(A) \neq \emptyset$. That is, there exists $x \in \mathcal{F}^<(A)$ such that conditions of Theorem 3.4 hold true. Condition $a_{ij} < 1$ follows directly from (i) and $a_{nn} = 1$ from (iii) of Theorem 3.4. □

Remark 3.6. Generally speaking, the conditions in Theorem 3.5 are only necessary. It can be easily seen that in the case $n = 2$ the necessary conditions in Theorem 3.5 are also sufficient. Namely, if $n = 2$, then we have two conditions: $a_{12} < 1$ and $a_{22} = 1$. Then, arbitrary vector $x_1 = 0$ and $0 < x_2 \leq 1 - a_{12}$ fulfills the conditions of Theorem 3.4. Hence the vector $x = (x_1, x_2)$ with $x_1 = 0$ and $0 < x_2 \leq 1 - a_{12}$ is a strictly increasing eigenvector of A. In the particular case when $a_{11} = 1$, the variable x_1 can even take arbitrary values from the interval $(0, 1)$. The result is completely formulated in the following theorem.

Theorem 3.7. Let $A \in \mathcal{I}(2,2)$. Then $\mathcal{F}^{<}(A) \neq \emptyset$ if and only if the $a_{12} < 1$ and $a_{22} = 1$. If this is the case, then

(i) $\mathcal{F}^{<}(A) = \{\, (x_1, x_2) \in \mathcal{I}(2,2);\ x_1 \in (0,1),\ x_1 < x_2 \leq \min(1, 1 + x_1 - a_{12}) \,\}$, if $a_{11} = 1$,

(ii) $\mathcal{F}^{<}(A) = \{\, (x_1, x_2) \in \mathcal{I}(2,2);\ x_1 \in (0, \min(a_{11}, a_{12})),\ x_2 = 1 + x_1 - a_{12} \leq 1 \,\}$, if $a_{11} < 1$.

Proof. The statement follows from the arguments in Remark 3.6. □

In the next theorem, a necessary and sufficient condition for the existence of a non-zero constant eigenvector is presented. The set of all constant eigenvectors of a matrix A is denoted by $\mathcal{F}^{=}(A)$.

Theorem 3.8. Let $A \in \mathcal{I}(n,n)$, then there is a non-zero constant eigenvector $x \in \mathcal{F}^{=}(A)$ if and only if

(i) $\max\{\, a_{ij};\ j \in N \,\} = 1$ for every $i \in N$.

Proof. Let $x = (c, c, \ldots, c) \in \mathcal{I}(n)$ with $c > 0$ be a constant eigenvector of A. Then (i) follows from the conditions (1), (2) in Proposition 3.2. Conversely, if (i) is satisfied, then clearly the conditions (1), (2) hold true for the unit constant vector $u = (1, 1, \ldots, 1) \in \mathcal{I}(n)$. Hence $u \in \mathcal{F}^{=}(a)$. □

Theorem 3.9. Let $A \in \mathcal{I}(n,n)$. If the condition (i) of Theorem 3.8 is satisfied, then $\mathcal{F}^{=}(A) = \{(c, c, , \ldots, c);\ c \in \mathcal{I}\}$. If A does not satisfy the condition (i), then $\mathcal{F}^{=}(A) = \{(0, 0, , \ldots, 0)\}$.

Proof. It is easy to verify that if A satisfies the condition (i), then the conditions (1), (2) hold true for an arbitrary constant vector $(c, c, \ldots, c)$ with $c \in \mathcal{I}$, hence $\mathcal{F}^{=}(A) = \{(c, c, , \ldots, c);\ c \in \mathcal{I}\}$.

On the other hand, let us assume that the condition (i) is not satisfied. Clearly, the zero constant vector $(0, 0, , \ldots, 0)$ fulfills the conditions (1), (2). Hence we have $\{(0, 0, , \ldots, 0)\} \subseteq \mathcal{F}^{=}(A)$. The equality $\mathcal{F}^{=}(A) = \{(0, 0, , \ldots, 0)\}$ follows from Theorem 3.8. □

Remark 3.10. Description of non-strictly increasing eigenvectors is necessary for computing of the general eigenspace of dimension $n = 3$ and higher. This extension of the presented theory is given in the next two sections for three-dimensional matrices. Moreover, the necessary conditions from Theorem 3.5 are extended to necessary and sufficient ones.

Remark 3.11. The max-Ł fuzzy algebra is closely related to the max-plus algebra (also called: tropical algebra) with operations $\oplus = \max$ and $\otimes = +$ defined on the set of all real numbers. If we denote by E the $n \times n$ matrix with all inputs equal to 1, and symbols $\mathcal{F}_{+}$ ($\mathcal{F}_{l}$) denote the eigenspace of a given matrix in max-plus (max-Ł) algebra, then the following theorem holds true.

Theorem 3.12. Let $A \in \mathcal{I}(n,n)$. If a vector $x \in \mathcal{I}(n)$ is max-plus eigenvector of the matrix $A - E$, then x is max-Ł eigenvector of A. In formal notation,

$$\mathcal{F}_+(A-E) \cap \mathcal{I}(n) \subseteq \mathcal{F}_l(A)\,.$$

Proof. Assume that $x \in \mathcal{I}(n)$ satisfies the equation $(A-E) \otimes x = x$, i.e. the max-plus equality

$$\bigoplus_{j \in N} (a_{ij} - 1) \otimes x_j = x_i$$

holds for every $i \in N$. Adding the expression $\oplus\, 0$ to both sides, and expressing the operation $\otimes$ explicitly as addition, we get by easy computation

$$\bigoplus_{j \in N} (a_{ij} - 1 + x_j) \oplus 0 = \bigoplus_{j \in N} \big((a_{ij} - 1 + x_j) \oplus 0\big) = \bigoplus_{j \in N} a_{ij} \otimes_l x_j = x_i \oplus 0 = x_i\,.$$

Hence, $A \otimes_l x = x$. □

4. EIGENVECTORS IN THE THREE-DIMENSIONAL CASE

In this section we consider the three-dimensional eigenproblem in the max-Łukasiewicz fuzzy algebra. In other words, we assume $n = 3$, hence we work with matrices in $\mathcal{I}(3,3)$ and vectors in $\mathcal{I}(3)$. The results from the previous sections will be extended and a complete description of the eigenspace will be given.

The following theorem describes the necessary and sufficient conditions under which a three-dimensional fuzzy matrix has a strictly increasing eigenvector.

Theorem 4.1. Let $A \in \mathcal{I}(3,3)$. Then $\mathcal{F}^<(A) \neq \emptyset$ if and only if the following conditions are satisfied

(i) $a_{12} < 1\,, \quad a_{13} < 1\,, \quad a_{23} < 1\,,$

(ii) $a_{22} = 1\,,$ or $a_{13} < a_{23}\,,$

(iii) $a_{33} = 1\,.$

Proof. Let $\mathcal{F}^<(A) \neq \emptyset$, i.e. there exists $x \in \mathcal{F}^<(A)$. The conditions (i) and (iii) follow directly from Theorem 3.4. To prove the condition (ii), let us assume that $a_{22} < 1$. Then by (iii) of Theorem 3.4 we get $a_{22} = 1 + x_2 - x_2$ or $a_{23} = 1 + x_2 - x_3$. The first equation implies $a_{22} = 1$, which is a contradiction. Therefore we must have $a_{23} = 1 + x_2 - x_3 < 1$. By (i) of Theorem 3.4, we have $a_{13} \leq 1 + x_1 - x_3 = 1 + (x_1 - x_2) + (x_2 - x_3) = (1 + x_2 - x_3) + (x_1 - x_2) < 1 + x_2 - x_3 = a_{23}$. This implies $a_{13} < a_{23}$.

Conversely, suppose that conditions (i), (ii) and (iii) are satisfied. To show that $\mathcal{F}^<(A) \neq \emptyset$, we consider two cases.

Case 1. If $a_{22} < 1$, then put $x_1 = 0$ and choose $x_2 \leq \min(1 - a_{12}, a_{23} - a_{13})$ and $0 < x_2 < 1 - a_{13}$, and put $x_3 = x_2 + (1 - a_{23})$. By our assumption, $1 - a_{12} > 0$, $a_{23} - a_{13} > 0$ and $1 - a_{13} > 0$. Therefore the choice of x_2 fulfilling the conditions $x_2 \leq$

$\min(1-a_{12}, a_{23}-a_{13})$ and $0 < x_2 < 1-a_{13}$ is always possible. Also by assumption, $a_{23} < 1$ implies that $1-a_{23} > 0$. Therefore $x_3 = x_2 + (1-a_{23}) > x_2$. Moreover, $x_2 \leq \min(1-a_{12}, a_{23}-a_{13}) \leq a_{23}-a_{13} \leq a_{23}$, i.e. $x_2 \leq a_{23}$. From this we have $x_3 = x_2 + (1-a_{23}) \leq 1$, i.e. $x_3 \leq 1$. This shows that $x \in \mathcal{I}^<(3)$. To show that $x \in \mathcal{F}^<(A)$, consider $x_2 \leq \min(1-a_{12}, a_{23}-a_{13})$ and $0 < x_2 < 1-a_{13}$. This implies that $x_2 \leq 1-a_{12}$ and $0 < x_2 < 1-a_{13}$, or equivalently, $a_{12} \leq 1-x_2$ and $a_{13} < 1-x_2$, which implies condition (i) of Theorem 3.4. Choice of $x_1 = 0$ satisfies the condition (ii) of Theorem 3.4. Also, $x_3 = x_2 + (1-a_{23})$, $a_{23} = 1 + x_2 - x_3$ imply condition (iii) of Theorem 3.4. Hence $\mathcal{F}^<(A) \neq \emptyset$.

