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GENERALIZED SYNCHRONIZATION AND CONTROL FOR INCOMMENSURATE FRACTIONAL UNIFIED CHAOTIC SYSTEM AND APPLICATIONS IN SECURE COMMUNICATION

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A fractional differential controller for incommensurate fractional unified chaotic system is described and proved by using the Gershgorin circle theorem in this paper. Also, based on the idea of a nonlinear observer, a new method for generalized synchronization (GS) of this system is proposed. Furthermore, the GS technique is applied in secure communication (SC), and a chaotic masking system is designed. Finally, the proposed fractional differential controller, GS and chaotic masking scheme are showed by using numerical and experimental simulations.

Keywords: fractional chaotic systems, fractional differential controller, GS, state observer, Gershgorin circle theorem, pole assignment algorithm, SC, chaotic masking

Classification: 65P20, 94A05, 11T71

1. INTRODUCTION

Chaotic behaviors have been observed and studied in different areas of science and engineering such as mechanics, electronics, physics, medicine, ecology, biology, economy and so on [1, 29, 36]. Due to troubles that may arise from unusual behaviors of a chaotic system, chaos control and synchronization have gained increasing attention in the past few decades. Nowadays, many control techniques have been described and found, such as open-loop control methods, traditional linear and nonlinear control methods, adaptive control methods, optimal control methods and fuzzy control methods [8, 32]. So have different kinds of synchronization: complete synchronization (CS, i. e. Identical synchronization (IS)), phase synchronization (PS), lag synchronization (LS), anticipatory synchronization (AS), GS, multiplexing synchronization (MS) etc. [3, 30, 35]. On the other hand, the chaotic dynamics and synchronization control techniques of fractional differential equations (FDEs) have attracted much attention in recent years. A number of chaotic synchronization methods for FDEs have been developed. However, overviewing these synchronization methods, most of them have concentrated on studying CS. In comparison, the number of GS studied is far fewer than CS. Furthermore, it is well known that CS is difficult to achieve except under ideal conditions, and there always exists parameter mismatches and distortions in the physical world [11, 38, 39]. Therefore, the control and GS for FDEs will become a very important issue.

In recent years, the applications of fractional calculus to physics, engineering and control processing are more interesting and widely. Some fractional PID controller, fractional PI controller, fractional PD controller and fractional lead-lag compensator have been constructed and studied [21, 25, 32]. In [40], the GS of FDEs systems is investigated by using different scaling factors for the system state variables and a new synchronization scheme is obtained. Ref. [15] described an application of differential evolution to the design of fractional-order PID controllers which are involving fractional-order integrator and fractional-order differentiator. A drive-response synchronization method with linear output error feedback is presented for GS of the fractional-order chaotic systems via a scalar transmitted signal in [27]. In some of these works, it is verified that the fractional-order controllers can have better disturbance rejection ratios and less sensitivity to plant parameter variations compared to the traditional controllers. However, the systems are discussed in those articles are ODEs or commensurate FDEs systems [23, 26, 28]. It is an ideal candidate for examining fractional controller for incommensurate FDEs systems.

From the above point of view, we can see that the study of GS and fractional controller for incommensurate FDEs systems is of high practical importance. A fractional controller is constructed based on the fractional differentiator in this paper. The paper is organized as follows. Fractional order derivatives, numerical algorithm for FDE and stability theorem in incommensurate FDEs system are presented in Section 2. In Section 3, the fractional differential controller is designed and proved by using the Gershgorin circle theorem. Based on the nonlinear observer and the pole assignment technique, a GS scheme of the chaotic system is also proposed in this section. The GS method is applied in SC and chaotic masking scheme is presented in Section 4. In Section 5, numerical simulations are provided to illustrate the performance of the proposed control strategy together with GS and SC. Finally, some concluding remarks are presented in the final section.

2. FRACTIONAL CALCULUS

2.1. Basic concepts

The fractional order differentiator can be denoted by a general fundamental operator ${}_aD_t^q$ as a generalization of the differential and integral operators [24], which is defined as follows,

$$
{}_aD_t^q = \begin{cases} \frac{\mathrm{d}^q}{\mathrm{d}t^q}, & R(q) > 0 \\ 1, & R(q) = 0 \\ \int_a^t (\mathrm{d}\tau)^{-q}, & R(q) < 0 \end{cases} \tag{1}
$$

where q is the fractional order which can be a complex number and the constant a is related to the initial conditions. There are two commonly used definitions for the general fractional differentiation and integration, i. e., the Grünwald–Letnikov (GL) and the Riemann Liouville (RL). The GL definition is as:

$$
{}_aD_t^q f(t) = \lim_{h\to 0} \frac{1}{h^q} \sum_{j=0}^{[(t-q)/h]} (-1)^j \begin{pmatrix} q \\ j \end{pmatrix} f(t-jh). \tag{2}
$$

While the RL definition is given by

$$ {}_aD_t^q f(t) = \frac{1}{\Gamma(n-q)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{q-n+1}} \,\mathrm{d}\tau. \tag{3}$$

