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# MAX-MIN INTERVAL SYSTEMS OF LINEAR EQUATIONS WITH BOUNDED SOLUTION

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Max-min algebra is an algebraic structure in which classical addition and multiplication are replaced by $\oplus$ and $\otimes$, where $a \oplus b = \max\{a, b\},\ a \otimes b = \min\{a, b\}$.

The notation $\boldsymbol{A} \otimes \boldsymbol{x} = \boldsymbol{b}$ represents an interval system of linear equations, where $\boldsymbol{A} = [\underline{A}, \overline{A}]$, $\boldsymbol{b} = [\underline{b}, \overline{b}]$ are given interval matrix and interval vector, respectively, and a solution is from a given interval vector $\boldsymbol{x} = [\underline{x}, \overline{x}]$. We define six types of solvability of max-min interval systems with bounded solution and give necessary and sufficient conditions for them.

*Keywords:* max-min algebra, interval system, T6-vector, weak T6 solvability, strong T6 solvability, T7-vector, weak T7 solvability, strong T7 solvability

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## 1. INTRODUCTION AND MOTIVATING EXAMPLE

Max-min (fuzzy relational) equations have found a broad area of applications in causal models which emphasize relationships between input and output variables. They are used in diagnosis models [1, 11, 13, 14] or models of nondeterministic systems [15]. Diagnosis models are of particular interest since they cope with uncertainty existing in many real-life situations either concerning medical diagnosis or diagnosis of technical devices. In the simplest formulation we are faced with a space of symptoms and a space of faults . Elements of faults are related with elements of symptoms by means of a fuzzy relation. In this framework $R(x_i, y_j) = r_{ij}$ stands for the degree to which the symptom $x_i$ is related to the fault $y_j$.

As usual, the higher the value of the relation for a certain pair of arguments means the stronger the relevant relationship between the symptom and the fault is. In the situation when a set of symptoms is represented as a fuzzy set $X$ where the degree of membership $a(x_i) = a_i$ refers to the strength of evidence of $i$th symptom, by performing max-min composition $(a \circ R)$ we obtain the fuzzy set $Y$ of faults which indicates degrees of faults $(b(y_j) = b_j)$. In this context we get not only an indication of the fault element in the structure but a list of elements that are fault to a certain degree. The solution of the equation $a \circ R = b$ provides a maximal set of symptoms that produce the given effect (fault).

In practice it may often happen that a given system of max-min linear equations does not have a solution. One of the methods of restoring solvability is to replace the input data by intervals of possible values. The resulting systems are the so-called interval systems of linear equations, for which several solvability concepts can be defined. J. Rohn [12] dealt with solvability of interval systems of linear equations over the classical algebra. An interesting approach to interval computations was published in [4]. In [5] the authors dealt with interval systems of linear equations in max-plus and max-min algebra over the set of integers with added $-\infty, +\infty$. In the max-min algebra and max-plus algebra, interval systems of linear equations have been studied by K. Cechlárová and R. A. Cuninghame–Green [2, 3]. They dealt with the weak, strong and tolerance solvability. In [6, 7, 8], we studied other solvability concepts in the max-plus and max-min algebra. In this paper, we shall deal with interval systems of linear equations with bounded solution.

There is also a motivation coming from applications for the use of interval systems. One of applications is presented in the following example.

**Example 1.1.** Suppose that there are $m$ producers of a new product and $n$ customers (for example warehouses) which are interested to purchase this product. If the price offer of producer $i$ to customer $j$ is $a_{ij}$ and the purchasing power of customer $j$ is $x_j$ then the sale can be realized for a price $\min\{a_{ij}, x_j\}$. Since the producer $i$ wants to sell the product for the maximal price, the price for which the product is sold is $\max_{j\in N} \min\{a_{ij}, x_j\}$. If producer $i$ wants to sell the product for a price $b_i$ (acceptable selling price) we get the equation

$$\max_{j\in N} \min\{a_{ij}, x_j\} = b_i, \tag{1}$$

for each $i \in M$.

In the following we shall write the system of equations of the form (1) in a matrix form using max-min algebra.

## 2. INTERVAL SYSTEMS WITH BOUNDED SOLUTION

Max-min algebra $\mathcal{B}$ is a triple $(B, \oplus, \otimes)$, where $(B, \leq)$ is a bounded linearly ordered set with binary operations *maximum* and *minimum*, denoted by $\oplus$ and $\otimes$, respectively. The least element in $B$ will be denoted by $O$, the greatest one by $I$.