Case 2. If $a_{22} = 1$, then put $x_1 = 0$, choose $0 < x_2 < \min(1-a_{12}, 1-a_{13})$ and choose x_3 such that $x_2 < x_3 \leq \min(1-a_{13}, x_2 + (1-a_{23}))$. The choice $0 < x_2 < \min(1-a_{12}, 1-a_{13})$ is always possible because by our assumption $1-a_{12} > 0$ and $1-a_{13} > 0$. Also $1-a_{23} > 0$ by the same argument and therefore $x \in \mathcal{I}^<(3)$. Consider $x_2 < \min(1-a_{12}, 1-a_{13}) \leq 1-a_{12}$. Then $x_2 < 1-a_{12}$ implies that $a_{12} < 1-x_2$. Similarly $a_{13} < 1-x_2$ and by $x_3 \leq \min(1-a_{13}, x_2+(1-a_{23}))$ we have $a_{23} \leq 1+x_2-x_3$, showing that condition (i) of Theorem 3.4 is satisfied. Conditions (ii) and (iii) of Theorem 3.4 are satisfied by the choice of $x_1 = 0$ and assumption $a_{22} = 1$, respectively. Hence $\mathcal{F}^<(A) \neq \emptyset$. □

Theorem 4.2. Let $A \in \mathcal{I}(3,3)$ satisfy conditions (i), (ii) and (iii) of Theorem 4.1. Then $x \in \mathcal{F}^<(A)$ if and only if $x \in \mathcal{I}^<(3)$ and either $x_1 = 0$ and conditions

$$\begin{aligned}&\text{if}\quad a_{22} < 1,\ \text{then}\\ &\quad 0 < x_2 \leq \min(1-a_{12}, a_{23}-a_{13}),\ x_2 < 1-a_{13},\ x_3 = x_2 + (1-a_{23}),\end{aligned} \tag{5}$$

$$\begin{aligned}&\text{if}\quad a_{22} = 1,\ \text{then}\\ &\quad 0 < x_2 < \min(1-a_{12}, 1-a_{13}),\ x_2 < x_3 \leq \min\big(1-a_{13}, x_2 + (1-a_{23})\big),\end{aligned} \tag{6}$$

are satisfied, or $x_1 > 0$ and conditions

$$\begin{aligned}&\text{if}\quad a_{11} = 1,\, a_{22} = 1,\ \text{then}\ x_1 < x_2 < 1,\\ &\quad x_2 \leq x_1 + (1-a_{12}),\, x_3 \leq \min\big(x_1 + (1-a_{13}), x_2 + (1-a_{23}), 1\big),\end{aligned} \tag{7}$$

$$\begin{aligned}&\text{if}\quad a_{11} = 1,\, a_{22} < 1,\ \text{then}\ x_1 < a_{23},\\ &\quad x_2 \leq \min\big(a_{23}, x_1 + (1-a_{12}), x_1 + (a_{23}-a_{13})\big),\, x_3 = x_2 + (1-a_{23}),\end{aligned} \tag{8}$$

$$\begin{aligned}&\text{if}\quad a_{11} < 1,\, a_{22} = 1,\ \text{then}\\ &\quad 0 < x_1 < a_{12},\, x_2 = x_1 + (1-a_{12}),\\ &\qquad\qquad x_3 \leq \min\big(x_1 + 2 - (a_{12}+a_{23}), x_1 + (1-a_{13}), 1\big),\ \text{or}\\ &\quad 0 < x_1 \leq a_{13},\, x_1 + (a_{23}-a_{13}) \leq x_2 < x_1 + (1-a_{12}),\\ &\qquad\qquad x_3 = x_1 + (1-a_{13}),\end{aligned} \tag{9}$$

$$\begin{aligned}&\text{if}\quad a_{11} < 1,\, a_{22} < 1,\ \text{then}\\ &\quad a_{12} - a_{13} + a_{23} \geq 1,\, 0 < x_1 \leq a_{12} + a_{23} - 1,\, x_2 = x_1 + (1-a_{12}),\\ &\qquad\qquad x_3 = x_1 + 2 - (a_{12}+a_{23}),\ \text{or}\\ &\quad a_{12} - a_{13} + a_{23} \leq 1,\, 0 < x_1 \leq a_{13},\, x_2 = x_1 + (a_{23}-a_{13}),\\ &\qquad\qquad x_3 = x_1 + (1-a_{13})\end{aligned} \tag{10}$$

are satisfied.

Proof. For convenience, we shall use notation $\mathcal{F}_0^<(A) = \{x \in \mathcal{F}^<(A);\, x_1 = 0\}$, $\mathcal{F}_1^<(A) = \{x \in \mathcal{F}^<(A);\, x_1 > 0\}$. Thus, we have $\mathcal{F}^<(A) = \mathcal{F}_0^<(A) \cup \mathcal{F}_1^<(A)$. The cases $x \in \mathcal{F}_0^<(A)$ and $x \in \mathcal{F}_1^<(A)$ will be considered separately.

Let $A \in \mathcal{I}(3,3)$ satisfy conditions of Theorem 4.1, let $x = (0, x_2, x_3) \in \mathcal{I}^<(3)$ satisfy conditions (5) and (6). Using the same arguments as in the converse part of Theorem 4.1 we can easily show that $x \in \mathcal{F}_0^<(A)$.

Conversely, suppose that $x \in \mathcal{F}_0^<(A)$. This implies that conditions of Theorem 3.4 are fulfilled. We consider two cases $a_{22} < 1$ and $a_{22} = 1$.

Case 1. If $a_{22} < 1$, condition (i) of Theorem 3.4 gives $a_{12} \le 1 + x_1 - x_2$, $a_{13} \le 1 + x_1 - x_3$ and $a_{23} \le 1 + x_2 - x_3$. Since $x_1 = 0$ by our assumption, therefore we have $x_2 \le 1 - a_{12}$, $x_3 \le 1 - a_{13}$ and $x_3 \le 1 + x_2 - a_{23}$. Condition (iii) of Theorem 3.4 gives $a_{23} = 1 + x_2 - x_3$, which implies that $x_3 = x_2 + 1 - a_{23}$. By $x_3 = x_2 + 1 - a_{23}$ and $x_3 \le 1 - a_{13}$, we have $x_2 \le a_{23} - a_{13}$. Also, $x_2 < x_3 \le 1 - a_{13}$ implies that $x_2 < 1 - a_{13}$. Therefore we have $0 < x_2 \le \min(1 - a_{12}, a_{23} - a_{13}), x_2 < 1 - a_{13}, x_3 = x_2 + 1 - a_{23}$.

Case 2. If $a_{22} = 1$, condition (i) of Theorem 3.4 gives $x_2 \le 1 - a_{12}$, $x_3 \le 1 - a_{13}$ and $x_3 \le 1 + x_2 - a_{23}$. Since $0 < x_2 < 1$, then we must have $x_2 < 1 - a_{12}$. Also $x_2 < x_3 \le 1 - a_{13}$ implies that $x_2 < 1 - a_{13}$. Therefore we have $0 < x_2 < \min(1 - a_{12}, 1 - a_{13}), x_2 < x_3 \le \min(1 - a_{13}, x_2 + 1 - a_{23})$.

Let us assume now that a vector $x = (x_1, x_2, x_3) \in \mathcal{I}^<(3)$ with $x_1 > 0$ satisfies conditions (7) – (10). There are four possible cases and we show that the conditions (i), (ii) and (iii) of Theorem 3.4 in each case are satisfied. In other words, we show that $x \in \mathcal{F}_1^<(A)$.

Let $a_{11} = 1$, $a_{22} = 1$. Then from $x_2 \le x_1 + 1 - a_{12}$ we have $a_{12} \le 1 + x_1 - x_2$. Also, $x_3 \le \min(x_1 + 1 - a_{13}, x_2 + 1 - a_{23}, 1) \le x_1 + 1 - a_{13}$ implies $a_{13} \le 1 + x_1 - x_3$, and $x_3 \le \min(x_1 + 1 - a_{13}, x_2 + 1 - a_{23}, 1) \le x_2 + 1 - a_{23}$ implies $a_{23} \le 1 + x_2 - x_3$. That is, (i) of Theorem 3.4 is satisfied. Conditions (ii) and (iii) of Theorem 3.4 are satisfied by the assumption $a_{11} = 1$ and $a_{22} = 1$, respectively.