For $n-1 < q < n$ and $\Gamma(x)$ is the well known Euler's Gamma function. In this paper, we will use another definition of differintegral introduced by Caputo. Caputo's definition can be written as

$$ {}_aD_t^q f(t) = \frac{1}{\Gamma(q-n)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{q-n+1}} \,\mathrm{d}\tau, \quad n-1 < q < n. \tag{4}$$

2.2. Algorithm for FDE

The numerical calculation of a FDE is not simple as that of an ODE. In the literatures of fractional chaos, two approximation methods have been proposed for numerical solution of a fractional differential equation. One is the frequency-domain method [5, 6]and another is the time-domain method based on the predictor-correctors scheme [9, 10]. Here we use a predictor-corrector algorithm for FDEs systems. The brief introduction of this algorithm is as following.

The Cauchy problem

$$ D_t^q x(t) = f(t, x(t)), \quad 0 < t \le T, \qquad x^{(i)}(0) = x_0^{(i)}, \quad i = 0, 1, \cdots, m-1 \tag{5}$$

where $m-1 < q \le m \in \mathbb{N}$, can be transformed into an equivalent Volterra integral equation

$$ x(t) = \sum_{i=0}^{m-1} \frac{t^i}{i!} x_0^{(i)} + \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} f(\tau, x(\tau)) \,\mathrm{d}\tau. \tag{6}$$

Set $h = \frac{T}{N-1}$, $N \in \mathbb{N}$, $t_n = nh$, $n = 0, \cdots N-1$. Then (6) can be discretized as follows

$$ x_h(t_{n+1}) = \sum_{i=0}^{m-1} \frac{t_{n+1}^i}{i!} x_0^{(i)} + \frac{h^q}{\Gamma(q+2)} [f(t_{n+1}, x_h^p(t_{n+1})) + \sum_{j=0}^{n} a_{j,n+1} f(t_j, x_h(t_j))] \tag{7}$$

where

$$ a_{j,n+1} = \begin{cases} n^{q+1} - (n-q)(n+1)^q, & j = 0 \\ (n-j+2)^{q+1} + (n-j)^{q+1} - 2(n-j+1)^{q+1}, & 1 \le j < n \end{cases}$$

$$ x_h^p(t_{n+1}) = \sum_{i=0}^{n-1} \frac{t_{n+1}^i}{i!} x_0^{(i)} + \frac{1}{\Gamma(q)} \sum_{j=0}^{n} b_{j,n+1} f(t_j, x_h(t_j))$$

$$ b_{j \cdot n+1} = \frac{h^q}{q} [(n+1-j)^q - (n-j)^q], \quad 0 \le j \le n.$$

2.3. Stability of FDEs system

Lemma 2.1. Linear incommensurate FDEs system [22]

$$\begin{cases} \frac{\mathrm{d}^{\alpha}X}{\mathrm{d}t^{\alpha}} = AX, X \in R^n, A \in R^{n\times n} \\ \frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}} = \left[\frac{\mathrm{d}^{\alpha_1}}{\mathrm{d}t^{\alpha_1}}, \frac{d^{\alpha_2}}{dt^{\alpha_2}}, \cdots, \frac{d^{\alpha_n}}{dt^{\alpha_n}}\right]^{\mathrm{T}}, 0 < \alpha_i < 1. \end{cases} \tag{8}$$

Let $\alpha_i = \frac{v_i}{u_i}$, $(v_i, u_i) = 1$, $v_i, u_i \in \mathbb{Z}^+$ for $i = 1, 2, \cdots, n$, and assume M to be the lowest common multiple of all the denominator u_i. Define

$$\Delta(\lambda) = diag(\lambda^{M_{\alpha_1}}, \lambda^{M_{\alpha_2}}, \cdots, \lambda^{M_{\alpha_n}}) - A. \tag{9}$$

Then the zero solution of system (8) is globally asymptotically stable in the sense of Lyapunov if all roots λ of equation $\det(\Delta(\lambda)) = 0$ satisfy $|\arg(\lambda)| > \frac{\pi}{2M}$ or $|\arg(\lambda)| > \frac{\Lambda\pi}{2}$, where $\Lambda = \max\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$.

Lemma 2.2. (Gershgorin circle theorem, Varga [34]) Let $A = (a_{ij})_{n\times n} \in \mathbb{C}^{n\times n}$, then the eigenvalue λ lies in one of the circles $|t - a_{ii}| \le \sum_{j\ne i} |a_{ij}|$.