Denote by $M$ and $N$ the sets of indices $\{1, 2, \ldots, m\}$, $\{1, 2, \ldots, n\}$, respectively. The set of all $m\times n$ *matrices* over $B$ is denoted by $B(m, n)$ and the set of all column *n-vectors* over $B$ by $B(n)$.

Operations $\oplus$ and $\otimes$ are extended to matrices and vectors in the same way as in classical algebra. Particularly, for a given matrix $A \in B(m, n)$ and vector $x \in B(n)$ we get $[A \otimes x]_i = \max_{j\in N}\{\min\{a_{ij}, x_j\}\}$.
We extend the ordering $\leq$ to the sets $B(m, n)$ and $B(n)$ as follows:

- for $A, C \in B(m, n):\ A \leq C$ if $a_{ij} \leq c_{ij}$ for each $i \in M,\ j \in N$,
- for $x, y \in B(n):\ x \leq y$ if $x_j \leq y_j$ for each $j \in N$.

It is easy to see that for each $A, C \in B(m, n)$ and for each $x, y \in B(n)$ the implication

$$\text{if } A \leq C \text{ and } x \leq y, \text{ then } A \otimes x \leq C \otimes y$$

holds true. We call this property the *monotonicity of* $\otimes$.

In max-min algebra we can rewrite the system of equations (1) in the form

$$A \otimes x = b, \tag{2}$$

which represents a *system of max-min linear equations.*

In Example 1.1, the values $a_{ij},\ x_j$ and $b_i$ may be not exact, but given by intervals of possible values $[\underline{a}_{ij}, \overline{a}_{ij}]$, $[\underline{x}_j, \overline{x}_j]$ and $[\underline{b}_i, \overline{b}_i]$. Similarly to [2, 4, 6, 7] we define an *interval matrix* $\boldsymbol{A}$ and *interval vectors* $\boldsymbol{b}, \boldsymbol{x}$ as follows:

$$\begin{aligned}
\boldsymbol{A} &= [\underline{A}, \overline{A}] = \left\{ A \in B(m,n);\ \underline{A} \le A \le \overline{A} \right\},\\
\boldsymbol{b} &= [\underline{b}, \overline{b}] = \left\{ b \in B(n);\ \underline{b} \le b \le \overline{b} \right\},\\
\boldsymbol{x} &= [\underline{x}, \overline{x}] = \{ x \in B(n);\ \underline{x} \le x \le \overline{x} \}.
\end{aligned}$$

Denote by

$$\boldsymbol{A} \otimes \boldsymbol{x} = \boldsymbol{b} \tag{3}$$

the set of all systems of max-min linear equations of the form (2) where $A \in \boldsymbol{A},\ b \in \boldsymbol{b}$ and $x \in \boldsymbol{x}$. We shall call (3) a *max-min interval system of linear equations with bounded solution.*

A special case of interval system (3) is an interval system in the form

$$\boldsymbol{A} \otimes x = \boldsymbol{b} \tag{4}$$

which represents the set of all systems of linear max-min equations of the form (2) where $A \in \boldsymbol{A},\ b \in \boldsymbol{b}$ and $x \in B(n)$.

## 3. SOLVABILITY CONCEPTS

We can define several conditions which the given interval system is required to fulfill. According to them we shall define several solvability concepts. Table 1 contains the list of all up to now studied types of solvability of (3) in max-min algebra. There are omitted solvability concepts which lead to trivial conditions. We also missed solvability concepts which arise by involving $x$ with quantifier $\exists$ because they have been studied in [2, 3, 6, 7, 8] for interval system (4). Necessary and sufficient conditions for the solvability concepts of interval system (4) can be easily modified for interval system (3).