Let $a_{11} = 1$, $a_{22} < 1$. Then from $x_3 = x_2 + 1 - a_{23}$ we have $a_{23} = 1 + x_2 - x_3$. Also, $x_2 \le \min(a_{23}, x_1 + 1 - a_{12}, x_1 + (a_{23} - a_{13}))$ implies $x_2 \le x_1 + 1 - a_{12}$ and $x_2 \le x_1 + (a_{23} - a_{13})$. The first inequality gives $a_{12} \le 1 + x_1 - x_2$. By the second inequality $x_2 \le x_1 + (a_{23} - a_{13})$ and $x_3 = x_2 + 1 - a_{23}$, we have $a_{13} \le 1 + x_1 - x_3$. That is, (i) of Theorem 3.4 is satisfied. Conditions (ii) and (iii) of Theorem 3.4 are satisfied by the assumption $a_{11} = 1$ and by $x_3 = x_2 + 1 - a_{23}$, respectively.

Let $a_{11} < 1$, $a_{22} = 1$. If the first part of the disjunction in (9) is satisfied, then $x_2 = x_1 + 1 - a_{12}$ implies $a_{12} = 1 + x_1 - x_2$. Also, $x_3 \le \min(x_1 + 2 - (a_{12} + a_{23}), x_1 + 1 - a_{13}, 1)$ implies that $x_3 \le x_1 + 1 - a_{13}$ and $x_3 \le x_1 + 2 - (a_{12} + a_{23})$. The first inequality implies $a_{13} \le 1 + x_1 - x_3$. By $x_2 = x_1 + 1 - a_{12}$ and $x_3 \le x_1 + 2 - (a_{12} + a_{23})$, we have $a_{23} \le 1 + x_2 - x_3$. That is, (i) of Theorem 3.4 is satisfied. Conditions (ii) and (iii) of Theorem 3.4 are satisfied by $x_2 = x_1 + 1 - a_{12}$ and by the assumption $a_{22} = 1$, respectively.

If the second part of the disjunction in (9) is satisfied, then $x_3 = x_1 + 1 - a_{13}$ implies $a_{13} = 1 + x_1 - x_3$. Also, $x_1 + (a_{23} - a_{13}) \le x_2 < x_1 + 1 - a_{12}$ implies the inequalities

$x_1 + (a_{23} - a_{13}) \leq x_2$ and $x_2 < x_1 + 1 - a_{12}$. Using $x_3 = x_1 + 1 - a_{13}$ in the first inequality we have $a_{23} \leq 1 + x_2 - x_3$, the second inequality gives $a_{12} \leq 1 + x_1 - x_2$. That is, (i) of Theorem 3.4 is satisfied. Conditions (ii) and (iii) of Theorem 3.4 are satisfied by $x_3 = x_1 + 1 - a_{13}$ and by the assumption $a_{22} = 1$, respectively.

Let $a_{11} < 1$, $a_{22} < 1$. If the first part of the disjunction in (10) is satisfied, then $a_{12} - a_{13} + a_{23} \geq 1$, and $x_2 = x_1 + 1 - a_{12}$ implies $a_{12} = 1 + x_1 - x_2$. By substituting in $x_3 = x_1 + 2 - (a_{12} + a_{23})$, we get $a_{23} = 1 + x_2 - x_3$. Also, $x_3 = x_1 + 2 - (a_{12} + a_{23}) = x_1 + 1 - (a_{12} + a_{23} - 1) \leq x_1 + 1 - a_{13}$. This implies that $x_3 \geq x_1 + 1 - a_{13}$, or $a_{13} \leq 1 + x_1 - x_3$. That is, (i) of Theorem 3.4 is satisfied. Conditions (ii) and (iii) of Theorem 3.4 are satisfied by $x_2 = x_1 + 1 - a_{12}$ and $a_{23} = 1 + x_2 - x_3$, respectively.

If the second part of the disjunction in (10) is satisfied, then $a_{12} - a_{13} + a_{23} \leq 1$, and $x_3 = x_1 + 1 - a_{13}$ gives $a_{13} = 1 + x_1 - x_3$. We have, $x_2 = x_1 + (a_{23} - a_{13}) \leq x_1 + 1 - a_{12}$, that is $a_{12} \leq 1 + x_1 - x_2$. Also, $x_2 = x_1 + (a_{23} - a_{13}) = (x_1 + 1 - a_{13}) - (1 - a_{23}) = x_3 - (1 - a_{23})$ implies that $a_{23} = 1 + x_2 - x_3$. That is, (i) of Theorem 3.4 is satisfied. Conditions (ii) and (iii) of Theorem 3.4 are satisfied by $x_3 = x_1 + 1 - a_{13}$ and $a_{23} = 1 + x_2 - x_3$, respectively.

Conversely, suppose that $x \in \mathcal{F}_1^<(A)$. Then conditions (i), (ii) and (iii) of Theorem 3.4 hold true and we show that conditions (7)–(10) are satisfied in all four cases. We start with an easy observation that in each case, condition (i) of Theorem 3.4 implies $x_2 \leq x_1 + 1 - a_{12}$, $x_3 \leq x_1 + 1 - a_{13}$ and $x_3 \leq x_2 + 1 - a_{23}$.

When $a_{11} = 1$, $a_{22} = 1$, then the inequalities in condition (7) follow directly from the assumption $x \in \mathcal{I}^<(3)$ and from the above observation.

When $a_{11} = 1$, $a_{22} < 1$, then condition (iii) of Theorem 3.4 gives $x_3 = x_2 + 1 - a_{23}$. Since $x \in \mathcal{I}^<(3)$ then $x_3 = x_2 + 1 - a_{23} \leq 1$ implies $x_1 < x_2 \leq a_{23}$. Using $x_3 = x_2 + 1 - a_{23}$ in $x_3 \leq x_1 + 1 - a_{13}$ we get $x_2 \leq x_1 + (a_{23} - a_{13})$. Thus, we have verified all relations in condition (8).

When $a_{11} < 1$, $a_{22} = 1$, then by condition (ii) of Theorem 3.4 either $x_2 = x_1 + 1 - a_{12}$, $x_3 \leq x_1 + 1 - a_{13}$ and $x_3 \leq x_2 + 1 - a_{23}$, or $x_2 < x_1 + 1 - a_{12}$, $x_3 = x_1 + 1 - a_{13}$ and $x_3 \leq x_2 + 1 - a_{23}$ hold true.

Let us consider the first possibility. Since $x \in \mathcal{I}^<(3)$ then $x_2 = x_1 + 1 - a_{12} < 1$, which implies $x_1 < a_{12}$. By $x_2 = x_1 + 1 - a_{12}$ and $x_3 \leq x_2 + 1 - a_{23}$ we have $x_3 \leq x_1 + 2 - (a_{12} + a_{23})$. The remaining relations in the first part of the disjunction in (9) follow from the starting observation.

In the second possibility, we have $x_3 = x_1 + 1 - a_{13} \leq 1$ that is, $x_1 \leq a_{13}$. By $x_3 = x_1 + 1 - a_{13}$ and $x_3 \leq x_2 + 1 - a_{23}$, we have $x_2 \geq x_1 + (a_{23} - a_{13})$. We have verified all relations in the second part of the disjunction in (9) .

When $a_{11} < 1$, $a_{22} < 1$, then condition (ii) of Theorem 3.4 implies $x_2 = x_1 + 1 - a_{12}$, or $x_3 = x_1 + 1 - a_{13}$. Moreover, condition (iii) of Theorem 3.4 gives $x_3 = x_2 + 1 - a_{23}$. We shall consider two cases.

In the first case we assume $x_2 = x_1 + 1 - a_{12}$, $x_3 \leq x_1 + 1 - a_{13}$ and $x_3 = x_2 + 1 - a_{23}$. Combining the two equalities we get $x_3 = x_1 + 2 - (a_{12} + a_{23})$, and using this value in the inequality assumption we get $1 \leq a_{12} - a_{13} + a_{23}$. Thus, all relations in the first part of the disjunction in condition (10) must hold true.