| **Solvability concept** | **Definition** |
|---|---|
| T1 solvability | $(\exists A \in \boldsymbol{A})(\forall x \in \boldsymbol{x})(\exists b \in \boldsymbol{b}) : A \otimes x = b$ |
| T2 solvability | $(\forall x \in \boldsymbol{x})(\exists A \in \boldsymbol{A})(\exists b \in \boldsymbol{b}) : A \otimes x = b$ |
| T3 solvability [9] | $(\forall x \in \boldsymbol{x})(\exists b \in \boldsymbol{b})(\forall A \in \boldsymbol{A}) : A \otimes x = b$ |
| T5 solvability [9] | $(\forall x \in \boldsymbol{x})(\forall A \in \boldsymbol{A})(\exists b \in \boldsymbol{b}) : A \otimes x = b$ |
| weak T6 solvability | $(\exists b \in \boldsymbol{b})(\forall x \in \boldsymbol{x})(\exists A \in \boldsymbol{A}) : A \otimes x = b$ |
| strong T6 solvability | $(\forall b \in \boldsymbol{b})(\forall x \in \boldsymbol{x})(\exists A \in \boldsymbol{A}) : A \otimes x = b$ |
| weak T7 solvability | $(\exists b \in \boldsymbol{b})(\exists A \in \boldsymbol{A})(\forall x \in \boldsymbol{x}) : A \otimes x = b$ |
| strong T7 solvability | $(\forall b \in \boldsymbol{b})(\exists A \in \boldsymbol{A})(\forall x \in \boldsymbol{x}) : A \otimes x = b$ |
| T8 solvability [9] | $(\forall A \in \boldsymbol{A})(\exists b \in \boldsymbol{b})(\forall x \in \boldsymbol{x}) : A \otimes x = b$ |
| T9 solvability [9] | $(\exists b \in \boldsymbol{b})(\forall A \in \boldsymbol{A})(\forall x \in \boldsymbol{x}) : A \otimes x = b$ |

**Table 1.** Solvability concepts of (3).

Each of solvability concepts can be adapted to the situation described in Example 1.1. For instance, the T1 solvability means that there are price offers such that for each customer's purchasing power, producers sell the product for some acceptable selling price; the T2 solvability corresponds to the case that for each customer's power, there are price offers such that producers sell the product for some acceptable selling price; and the strong T6 solvability means that for each acceptable selling prices and customer's purchasing powers there are producer's price offers such that the given selling prices are achieved.

## 4. KNOWN RESULTS

In this section, we introduce necessary and sufficient conditions for the solvability concepts of (3) which have been studied in other papers.

**Theorem 4.1.** [9] Interval system (3) is T5 solvable if and only if

$$\underline{A} \otimes \underline{x} \geq \underline{b}, \tag{5}$$

$$\overline{A} \otimes \overline{x} \leq \overline{b}. \tag{6}$$

For given indices $i \in M,\ j \in N$ denote the vector $x^{(j)} = (x_k^{(j)})$ and the matrix $A^{(ij)} = (a_{kl}^{(ij)})$ as follows:

$$x_k^{(j)} = \begin{cases} \overline{x}_k & \text{for } k = j, \\ \underline{x}_k & \text{otherwise,} \end{cases} \qquad a_{kl}^{(ij)} = \begin{cases} \overline{a}_{kl} & \text{for } k = i,\ l = j, \\ \underline{a}_{kl} & \text{otherwise.} \end{cases}$$

**Theorem 4.2.** [9] Interval system (3) is T3 solvable if and only if interval system (3) is T5 solvable and

$$\underline{A} \otimes x^{(j)} = \overline{A} \otimes x^{(j)} \tag{7}$$

for each $j \in N$.

**Theorem 4.3.** [9] Interval system (3) is T8 solvable if and only if interval system (3) is T5 solvable and

$$A^{(ij)} \otimes \underline{x} = A^{(ij)} \otimes \overline{x} \tag{8}$$

for each $i \in M,\ j \in N$.

**Theorem 4.4.** [9] Interval system (3) is T9 solvable if and only if interval system (3) is T5 solvable and

$$\underline{A} \otimes \underline{x} = \overline{A} \otimes \overline{x}.$$

## 5. T1 AND T2 SOLVABILITY

In this section, we prove necessary and sufficient conditions for the T1 and T2 solvability. Since a necessary and sufficient condition for the T2 solvability was proved in [10], which is not available on-line, we shall introduce Theorem 5.3 with the proof.

We shall use an earlier defined notion of a possible solution.

**Definition 5.1.** A vector $x \in B(n)$ is a possible solution of interval system (4) if there exist $A \in \boldsymbol{A}$ and $b \in \boldsymbol{b}$ such that $A \otimes x = b$.