In the second case we assume $x_2 \leq x_1 + 1 - a_{12}$, $x_3 = x_1 + 1 - a_{13}$ and $x_3 = x_2 + 1 - a_{23}$. The two equalities imply $x_2 = x_1 + (a_{23} - a_{13})$. Inserting this value into the inequality assumption we get $a_{12} - a_{13} + a_{23} \le 1$. Thus, all relations in the second part of the disjunction in condition (10) have been verified. □

5. NON-STRICTLY INCREASING EIGENVECTORS

In this section the three-dimensional eigenproblem is investigated for non-strictly increasing eigenvectors. Analogously to the notation introduced in [7] and [9] we denote

$$\mathcal{I}^<(D_{12},3) = \left\{ x \in \mathcal{I}(3);\ x_1 = x_2 < x_3 \right\}, \tag{11}$$

$$\mathcal{I}^<(D_{23},3) = \left\{ x \in \mathcal{I}(3);\ x_1 < x_2 = x_3 \right\}, \tag{12}$$

and

$$\mathcal{F}^<(D_{12},A) = \left\{ x \in \mathcal{I}^<(D_{12},3);\ A \otimes_l x = x \right\}, \tag{13}$$

$$\mathcal{F}^<(D_{23},A) = \left\{ x \in \mathcal{I}^<(D_{23},3);\ A \otimes_l x = x \right\}, \tag{14}$$

for a given matrix $A \in \mathcal{I}(3,3)$.

Proposition 5.1. Let $A \in \mathcal{I}(3,3)$, $x \in \mathcal{I}^<(D_{12},3)$. Then $x \in \mathcal{F}^<(D_{12},A)$ if and only if the following hold

$$1 - a_{13} \ge x_3 - x_1\,, \tag{15}$$

$$1 - a_{23} \ge x_3 - x_1\,, \tag{16}$$

$$x_1 = 0 \quad \text{or} \quad \max(a_{11},a_{12}) = 1 \quad \text{or} \quad 1 - a_{13} = x_3 - x_1\,, \tag{17}$$

$$x_1 = 0 \quad \text{or} \quad \max(a_{21},a_{22}) = 1 \quad \text{or} \quad 1 - a_{23} = x_3 - x_1\,, \tag{18}$$

$$a_{33} = 1\,. \tag{19}$$

P r o o f. Let $x \in \mathcal{F}^<(D_{12},A)$, then $0 \le x_1 = x_2 < x_3 \le x_3$. By (1) in Proposition 3.2, $a_{13} \otimes_l x_3 \le x_1$. Then by definition $0 \le a_{13} + x_3 - 1 \le x_1$ or $a_{13} + x_3 - 1 \le 0 \le x_1$, implies that $1 - a_{13} \ge x_3 - x_1$ holds in either case. Similarly, $1 - a_{23} \ge x_3 - x_2 = x_3 - x_1$. By (2) in Proposition 3.2 we have, $a_{11} \otimes_l x_1 = x_1$ or $a_{12} \otimes_l x_2 = x_1$ or $a_{13} \otimes_l x_3 = x_1$. The first two of the equalities are trivially fulfilled if $x_1 = x_2 = 0$. On the other hand, if $x_1 = x_2 > 0$, then we get $a_{11} = 1$ or $a_{12} = 1$ or $a_{13} + x_3 - 1 = x_1$. That is, $\max(a_{11},a_{12}) = 1$ or $1 - a_{13} = x_3 - x_1$. By the same argument we get $x_1 = 0$ or $\max(a_{21},a_{22}) = 1$ or $1 - a_{23} = x_3 - x_1$. In view of the assumption $x_1 = x_2 < x_3$, equation (19) follows from (2) in Proposition 3.2.

Conversely, conditions (15) and (16), in view of Proposition 2 are equivalent to $a_{13} \otimes_l x_3 \le x_1$ and $a_{23} \otimes_l x_3 \le x_1$ respectively. Three conditions (17), (18) and (19) together imply condition (2) in Proposition 3.2. Hence $x \in \mathcal{F}^<(D_{12},A)$. □

Proposition 5.2. Let $A \in \mathcal{I}(3,3)$, $x \in \mathcal{I}^<(D_{23},3)$. Then $x \in \mathcal{F}^<(D_{23},A)$ if and only if the following hold

$$1 - \max(a_{12},a_{13}) \ge x_2 - x_1\,, \tag{20}$$

$$x_1 = 0 \quad \text{or} \quad a_{11} = 1 \quad \text{or} \quad 1 - \max(a_{12},a_{13}) = x_2 - x_1\,, \tag{21}$$

$$\max(a_{22},a_{23}) = 1\,, \tag{22}$$

$$\max(a_{32},a_{33}) = 1\,. \tag{23}$$

Proof. Let $x \in \mathcal{I}^{<}(D_{23}, 3)$ then, $0 \leq x_1 < x_2 = x_3 \leq 1$. Similarly as in previous proof, it is easy to show that conditions (20), (21), (22), (23) are equivalent to conditions in Proposition 3.2. □

Theorem 5.3. Let $A \in \mathcal{I}(3, 3)$. Then $\mathcal{F}^{<}(D_{12}, A) \neq \emptyset$ if and only if the following conditions are satisfied

(i) $a_{13} < 1\,, \quad a_{23} < 1\,,$

(ii) $a_{33} = 1\,.$

Proof. Let there exist $x \in \mathcal{F}^{<}(D_{12}, A)$. In view of (15) of Proposition 5.1, $a_{13} - 1 \leq x_1 - x_3 < 0$ implies that $a_{13} < 1$. Similarly we have $a_{23} < 1$. Condition (ii) is the same as condition (19).

Conversely, suppose that conditions (i), (ii) hold true. We show that there exists $x = (x_1, x_2, x_3) \in \mathcal{F}^{<}(D_{12}, A)$. The choice of $x_1 = x_2 = 0$ and $x_3 = \min(1 - a_{13}, 1 - a_{23})$ satisfies conditions (15) − (19) of Proposition 5.1. Hence $\mathcal{F}^{<}(D_{12}, A) \neq \emptyset$. □

Theorem 5.4. Let $A \in \mathcal{I}(3, 3)$. Then $\mathcal{F}^{<}(D_{23}, A) \neq \emptyset$ if and only if the following conditions are satisfied

(i) $a_{12} < 1\,, \quad a_{13} < 1\,,$

(ii) $\max(a_{22}, a_{23}) = 1\,,$

(iii) $\max(a_{32}, a_{33}) = 1\,.$

Proof. The proof is analogous to the proof of Theorem 5.3. In the converse part we put $x_1 = 0$, and take an arbitrary value $x_2 = x_3$ in the interval $\langle 1 - \max(a_{12}, a_{13}), 1\rangle$. □

Theorem 5.5. Let $A \in \mathcal{I}(3, 3)$ and let conditions (i)–(ii) of Theorem 5.3 be satisfied. Denoting $\mathcal{F}_0^{<}(D_{12}, A) = \{x \in \mathcal{F}^{<}(D_{12}, A);\, x_1 = 0\}$, $\mathcal{F}_1^{<}(D_{12}, A) = \{x \in \mathcal{F}^{<}(D_{12}, A);\, x_1 > 0\}$ we have $\mathcal{F}^{<}(D_{12}, A) = \mathcal{F}_0^{<}(D_{12}, A) \cup \mathcal{F}_1^{<}(D_{12}, A)$, where $\mathcal{F}_0^{<}(D_{12}, A)$ consists exactly of all vectors $x = (x_1, x_2, x_3) \in \mathcal{I}(3)$ satisfying

$$0 = x_1 = x_2 < x_3 \leq 1 - \max(a_{13}, a_{23})\,, \tag{24}$$

and $\mathcal{F}_1^{<}(D_{12}, A)$ consists exactly of all vectors $x = (x_1, x_2, x_3) \in \mathcal{I}(3)$ satisfying $0 < x_1 = x_2 < x_3$ and conditions

$$\text{if } \max(a_{11}, a_{12}) = 1,\ \max(a_{21}, a_{22}) = 1, \text{ then } 1 - \max(a_{13}, a_{23}) \geq x_3 - x_1\,, \tag{25}$$
$$\text{if } \max(a_{11}, a_{12}) < 1,\ \max(a_{21}, a_{22}) = 1, \text{ then } 1 - a_{23} \geq 1 - a_{13} = x_3 - x_1\,, \tag{26}$$
$$\text{if } \max(a_{11}, a_{12}) = 1,\ \max(a_{21}, a_{22}) < 1, \text{ then } 1 - a_{13} \geq 1 - a_{23} = x_3 - x_1\,, \tag{27}$$
$$\text{if } \max(a_{11}, a_{12}) < 1,\ \max(a_{21}, a_{22}) < 1, \text{ then } 1 - a_{13} = 1 - a_{23} = x_3 - x_1\,. \tag{28}$$

Proof. Let $A \in \mathcal{I}(3,3)$ satisfy conditions (i) – (ii) of Theorem 5.3 and let $x = (0,0,x_3) \in \mathcal{I}^<(D_{12},3)$ satisfy condition (24). It is easy to see that conditions in Proposition 5.1 are fulfilled, i.e. $x \in \mathcal{F}^<(D_{12},A)$. Conversely, if $x \in \mathcal{F}_0^<(D_{12},A)$, then condition (24) follows from conditions (15) and (16).