**Theorem 5.2.** [2] A vector $x \in B(n)$ is a possible solution of interval system (4) if and only if

$$\overline{A} \otimes x \geq \underline{b}, \tag{9}$$

$$\underline{A} \otimes x \leq \overline{b}. \tag{10}$$

It is clear that a possible solution can be defined in the same manner for interval system (3) and Theorem 5.2 holds true in this case, too.

**Theorem 5.3.** [10] Interval system (3) is T2 solvable if and only if

$$\overline{A} \otimes \underline{x} \geq \underline{b}. \tag{11}$$

$$\underline{A} \otimes \overline{x} \leq \overline{b}, \tag{12}$$

Proof. According to Definition 5.1 interval system (3) is T2 solvable if and only if each vector $x \in \boldsymbol{x}$ is a possible solution of (3). Inequality (9) is fulfilled for each $x \in \boldsymbol{x}$ if and only if (9) holds true for $\underline{x}$, hence we get inequality (11). Similarly we get (12). □

**Lemma 5.4.** Interval system (3) is T1 solvable if and only if there exists a matrix $A \in \boldsymbol{A}$ such that

$$A \otimes \overline{x} \leq \overline{b}, \tag{13}$$

$$A \otimes \underline{x} \geq \underline{b}. \tag{14}$$

Proof. The T1 solvability means that there exists $A \in \boldsymbol{A}$ such that $A \otimes x \in [\underline{b}, \overline{b}]$ for each $x \in \boldsymbol{x}$, i.e., $A \otimes x \geq \underline{b}$ and $A \otimes x \leq \overline{b}$ for each $x \in \boldsymbol{x}$. The first (second) inequality is satisfied for each $x \in \boldsymbol{x}$ if and only if it holds for $\underline{x}$ ($\overline{x}$), which is equivalent to the system of inequalities (13), (14). □

**Theorem 5.5.** Interval system (3) is T1 solvable if and only if interval system (3) is T2 solvable.

Proof. It is easy to see, that the T1 solvability implies the T2 solvability.
For the converse implication suppose that interval system (3) is T2 solvable. We shall construct the matrix $A^* \in \boldsymbol{A}$ which satisfies the system of inequalities (13), (14). For any $i \in M$ denote $N_i = \{j \in N : \overline{x}_j > \overline{b}_i\}$.

Define the matrix $A^*$ as follows:

$$a^*_{ij} = \begin{cases} \min\{\overline{a}_{ij}, \overline{b}_i\} & \text{for } i \in M,\ j \in N_i, \\ \overline{a}_{ij} & \text{for } i \in M,\ j \notin N_i. \end{cases} \tag{15}$$

First we show that $A^* \in \boldsymbol{A}$. The inequality $A^* \leq \overline{A}$ trivially holds. Using (12) we get the inequality $\underline{a}_{ij} \otimes \overline{x}_j \leq \overline{b}_i$ for each $i \in M,\ j \in N$. Since for $j \in N_i$ we have $\overline{x}_j > \overline{b}_i$, we get $\underline{a}_{ij} \leq \overline{b}_i$. Thus $A^* \geq \underline{A}$.

Now we shall prove inequality (13) :
For $j \in N_i$ we have $a^*_{ij} \leq \overline{b}_i$ which implies $a^*_{ij} \otimes \overline{x}_j \leq \overline{b}_i$ and consequently $\bigoplus_{j \in N_i} a^*_{ij} \otimes \overline{x}_j \leq \overline{b}_i$.

For $j \notin N_i$ the inequality $\overline{x}_j \leq \overline{b}_i$ implies $a^*_{ij} \otimes \overline{x}_j \leq \overline{b}_i$. Hence $\bigoplus_{j \notin N_i} a^*_{ij} \otimes \overline{x}_j \leq \overline{b}_i$.

We have $[A^* \otimes \overline{x}]_i = (\bigoplus_{j \in N_i} a^*_{ij} \otimes \overline{x}_j) \oplus (\bigoplus_{j \notin N_i} a^*_{ij} \otimes \overline{x}_j) \leq \overline{b}_i$ for each $i \in M$. Thus the matrix $A^*$ satisfies inequality (13).

Inequality (14) follows from the following:
From inequality (11) it follows that for each $i \in M$ there exists $r \in N$ such that $\overline{a}_{ir} \otimes \underline{x}_r \geq \underline{b}_i$ and consequently $\overline{a}_{ir} \geq \underline{b}_i$ and $\underline{x}_r \geq \underline{b}_i$. Let $i \in M$ be arbitrary, but fixed. According to (15) we have either $a^*_{ir} = \overline{a}_{ir}$ or $a^*_{ir} = \overline{b}_i$. In both cases we get $a^*_{ir} \otimes \underline{x}_r \geq \underline{b}_i$, which implies $[A^* \otimes \underline{x}]_i \geq \underline{b}_i$, so $A^* \otimes \underline{x} \geq \underline{b}$.

As the matrix $A^*$ satisfies the system of inequalities (13), (14), interval system (3) is T1 solvable. □

**Remark 5.6.** The matrix $A^*$ defined by (15) is the maximum matrix satisfying the system of inequalities (13), (14).

**Example 5.7.** Let $B = [0, 1]$ and

$$\boldsymbol{A} = \begin{pmatrix} [0.5, 0.7] & [0.2, 0.5] & [0.7, 0.9] \\ [0.4, 1] & [0.8, 0.8] & [0.5, 0.8] \\ [0.3, 0.6] & [0.5, 0.8] & [0.4, 0.5] \end{pmatrix}, \ \boldsymbol{x} = \begin{pmatrix} [0.4, 0.9] \\ [0.2, 0.6] \\ [0.6, 0.7] \end{pmatrix}, \ \boldsymbol{b} = \begin{pmatrix} [0.4, 0.8] \\ [0.3, 0.6] \\ [0.4, 0.6] \end{pmatrix}.$$

First, we check the T2 solvability.

Since $\underline{A} \otimes \overline{x} = (0.7, 0.6, 0.5)^T \leq \overline{b}$ and $\overline{A} \otimes \underline{x} = (0.6, 0.6, 0.5)^T \geq \underline{b}$, the given interval system is T2 solvable. By Theorem 5.5 the given interval system is T1 solvable, too.

Moreover we can construct the matrix $A^*$ satisfying the system of inequalities (13), (14). By (15), we get

$$A^* = \begin{pmatrix} 0.7 & 0.5 & 0.9 \\ 0.6 & 0.8 & 0.6 \\ 0.6 & 0.8 & 0.5 \end{pmatrix}.$$

Since $A^* \otimes \underline{x} = (0.6, 0.6, 0.5)^T \geq \underline{b}$ and $A^* \otimes \overline{x} = (0.7, 0.6, 0.6)^T \leq \overline{b}$ the matrix $A^*$ satisfies the system of inequalities (13), (14).

## 6. T6 AND T7 SOLVABILITY

In this section, we introduce the notions of a T6-vector and T7-vector and bring equivalent conditions for the weak T6, strong T6, weak T7 and strong T7 solvability.

**Definition 6.1.**

a) A vector $b \in \boldsymbol{b}$ is a *T6-vector* of interval system (3) if for each $x \in \boldsymbol{x}$ there exists $A \in \boldsymbol{A}$ such that $A \otimes x = b$.

b) A vector $b \in \boldsymbol{b}$ is a *T7-vector* of interval system (3) if there exists $A \in \boldsymbol{A}$ such that for each $x \in \boldsymbol{x}$ the equality $A \otimes x = b$ holds.

From the above definition it follows that if a vector $b \in \boldsymbol{b}$ is a T7-vector of interval system (3) then the vector $b$ is a T6-vector of interval system (3). In the following we prove that the converse implication holds true, too.

**Theorem 6.2.** If a vector $b \in \boldsymbol{b}$ is a T6-vector of interval system (3) then

$$\underline{A} \otimes \overline{x} \leq b \leq \overline{A} \otimes \underline{x}. \tag{16}$$

Proof. If $b \in \boldsymbol{b}$ is a T6-vector of (3) then for $x = \overline{x}$ there exists a matrix $A \in \boldsymbol{A}$ such that $A \otimes \overline{x} = b$. Then $\underline{A} \otimes \overline{x} \leq A \otimes \overline{x} = b$, so the first inequality in (16) is satisfied.

Similarly, the second inequality in (16) follows from the fact that for $x = \underline{x}$ there exists $C \in \boldsymbol{A}$ such that $C \otimes \underline{x} = b$ and from monotonicity of $\otimes$. □

**Theorem 6.3.** If a vector $b \in \boldsymbol{b}$ fulfills system of inequalities (16) then the vector $b$ is a T7-vector of (3).