Let us assume now that a vector $x = (x_1,x_2,x_3) \in \mathcal{I}(3)$ satisfies $0 < x_1 = x_2 < x_3$ and conditions (25) – (28). Then $x \in \mathcal{I}^<(D_{12},3)$ and conditions (25) – (28) imply conditions (15) – (18) in Proposition 5.1. The condition (19) of Proposition 5.1 is identical with (ii) in Theorem 5.3. Hence $x = (x_1,x_2,x_3) \in \mathcal{F}^<(D_{12},A)$, in view of Proposition 5.1.

Conversely, let $x \in \mathcal{F}_1^<(D_{12},A)$. Then $x_1 > 0$ and conditions (15) – (19) of Proposition 5.1 are satisfied. Conditions (25) – (28) then follow immediately. □

Theorem 5.6. Let $A \in \mathcal{I}(3,3)$ and let conditions (i) – (ii) of Theorem 5.4 be satisfied. Denoting $\mathcal{F}_0^<(D_{23},A) = \{x \in \mathcal{F}^<(D_{23},A);\, x_1 = 0\}$, $\mathcal{F}_1^<(D_{23},A) = \{x \in \mathcal{F}^<(D_{23},A);\, x_1 > 0\}$ we have $\mathcal{F}^<(D_{23},A) = \mathcal{F}_0^<(D_{23},A) \cup \mathcal{F}_1^<(D_{23},A)$, where $\mathcal{F}_0^<(D_{23},A)$ consists exactly of all vectors $x = (x_1,x_2,x_3) \in \mathcal{I}(3)$ satisfying

$$0 = x_1 < x_2 = x_3 \leq 1 - \max(a_{12},a_{13})\,, \tag{29}$$

and $\mathcal{F}_1^<(D_{23},A)$ consists exactly of all vectors $x = (x_1,x_2,x_3) \in \mathcal{I}(3)$ satisfying $0 < x_1 < x_2 = x_3$ and conditions

$$\text{if } a_{11} = 1, \text{ then } 1 - \max(a_{12},a_{13}) \geq x_2 - x_1\,, \tag{30}$$

$$\text{if } a_{11} < 1, \text{ then } 1 - \max(a_{12},a_{13}) = x_2 - x_1\,. \tag{31}$$

Proof. The proof is analogous to the proof of Theorem 5.5. □

6. EXAMPLES

In this section the above considerations will be illustrated by computing the complete eigenspace in max-Ł fuzzy algebra of two given three-dimensional matrices.

Example 6.1. Let us consider the matrix

$$A = \begin{pmatrix} 0.6 & 0.8 & 0.3 \\ 0.5 & 0.9 & 0.4 \\ 0.3 & 0.7 & 1 \end{pmatrix}$$

Matrix A satisfies the conditions (i), (ii) and (iii) of Theorem 4.1, hence $\mathcal{F}^<(A) \neq \emptyset$. By Theorem 4.2, the strictly increasing eigenspace of A is $\mathcal{F}^<(A) = \mathcal{F}_0^<(A) \cup \mathcal{F}_1^<(A)$. Since $a_{22} = 0.9 < 1$, then by condition (5) of Theorem 4.2, $\mathcal{F}_0^<(A)$ consists exactly of the vectors $(0,x_2,x_3) \in \mathcal{I}(3)$ fulfilling the conditions

$$0 < x_2 \leq \min(1 - a_{12}, a_{23} - a_{13}),\ x_2 < 1 - a_{13},\ x_3 = x_2 + (1 - a_{23}), \text{ i.e.}$$
$$0 < x_2 \leq \min(1 - 0.8, 0.4 - 0.3),\ x_2 < 1 - 0.3,\ x_3 = x_2 + (1 - 0.4)\,.$$

Hence,

$$\mathcal{F}_0^<(A) = \big\{(0,x_2,x_3) \in \mathcal{I}(3) : 0 < x_2 \leq 0.1,\, x_3 = x_2 + 0.6\big\}\,.$$

Since $a_{11} = 0.6 < 1$, $a_{22} = 0.9 < 1$, and $a_{12} - a_{13} + a_{23} = 0.8 - 0.3 + 0.4 = 0.9 \le 1$, then we get in a similar way by the condition (10) of Theorem 4.2

$$\mathcal{F}_1^<(A) = \big\{(x_1, x_2, x_3) \in \mathcal{I}(3) : 0 < x_1 \le 0.3, x_2 = x_1 + 0.1, x_3 = x_1 + 0.7\big\}\,.$$

Further, Theorem 5.5 implies that

$$\mathcal{F}^<(D_{12}, A) = \mathcal{F}_0^<(D_{12}, A) = \big\{(0, 0, x_3) \in \mathcal{I}(3) : 0 < x_3 \le 0.6\big\}\,.$$

By Theorem 5.4 we get $\mathcal{F}^<(D_{23}, A) = \emptyset$ and according to Theorem 3.9, A has exactly one constant eigenvector $(0, 0, 0)$. By analogous considerations of all matrices $A_{\varphi\varphi}$ for permutations $\varphi \in P_3$, we get that, in view of Theorem 2.1, A has no other eigenvectors. Summarizing we get

$$\begin{aligned}
\mathcal{F}(A) = \mathcal{F}^\le(A) &= \mathcal{F}^=(A) \cup \mathcal{F}^<(D_{12}, A) \cup \mathcal{F}_0^<(A) \cup \mathcal{F}_1^<(A)\\
&= \big\{(0, 0, 0)\big\} \cup \big\{(0, 0, x_3) \in \mathcal{I}(3) : 0 < x_3 \le 0.6\big\}\\
&\quad \cup \big\{(0, x_2, x_3) \in \mathcal{I}(3) : 0 < x_2 \le 0.1,\ x_3 = x_2 + 0.6\big\}\\
&\quad \cup \big\{(x_1, x_2, x_3) \in \mathcal{I}(3) : 0 < x_1 \le 0.3, x_2 = x_1 + 0.1, x_3 = x_1 + 0.7\big\}\,.
\end{aligned}$$

Example 6.2. In this example we change the entry $a_{22} = 0.9$ to $b_{22} = 1$, and leave the remaining entries unchanged. We start with computing the strictly increasing eigenspace of matrix B

$$B = \begin{pmatrix} 0.6 & 0.8 & 0.3\\ 0.5 & 1 & 0.4\\ 0.3 & 0.7 & 1 \end{pmatrix}$$

Matrix B satisfies conditions (i), (ii), (iii) in Theorem 4.1, hence $\mathcal{F}^<(B) \ne \emptyset$. By Theorem 4.2, the strictly increasing eigenspace of B is $\mathcal{F}^<(B) = \mathcal{F}_0^<(B) \cup \mathcal{F}_1^<(B)$. Since $b_{22} = 1$, then by condition (6) in Theorem 4.2, $\mathcal{F}_0^<(B)$ consists exactly of the vectors $(0, x_2, x_3) \in \mathcal{I}(3)$ fulfilling the conditions

$$\begin{aligned}
&0 < x_2 < \min(1 - b_{12}, 1 - b_{13}),\ x_2 < x_3 \le \min\big(1 - b_{13}, x_2 + (1 - b_{23})\big), \text{ i.e.}\\
&0 < x_2 \le \min(1 - 0.8, 1 - 0.3),\ x_2 < x_3 \le \min\big(1 - 0.3, x_2 + (1 - 0.4)\big)\,.
\end{aligned}$$

Hence,

$$\mathcal{F}_0^<(B) = \big\{(0, x_2, x_3) \in \mathcal{I}(3) : 0 < x_2 \le 0.2,\ x_2 < x_3 \le \min\big(0.7, x_2 + 0.6\big)\big\}\,.$$

Since $b_{11} = 0.6 < 1$, $b_{22} = 1$, then we get in a similar way by the condition (9) of Theorem 4.2 that $\mathcal{F}_1^<(B)$ consists exactly of the vectors $(0, x_2, x_3) \in \mathcal{I}^<(3)$ fulfilling the conditions

$$\begin{aligned}
&0 < x_1 < b_{12},\ x_2 = x_1 + (1 - b_{12})\,,\\
&\qquad\qquad x_3 \le \min\big(x_1 + 2 - (b_{12} + b_{23}), x_1 + (1 - b_{13}), 1\big), \text{ or}\\
&0 < x_1 \le b_{13},\ x_1 + (b_{23} - b_{13}) \le x_2 < x_1 + (1 - b_{12}),\ x_3 = x_1 + (1 - b_{13})\,.
\end{aligned}$$

Thus, $\mathcal{F}_1^<(B) =$

$$\begin{aligned}
&= \big\{(x_1, x_2, x_3) \in \mathcal{I}(3) : 0 < x_1 < 0.8, x_2 = x_1 + 0.2, x_2 < x_3 \le \min(x_1 + 0.7, 1)\big\}\\
&\quad \cup\ \big\{(x_1, x_2, x_3) \in \mathcal{I}(3) : 0 < x_1 \le 0.3, x_1 + 0.1 \le x_2 < x_1 + 0.2, x_3 = x_1 + 0.7\big\}\,.
\end{aligned}$$