Proof. Suppose that a vector $b \in \boldsymbol{b}$ fulfills the system of inequalities (16). We shall construct the matrix $A^*$ such that for each $x \in \boldsymbol{x}$ the equality $A^* \otimes x = b$ holds true.

For any $i \in M$ denote $N_i = \{j \in N : \overline{a}_{ij} \otimes \underline{x}_j \geq b_i\}$. From $\overline{A} \otimes \underline{x} \geq b$ it follows that for each $i \in M$ there exists at least one $j \in N$ such that $j \in N_i$, hence $N_i \neq \emptyset$ for each $i \in M$.

Define the matrix $A^* = (a^*_{ij})$ as follows:

$$a^*_{ij} = \begin{cases} \max\{\underline{a}_{ij}, b_i\} & \text{for } j \in N_i, \\ \underline{a}_{ij} & \text{for } j \notin N_i. \end{cases} \tag{17}$$

For $j \in N_i$ we have $b_i \leq \overline{a}_{ij}$, so $A^* \in \boldsymbol{A}$.

We prove that

- $a^*_{ij} \otimes x_j = b_i$ for $j \in N_i$,
- $a^*_{ij} \otimes x_j \leq b_i$ for $j \notin N_i$

holds true for each $x \in \boldsymbol{x}$.

For $j \in N_i$ we shall distinguish two possibilities:

$$a^*_{ij} = b_i \qquad \text{or} \qquad a^*_{ij} = \underline{a}_{ij} > b_i\ .$$

In the case $a^*_{ij} = b_i$ the inequality $\overline{a}_{ij} \otimes \underline{x}_j \geq b_i$ implies $\underline{x}_j \geq b_i$ and consequently $x_j \geq b_i$ for each $x \in \boldsymbol{x}$ which gives $a^*_{ij} \otimes x_j = b_i \otimes x_j = b_i$.

In the second case the inequality $\underline{A} \otimes \overline{x} \leq b$ implies $\overline{x}_j \leq b_i$. The definition of the set $N_i$ implies the inequality $\underline{x}_j \geq b_i$. We get $b_i \leq \underline{x}_j \leq \overline{x}_j \leq b_i$ which implies $\underline{x}_j = \overline{x}_j = b_i$ and consequently $a^*_{ij} \otimes x_j = \underline{a}_{ij} \otimes b_i = b_i$ for each $x \in \boldsymbol{x}$.

For $j \notin N_i$ the inequality $\underline{a}_{ij} \otimes \overline{x}_j \leq b_i$ implies the inequality $a^*_{ij} \otimes x_j \leq b_i$ for each $x \in \boldsymbol{x}$.

From $\bigoplus_{j \in N_i} a^*_{ij} \otimes x_j = b_i$, $\bigoplus_{j \notin N_i} a^*_{ij} \otimes x_j \leq b_i$ and $N_i \neq \emptyset$ we get
$[A^* \otimes x]_i = \bigoplus_{j \in N_i} (a^*_{ij} \otimes x_j) \oplus \bigoplus_{j \notin N_i} (a^*_{ij} \otimes x_j) = b_i$.

As there exists the matrix $A^*$ such that for each $x \in \boldsymbol{x}$ the equality $A^* \otimes x = b$ holds true, the vector $b$ is a T7-vector of interval system (3). □

**Theorem 6.4.** Let $b \in \boldsymbol{b}$ be an arbitrary vector. The following conditions are equivalent

i) $b$ is a T6-vector of (3),

ii) $\underline{A} \otimes \overline{x} \leq b \leq \overline{A} \otimes \underline{x}$,

iii) $b$ is a T7-vector of (3).

Proof. It follows directly from the above introduced assertions. □

**Theorem 6.5.** Interval system (3) is weakly T6 solvable if and only if interval system (3) is T2 solvable and

$$\underline{A} \otimes \overline{x} \leq \overline{A} \otimes \underline{x}. \tag{18}$$

Proof. First, we prove the necessary condition. The existence of a T6-vector $b \in \boldsymbol{b}$ implies (16) which implies (18). According to the definitions of the T2 and weak T6 solvability the weak T6 solvability implies the T2 solvability.