By Theorem 3.9, B has exactly one constant eigenvector $(0,0,0)$, by Theorem 5.5 and Theorem 5.6 we get $\mathcal{F}^<(D_{12},B) = \mathcal{F}_0^<(D_{12},B)$, $\mathcal{F}^<(D_{23},B) = \mathcal{F}_0^<(D_{12},B) \cup \mathcal{F}_1^<(D_{23},B)$, where

$$\mathcal{F}_0^<(D_{12},B) = \big\{(0,0,x_3) \in \mathcal{I}(3) : 0 < x_3 \le 0.6\big\}\,,$$

$$\mathcal{F}_0^<(D_{23},B) = \big\{(0,x_2,x_3) \in \mathcal{I}(3) : 0 < x_2 = x_3 \le 0.2\big\}\,,$$

$$\mathcal{F}_1^<(D_{23},B) = \big\{(x_1,x_2,x_3) \in \mathcal{I}(3) : 0 < x_1 \le 0.8, x_2 = x_3 = x_1 + 0.2\big\}\,.$$

Summarizing we get the increasing eigenspace of B in the following form

$$\begin{aligned}
&\mathcal{F}^{\le}(B) = \mathcal{F}^=(B) \cup \mathcal{F}_0^<(D_{12},B) \cup \mathcal{F}_0^<(D_{23},B) \cup \mathcal{F}_1^<(D_{23},B) \cup \mathcal{F}_0^<(B) \cup \mathcal{F}_1^<(B)\\
&= \big\{(0,0,0)\big\} \cup \big\{(0,0,x_3) \in \mathcal{I}(3) : 0 < x_3 \le 0.6\big\}\\
&\quad \cup \big\{(0,x_2,x_3) \in \mathcal{I}(3) : 0 < x_2 = x_3 \le 0.2\big\}\\
&\quad \cup \big\{(x_1,x_2,x_3) \in \mathcal{I}(3) : 0 < x_1 \le 0.8, x_2 = x_3 = x_1 + 0.2\big\}\\
&\quad \cup \big\{(0,x_2,x_3) \in \mathcal{I}(3) : 0 < x_2 \le 0.2,\ x_2 < x_3 \le \min\big(0.7, x_2 + 0.6\big)\big\}\\
&\quad \cup \big\{(x_1,x_2,x_3) \in \mathcal{I}(3) : 0 < x_1 < 0.8, x_2 = x_1 + 0.2, x_2 < x_3 \le \min(x_1 + 0.7, 1)\big\}\\
&\quad \cup \big\{(x_1,x_2,x_3) \in \mathcal{I}(3) : 0 < x_1 \le 0.3, x_1 + 0.1 \le x_2 < x_1 + 0.2, x_3 = x_1 + 0.7\big\}\,.
\end{aligned}$$

Further eigenvectors of matrix B will be found using Theorem 2.1 with all possible permutations in P_3. E. g., applying permutation $\varphi = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ to rows and columns of B we get matrix $B_{\varphi\varphi}$ which satisfies conditions (i), (ii) and (iii) of Theorem 4.1

$$B_{\varphi\varphi} = \begin{pmatrix} 0.6 & 0.3 & 0.8 \\ 0.3 & 1 & 0.7 \\ 0.5 & 0.4 & 1 \end{pmatrix}$$

For simplicity, we use the notation in which the permuted vector x_φ is denoted as $y = (y_1,y_2,y_3) = (x_{\varphi(1)}, x_{\varphi(2)}, x_{\varphi(3)}) = (x_1,x_3,x_2)$. Analogously as above we compute

$$\begin{aligned}
\mathcal{F}_0^<(B_{\varphi\varphi}) &= \big\{(0,y_2,y_3) \in \mathcal{I}(3) : 0 < y_2 < \min(1-a_{12}, 1-a_{13}),\\
&\qquad y_2 < y_3 \le \min\big(1-a_{13}, y_2 + (1-a_{23})\big)\big\}\,,\\
\mathcal{F}_1^<(B_{\varphi\varphi}) &= \big\{(y_1,y_2,y_3) \in \mathcal{I}(3) : 0 < y_1 < a_{12},\ y_2 = y_1 + (1-a_{12}),\\
&\qquad y_3 \le \min\big(y_1 + 2 - (a_{12}+a_{23}), y_1 + (1-a_{13}), 1\big)\big\}\\
&\quad \cup \big\{(y_1,y_2,y_3) \in \mathcal{I}(3) : 0 < y_1 \le a_{13},\ y_1 + (a_{23}-a_{13}) \le y_2 < y_1 + (1-a_{12}),\\
&\qquad y_3 = y_1 + (1-a_{13})\big\}\,,\\
\mathcal{F}_0^<(D_{12},B_{\varphi\varphi}) &= \big\{(0,0,y_3) \in \mathcal{I}(3) : 0 = y_1 = y_2 < y_3 \le 1 - \max(a_{13},a_{23})\big\}\,,\\
\mathcal{F}_1^<(D_{12},B_{\varphi\varphi}) &= \big\{(y_1,y_2,y_3) \in \mathcal{I}(3) : 0 < y_1 = y_2 < y_3,\ 1 - a_{23} \ge 1 - a_{13} = y_3 - y_1\big\}\,,\\
\mathcal{F}_0^<(D_{23},B_{\varphi\varphi}) &= \big\{(0,y_2,y_3) \in \mathcal{I}(3) : 0 = y_1 < y_2 = y_3 \le 1 - \max(a_{12},a_{13})\big\}\,,\\
\mathcal{F}_1^<(D_{23},B_{\varphi\varphi}) &= \big\{(y_1,y_2,y_3) \in \mathcal{I}(3) : 0 < y_1 < y_2 = y_3,\ 1 - \max(a_{12},a_{13}) = y_2 - y_1\big\}\,,\\
\mathcal{F}^=(B_{\varphi\varphi}) &= \big\{(0,0,0)\big\}\,.
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}_0^<(B_{\varphi\varphi}) &= \big\{(0, y_2, y_3) \in \mathcal{I}(3) : 0 < y_2 < \min(1-0.3, 1-0.8),\\
&\qquad y_2 < y_3 \le \min\big(1-0.8, y_2 + (1-0.7)\big)\big\}\,,\\
\mathcal{F}_1^<(B_{\varphi\varphi}) &= \big\{(y_1, y_2, y_3) \in \mathcal{I}(3) : 0 < y_1 < 0.3,\ y_2 = y_1 + (1-0.3),\\
&\qquad y_3 \le \min\big(y_1 + 2 - (0.3+0.7), y_1 + (1-0.8), 1\big)\big\}\\
&\quad \cup \big\{(y_1, y_2, y_3) \in \mathcal{I}(3) : 0 < y_1 \le 0.8,\ y_1 + (0.7-0.8) \le y_2 < y_1 + (1-0.3),\\
&\qquad y_3 = y_1 + (1-0.8)\big\}\,,\\
\mathcal{F}_0^<(D_{12}, B_{\varphi\varphi}) &= \big\{(0, 0, y_3) \in \mathcal{I}(3) : 0 = y_1 = y_2 < y_3 \le 1 - \max(0.8, 0.7)\big\}\,,\\
\mathcal{F}_1^<(D_{12}, B_{\varphi\varphi}) &= \big\{(y_1, y_2, y_3) \in \mathcal{I}(3) : 0 < y_1 = y_2 < y_3,\ 1-0.7 \ge 1-0.8 = y_3 - y_1\big\}\,,\\
\mathcal{F}_0^<(D_{23}, B_{\varphi\varphi}) &= \big\{(0, y_2, y_3) \in \mathcal{I}(3) : 0 = y_1 < y_2 = y_3 \le 1 - \max(0.3, 0.8)\big\}\,,\\
\mathcal{F}_1^<(D_{23}, B_{\varphi\varphi}) &= \big\{(y_1, y_2, y_3) \in \mathcal{I}(3) : 0 < y_1 < y_2 = y_3,\ 1 - \max(0.3, 0.8) = y_2 - y_1\big\}\,,\\
\mathcal{F}^=(B_{\varphi\varphi}) &= \big\{(0,0,0)\big\}\,.
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}_0^<(B_{\varphi\varphi}) &= \big\{(0, y_2, y_3) \in \mathcal{I}(3) : 0 < y_2 < 0.2,\ \ y_2 < y_3 \le \min\big(0.2, y_2 + 0.3)\big)\big\}\,,\\
\mathcal{F}_1^<(B_{\varphi\varphi}) &= \big\{(y_1, y_2, y_3) \in \mathcal{I}(3) : 0 < y_1 < 0.3,\ y_2 = y_1 + 0.7,\ y_3 \le \min\big(y_1+1, y_1+0.2, 1\big)\big\}\\
&\quad \cup \big\{(y_1, y_2, y_3) \in \mathcal{I}(3) : 0 < y_1 \le 0.8,\ y_1 + (-0.1) \le y_2 < y_1 + 0.7,\ y_3 = y_1 + 0.2)\big\}\,,\\
\mathcal{F}_0^<(D_{12}, B_{\varphi\varphi}) &= \big\{(0, 0, y_3) \in \mathcal{I}(3) : 0 = y_1 = y_2 < y_3 \le 0.2\big\}\,,\\
\mathcal{F}_1^<(D_{12}, B_{\varphi\varphi}) &= \big\{(y_1, y_2, y_3) \in \mathcal{I}(3) : 0 < y_1 = y_2 < y_3,\ 0.3 \ge 0.2 = y_3 - y_1\big\}\,,\\
\mathcal{F}_0^<(D_{23}, B_{\varphi\varphi}) &= \big\{(0, y_2, y_3) \in \mathcal{I}(3) : 0 = y_1 < y_2 = y_3 \le 0.2\big\}\,,\\
\mathcal{F}_1^<(D_{23}, B_{\varphi\varphi}) &= \big\{(y_1, y_2, y_3) \in \mathcal{I}(3) : 0 < y_1 < y_2 = y_3,\ 0.2 = y_2 - y_1\big\}\,,\\
\mathcal{F}^=(B_{\varphi\varphi}) &= \big\{(0,0,0)\big\}\,.
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}_0^<(B_{\varphi\varphi}) &= \big\{(0, y_2, y_3) \in \mathcal{I}(3) : 0 < y_2 < y_3 \le 0.2)\big\}\,,\\
\mathcal{F}_1^<(B_{\varphi\varphi}) &= \big\{(y_1, y_2, y_3) \in \mathcal{I}(3) : 0 < y_1 \le 0.8,\ y_1 < y_2 < y_3 = y_1 + 0.2)\big\}\,,\\
\mathcal{F}_0^<(D_{12}, B_{\varphi\varphi}) &= \big\{(0, 0, y_3) \in \mathcal{I}(3) : 0 < y_3 \le 0.2\big\}\,,\\
\mathcal{F}_1^<(D_{12}, B_{\varphi\varphi}) &= \big\{(y_1, y_2, y_3) \in \mathcal{I}(3) : 0 < y_1 = y_2 \le 0.8,\ y_3 = y_1 + 0.2\big\}\,,\\
\mathcal{F}_0^<(D_{23}, B_{\varphi\varphi}) &= \big\{(0, y_2, y_3) \in \mathcal{I}(3) : 0 < y_2 = y_3 \le 0.2\big\}\,,\\
\mathcal{F}_1^<(D_{23}, B_{\varphi\varphi}) &= \big\{(y_1, y_2, y_3) \in \mathcal{I}(3) : 0 < y_1 \le 0.8,\ y_2 = y_3 = y_1 + 0.2\big\}\,,\\
\mathcal{F}^=(B_{\varphi\varphi}) &= \big\{(0,0,0)\big\}\,.
\end{aligned}$$