For the converse implication suppose that interval system (3) is T2 solvable, inequality (18) is satisfied and (3) is not weakly T6 solvable, i. e., there is no vector $b \in \boldsymbol{b}$ fulfilling inequality (16). This means that there exists $i \in M$ such that $[[\underline{A} \otimes \overline{x}]_i, [\overline{A} \otimes \underline{x}]_i] \cap [\underline{b}_i, \overline{b}_i] = \emptyset$. We have two possibilities:

$$[\underline{A} \otimes \overline{x}]_i > \overline{b}_i \qquad \text{or} \qquad [\overline{A} \otimes \underline{x}]_i < \underline{b}_i.$$

In the first case we get a contradiction with inequality (12), the second case results in a contradiction with inequality (11). □

**Theorem 6.6.** Let interval system (3) be weakly T6 solvable. A vector $b$ is a T6-vector of (3) if and only if $b \in [b^*, b^{**}]$ where $b_i^* = \max\{[\underline{A} \otimes \overline{x}]_i, \underline{b}_i\}$, $b_i^{**} = \min\{[\overline{A} \otimes \underline{x}]_i, \overline{b}_i\}$ for each $i \in M$.

Proof. The proof follows from Theorem 6.2 and from the second part of the proof of Theorem 6.5, as $[[\underline{A} \otimes \overline{x}]_i, [\overline{A} \otimes \underline{x}]_i] \cap [\underline{b}_i, \overline{b}_i] = [b_i^*, b_i^{**}]$. □

**Theorem 6.7.** Interval system (3) is weakly T7 solvable if and only if (3) is weakly T6 solvable.

Proof. From Theorem 6.4 if follows that the existence of a T6-vector of (3) is equivalent to the existence of a T7-vector of interval system (3). □

**Theorem 6.8.** Interval system (3) is strongly T6 solvable if and only if

$$\underline{A} \otimes \overline{x} \leq \underline{b}, \tag{19}$$

$$\overline{A} \otimes \underline{x} \geq \overline{b}. \tag{20}$$

Proof. Interval system (3) is strongly T6 solvable if and only if $[\underline{b}_i, \overline{b}_i] \subseteq [[\underline{A} \otimes \overline{x}]_i, [\overline{A} \otimes \underline{x}]_i]$ for each $i \in M$ which is equivalent to the system of inequalities (19), (20). □

**Theorem 6.9.** Interval system (3) is strongly T7 solvable if and only if interval system (3) is strongly T6 solvable.

Proof. According to Theorem 6.4 the set of all T7-vectors is equal to the set of all T6-vectors. Then each vector $b \in \boldsymbol{b}$ is a T6-vector of interval system (3) if and only if each vector $b \in \boldsymbol{b}$ is a T7-vector of interval system (3). □

**Example 6.10.** Let $B = [0, 1]$. Check the solvability concepts for interval system $\boldsymbol{A} \otimes \boldsymbol{x} = \boldsymbol{b}$, where

$$\boldsymbol{A} = \begin{pmatrix} [0.7, 0.8] & [0.5, 0.9] & [0.5, 0.8] \\ [0.4, 0.5] & [0.2, 0.5] & [0.3, 0.4] \\ [0.2, 0.9] & [0.3, 0.8] & [0.3, 0.8] \end{pmatrix}, \ \boldsymbol{x} = \begin{pmatrix} [0.6, 0.6] \\ [0.4, 0.6] \\ [0.5, 0.7] \end{pmatrix}, \ \boldsymbol{b} = \begin{pmatrix} [0.6, 0.7] \\ [0.4, 0.5] \\ [0.4, 0.8] \end{pmatrix}.$$

As $\underline{A} \otimes \overline{x} = (0.6, 0.4, 0.3)^T \leq \overline{b}$ and $\overline{A} \otimes \underline{x} = (0.6, 0.5, 0.6)^T \geq \underline{b}$, the given interval system is T2 solvable and consequently it is T1 solvable.

The inequality $\underline{A} \otimes \overline{x} \leq \overline{A} \otimes \underline{x}$ is satisfied, so the given interval system is weakly T6 solvable with $b^* = (0.6, 0.4, 0.4)^T$ and $b^{**} = (0.6, 0.5, 0.6)^T$. By Theorem 6.7 the given interval system is weakly T7 solvable, too.

We check the strong T6 and strong T7 solvability. Since $\overline{A} \otimes \underline{x} \not\geq \overline{b}$, the given interval system is not strongly T6 solvable and consequently it is not strongly T7 solvable.

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