Coming back to the original notation $x_\varphi = (x_1, x_3, x_2) = (y_1, y_2, y_3)$ we get the permuted increasing eigenspace $\mathcal{F}_\varphi^{\le}(B)$ of matrix B with $x_1 \le x_3 \le x_2$, in the form

$$\begin{aligned}
\mathcal{F}_\varphi^{\le}(B) &= \mathcal{F}_{0\varphi}^<(B_{\varphi\varphi}) \cup \mathcal{F}_{1\varphi}^<(B_{\varphi\varphi}) \cup \mathcal{F}_{0\varphi}^<(D_{12}, B_{\varphi\varphi}) \cup \mathcal{F}_{1\varphi}^<(D_{12}, B_{\varphi\varphi})\\
&\quad \cup \mathcal{F}_{0\varphi}^<(D_{23}, B_{\varphi\varphi}) \cup \mathcal{F}_{1\varphi}^<(D_{23}, B_{\varphi\varphi}) \cup \mathcal{F}_\varphi^=(B_{\varphi\varphi})\,,
\end{aligned}$$

where

$$\mathcal{F}^{<}_{0\varphi}(B_{\varphi\varphi}) = \{(0, x_2, x_3) \in \mathcal{I}(3) : 0 < x_3 < x_2 \le 0.2)\},$$
$$\mathcal{F}^{<}_{1\varphi}(B_{\varphi\varphi}) = \{(x_1, x_2, x_3) \in \mathcal{I}(3) : 0 < x_1 \le 0.8,\ x_1 < x_3 < x_2 = x_1 + 0.2)\},$$
$$\mathcal{F}^{<}_{0\varphi}(D_{12}, B_{\varphi\varphi}) = \{(0, x_2, 0) \in \mathcal{I}(3) : 0 < x_2 \le 0.2\},$$
$$\mathcal{F}^{<}_{1\varphi}(D_{12}, B_{\varphi\varphi}) = \{(x_1, x_2, x_3) \in \mathcal{I}(3) : 0 < x_1 = x_3 \le 0.8,\ x_2 = x_1 + 0.2\},$$
$$\mathcal{F}^{<}_{0\varphi}(D_{23}, B_{\varphi\varphi}) = \{(0, x_2, x_3) \in \mathcal{I}(3) : 0 < x_3 = x_2 \le 0.2\},$$
$$\mathcal{F}^{<}_{1\varphi}(D_{23}, B_{\varphi\varphi}) = \{(x_1, x_2, x_3) \in \mathcal{I}(3) : 0 < x_1 \le 0.8,\ x_3 = x_2 = x_1 + 0.2\},$$
$$\mathcal{F}^{=}_{\varphi}(B_{\varphi\varphi}) = \{(0, 0, 0)\}.$$

If another permutation $\psi = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ is applied to rows and columns of B, then conditions (i), (ii) and (iii) of Theorem 4.1 are satisfied by the obtained matrix

$$B_{\psi\psi} = \begin{pmatrix} 1 & 0.3 & 0.7 \\ 0.3 & 0.6 & 0.8 \\ 0.4 & 0.5 & 1 \end{pmatrix}$$

Analogously we compute in notation $z = (z_1, z_2, z_3) = (x_{\psi(1)}, x_{\psi(2)}, x_{\psi(3)}) = (x_3, x_1, x_2)$

$$\mathcal{F}^{<}_{0}(B_{\psi\psi}) = \{(0, z_2, z_3) \in \mathcal{I}(3) : 0 < z_2 \le \min(1 - a_{12}, a_{23} - a_{13}),\ z_2 < 1 - a_{13},\ z_3 = z_2 + (1 - a_{23})\},$$
$$\mathcal{F}^{<}_{1}(B_{\psi\psi}) = \{(z_1, z_2, z_3) \in \mathcal{I}(3) : z_1 < a_{23},\ z_2 \le \min\big(a_{23}, z_1 + (1 - a_{12}), z_1 + (a_{23} - a_{13})\big),\ z_3 = z_2 + (1 - a_{23})\},$$
$$\mathcal{F}^{<}_{0}(D_{12}, B_{\psi\psi}) = \{(0, 0, z_3) \in \mathcal{I}(3) : 0 = z_1 = z_2 < z_3 \le 1 - \max(a_{13}, a_{23})\},$$
$$\mathcal{F}^{<}_{1}(D_{12}, B_{\psi\psi}) = \{(z_1, z_2, z_3) \in \mathcal{I}(3) : 0 < z_1 = z_2 < z_3,\ 1 - a_{13} \ge 1 - a_{23} = z_3 - z_1\},$$
$$\mathcal{F}^{<}_{0}(D_{23}, B_{\psi\psi}) = \{(0, z_2, z_3) \in \mathcal{I}(3) : 0 = z_1 < z_2 = z_3 \le 1 - \max(a_{12}, a_{13})\},$$
$$\mathcal{F}^{<}_{1}(D_{23}, B_{\psi\psi}) = \{(z_1, z_2, z_3) \in \mathcal{I}(3) : 0 < z_1 < z_2 = z_3,\ 1 - \max(a_{12}, a_{13}) \ge z_2 - z_1\},$$
$$\mathcal{F}^{=}(B_{\psi\psi}) = \{(0, 0, 0)\}.$$

$$\mathcal{F}^{<}_{0}(B_{\psi\psi}) = \{(0, z_2, z_3) \in \mathcal{I}(3) : 0 < z_2 \le \min(1 - 0.3, 0.8 - 0.7),\ z_2 < 1 - 0.7,\ z_3 = z_2 + (1 - 0.8)\},$$
$$\mathcal{F}^{<}_{1}(B_{\psi\psi}) = \{(z_1, z_2, z_3) \in \mathcal{I}(3) : z_1 < 0.8,\ z_2 \le \min\big(0.8, z_1 + (1 - 0.3), z_1 + (0.8 - 0.7)\big),\ z_3 = z_2 + (1 - 0.8)\},$$
$$\mathcal{F}^{<}_{0}(D_{12}, B_{\psi\psi}) = \{(0, 0, z_3) \in \mathcal{I}(3) : 0 = z_1 = z_2 < z_3 \le 1 - \max(0.7, 0.8)\},$$
$$\mathcal{F}^{<}_{1}(D_{12}, B_{\psi\psi}) = \{(z_1, z_2, z_3) \in \mathcal{I}(3) : 0 < z_1 = z_2 < z_3,\ 1 - 0.7 \ge 1 - 0.8 = z_3 - z_1\},$$
$$\mathcal{F}^{<}_{0}(D_{23}, B_{\psi\psi}) = \{(0, z_2, z_3) \in \mathcal{I}(3) : 0 = z_1 < z_2 = z_3 \le 1 - \max(0.3, 0.7)\},$$
$$\mathcal{F}^{<}_{1}(D_{23}, B_{\psi\psi}) = \{(z_1, z_2, z_3) \in \mathcal{I}(3) : 0 < z_1 < z_2 = z_3,\ 1 - \max(0.3, 0.7) \ge z_2 - z_1\},$$
$$\mathcal{F}^{=}(B_{\psi\psi}) = \{(0, 0, 0)\}.$$

$$\begin{aligned}
&\mathcal{F}_0^<(B_{\psi\psi}) = \big\{(0, z_2, z_3) \in \mathcal{I}(3) : 0 < z_2 \le 0.1,\ z_2 < 0.3,\ z_3 = z_2 + 0.2)\big\}\,,\\
&\mathcal{F}_1^<(B_{\psi\psi}) = \big\{(z_1, z_2, z_3) \in \mathcal{I}(3) : z_1 < 0.8,\ z_2 \le \min\big(0.8, z_1 + 0.7, z_1 + 0.1\big),\\
&\qquad\qquad z_3 = z_2 + 0.2)\big\}\,,\\
&\mathcal{F}_0^<(D_{12}, B_{\psi\psi}) = \big\{(0, 0, z_3) \in \mathcal{I}(3) : 0 = z_1 = z_2 < z_3 \le 0.2\big\}\,,\\
&\mathcal{F}_1^<(D_{12}, B_{\psi\psi}) = \big\{(z_1, z_2, z_3) \in \mathcal{I}(3) : 0 < z_1 = z_2 < z_3,\ 0.3 \ge 0.2 = z_3 - z_1\big\}\,,\\
&\mathcal{F}_0^<(D_{23}, B_{\psi\psi}) = \big\{(0, z_2, z_3) \in \mathcal{I}(3) : 0 = z_1 < z_2 = z_3 \le 0.3\big\}\,,\\
&\mathcal{F}_1^<(D_{23}, B_{\psi\psi}) = \big\{(z_1, z_2, z_3) \in \mathcal{I}(3) : 0 < z_1 < z_2 = z_3,\ 0.3 \ge z_2 - z_1\big\}\,,\\
&\mathcal{F}^=(B_{\psi\psi}) = \big\{(0, 0, 0)\big\}\,.
\end{aligned}$$

$$\begin{aligned}
&\mathcal{F}_0^<(B_{\psi\psi}) = \big\{(0, z_2, z_3) \in \mathcal{I}(3) : 0 < z_2 \le 0.1,\ z_3 = z_2 + 0.2)\big\}\,,\\
&\mathcal{F}_1^<(B_{\psi\psi}) = \big\{(z_1, z_2, z_3) \in \mathcal{I}(3) : 0 < z_1 < z_2 \le \min\big(0.8, z_1 + 0.1\big),\ z_3 = z_2 + 0.2)\big\}\,,\\
&\mathcal{F}_0^<(D_{12}, B_{\psi\psi}) = \big\{(0, 0, z_3) \in \mathcal{I}(3) : 0 < z_3 \le 0.2\big\}\,,\\
&\mathcal{F}_1^<(D_{12}, B_{\psi\psi}) = \big\{(z_1, z_2, z_3) \in \mathcal{I}(3) : 0 < z_1 = z_2 < z_3 = z_1 + 0.2\big\}\,,\\
&\mathcal{F}_0^<(D_{23}, B_{\psi\psi}) = \big\{(0, z_2, z_3) \in \mathcal{I}(3) : 0 < z_2 = z_3 \le 0.3\big\}\,,\\
&\mathcal{F}_1^<(D_{23}, B_{\psi\psi}) = \big\{(z_1, z_2, z_3) \in \mathcal{I}(3) : 0 < z_1 < z_2 = z_3 \le z_1 + 0.3\big\}\,,\\
&\mathcal{F}^=(B_{\psi\psi}) = \big\{(0, 0, 0)\big\}\,.
\end{aligned}$$

Coming back to the original notation $x_\psi = (x_3, x_1, x_2) = (z_1, z_2, z_3)$ we get the permuted increasing eigenspace $\mathcal{F}_\psi^\le(B)$ of matrix B with $x_2 \le x_3 \le x_1$, in the form

$$\begin{aligned}
\mathcal{F}_\psi^\le(B) = {}& \mathcal{F}_{0\psi}^<(B_{\psi\psi}) \cup \mathcal{F}_{1\psi}^<(B_{\psi\psi}) \cup \mathcal{F}_{0\psi}^<(D_{12}, B_{\psi\psi}) \cup \mathcal{F}_{1\psi}^<(D_{12}, B_{\psi\psi})\\
&\cup \mathcal{F}_{0\psi}^<(D_{23}, B_{\psi\psi}) \cup \mathcal{F}_{1\psi}^<(D_{23}, B_{\psi\psi}) \cup \mathcal{F}_\psi^=(B_{\psi\psi})\,,
\end{aligned}$$

where

$$\begin{aligned}
&\mathcal{F}_0^<(B_{\psi\psi}) = \big\{(x_1, x_2, 0) \in \mathcal{I}(3) : 0 < x_1 \le 0.1,\ x_2 = x_1 + 0.2)\big\}\,,\\
&\mathcal{F}_1^<(B_{\psi\psi}) = \big\{(x_3, x_1, x_2) \in \mathcal{I}(3) : 0 < x_3 < x_1 \le \min\big(0.8, x_3 + 0.1\big),\ x_2 = x_1 + 0.2)\big\}\,,\\
&\mathcal{F}_0^<(D_{12}, B_{\psi\psi}) = \big\{(0, x_2, 0) \in \mathcal{I}(3) : 0 < x_2 \le 0.2\big\}\,,\\
&\mathcal{F}_1^<(D_{12}, B_{\psi\psi}) = \big\{(x_1, x_2, x_3) \in \mathcal{I}(3) : 0 < x_3 = x_1 < x_2 = x_3 + 0.2\big\}\,,\\
&\mathcal{F}_0^<(D_{23}, B_{\psi\psi}) = \big\{(x_1, x_2, 0) \in \mathcal{I}(3) : 0 < x_1 = x_2 \le 0.3\big\}\,,\\
&\mathcal{F}_1^<(D_{23}, B_{\psi\psi}) = \big\{(x_1, x_2, x_3) \in \mathcal{I}(3) : 0 < x_3 < x_1 = x_2 \le x_3 + 0.3\big\}\,,\\
&\mathcal{F}^=(B_{\psi\psi}) = \big\{(0, 0, 0)\big\}\,.
\end{aligned}$$

It is easy to verify that, of all six possible permutations in P_3, only the identical permutation and φ, ψ are such that the permuted matrix satisfies conditions (i), (ii) and (iii) of Theorem 4.1, hence there are no further permuted strictly increasing eigenvectors of B. By Theorem 5.3 and Theorem 5.4 we can also verify that no further permuted non-strictly increasing eigenvectors exist. As a consequence, the eigenspace of matrix B in max-Ł fuzzy algebra is equal to

$$\mathcal{F}(B) = \mathcal{F}^{\leq}(B) \cup \mathcal{F}^{\leq}_{\varphi}(B) \cup \mathcal{F}^{\leq}_{\psi}(B).$$

CONCLUSIONS

Presented results are part of the research aimed on investigation of stable states of systems with fuzzy transition matrix in max-t fuzzy algebras with various triangular norm t. Stable states correspond to eigenvectors of the transition fuzzy matrix. For matrices in max-min algebra and in max-drast algebra the eigenvectors were described in [7] and in [8].

In this paper we describe the structure of the eigenspace for three-dimensional fuzzy matrices in max-*Ł*ukasiewicz fuzzy algebra. Necessary and sufficient conditions are proved under which the monotone eigenspace of a given matrix is non-empty, which means that the corresponding system has at least one stable state. Further, the structure of the monotone eigenspace is studied and, using simultaneous row and column permutations of the matrix, complete characterization of the general eigenspace structure of a given three-dimensional fuzzy matrix is presented. In other words, all stable states of the corresponding system are described.

More complex cases of eigenspace structure for max-*Ł*ukasiewicz fuzzy matrices with higher dimension will be considered in future research.

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