3. SYNCHRONIZATION AND CONTROL

3.1. Design of fractional differential controller

The fractional unified chaotic system [18] can be written

$$\begin{cases} \frac{\mathrm{d}^{\alpha}x_1}{\mathrm{d}t^{\alpha}} = (25a + 10)(x_2 - x_1) \\ \frac{\mathrm{d}^{\beta}x_2}{\mathrm{d}t^{\beta}} = (28 - 35a)x_1 - x_1x_3 + (29a - 1)x_2 \\ \frac{\mathrm{d}^{\gamma}x_3}{\mathrm{d}t^{\gamma}} = x_1x_2 - \frac{(a+8)}{3}x_3 \end{cases} \tag{10}$$

where the parameter $a \in [0, 1]$, $0 < \alpha, \beta, \gamma < 1$. When $a \in [0, 0.8)$, equation (10) represents Lorenz fractional chaotic system, whose strange attractors is shown in Figure 1. When $a = 0.8$, it represents Lü fractional chaotic system. When $a \in (0.8, 1]$, it represents Chen fractional chaotic system. Also system (10) has three equilibrium points $O(0, 0, 0)$, $O_{\pm}(\pm\sqrt{(8 + a)(9 - 2a)}, \pm\sqrt{(8 + a)(9 - 2a)}, 27 - 6a)$.

Denote the equilibrium points $O_{\pm}$ as (η, η, ξ) , and let $\hat{x}_i = x_i - \eta$, ($i = 1, 2$),$\hat{x}_3 = x_3 - \xi$, since the Caputo derivative of a constant is zero, then the system (10) can be written as

$$\begin{cases} \frac{\mathrm{d}^{\alpha}\hat{x}_1}{\mathrm{d}t^{\alpha}} = (25a + 10)(\hat{x}_2 - \hat{x}_1) \\ \frac{\mathrm{d}^{\beta}\hat{x}_2}{\mathrm{d}t^{\beta}} = (1 - 29a)(\hat{x}_1 - \hat{x}_2) - \hat{x}_1\hat{x}_3 - \eta\hat{x}_3 \\ \frac{\mathrm{d}^{\gamma}\hat{x}_3}{\mathrm{d}t^{\gamma}} = \hat{x}_1\hat{x}_2 + \eta(\hat{x}_1 + \hat{x}_2) - \frac{(a+8)}{3}\hat{x}_3. \end{cases} \tag{11}$$

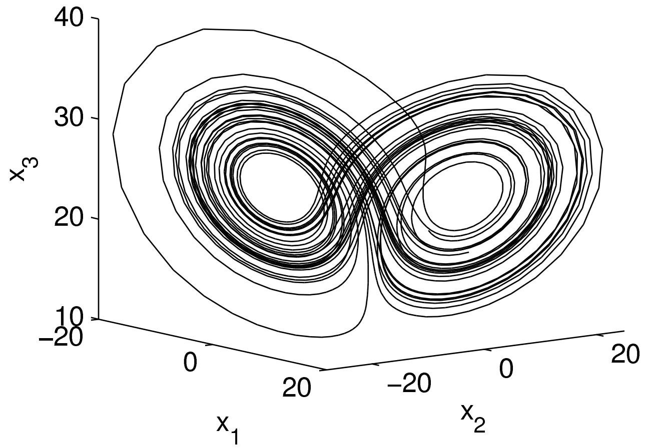

Fig. 1. Attractor of Fractional fractional unified chaotic system with $a = 0.7$, $(\alpha, \beta, \gamma) = (0.85, 0.9, 0.95)$.

Consider the control system

$$\begin{cases} \frac{d^\alpha \hat{x}_1}{dt^\alpha} = (25a+10)(\hat{x}_2 - \hat{x}_1) + u_1 - k_1\hat{x}_1 \\ \frac{d^\beta \hat{x}_2}{dt^\beta} = (1-29a)(\hat{x}_1 - \hat{x}_2) - \hat{x}_1\hat{x}_3 - \eta\hat{x}_3 + u_2 - k_2\hat{x}_2 \\ \frac{d^\gamma \hat{x}_3}{dt^\gamma} = \hat{x}_1\hat{x}_2 + \eta(\hat{x}_1 + \hat{x}_2) - \frac{(a+8)}{3}\hat{x}_3 + u_3 - k_3\hat{x}_3 \\ \frac{d^p u_1}{dt^p} = -u_1 - k_4\hat{x}_1 \\ \frac{d^q u_2}{dt^q} = -u_2 - k_5\hat{x}_2 \\ \frac{d^r u_3}{dt^r} = -u_3 - k_6\hat{x}_3 \end{cases} \tag{12}$$

where $u_i(0) = 0$, $k_i > 0$. Obviously, the system (12) can be transformed into the form of (8), and the coefficient matrix is

$$A(X) = \begin{pmatrix} -(25a+10+k_1) & 25a+10 & 0 & 1 & 0 & 0 \\ (1-29a) - \hat{x}_3 & -(1-29a+k_2) & -\eta & 0 & 1 & 0 \\ \hat{x}_2 + \eta & \eta & -\left(\frac{(a+8)}{3} + k_3\right) & 0 & 0 & 1 \\ -k_4 & & & -1 & & \\ & -k_5 & & & -1 & \\ & & -k_6 & & & -1 \end{pmatrix}.$$

Obviously, $A(X)$ is not a constant matrix, Lemma 2.1 can not be applied to fractional order nonlinear system directly. However, the zero solution of fractional order nonlinear system is asymptotically stable if the real part of all the eigenvalues of the coefficient matrix containing state variables less than zero regardless of the values of state variables (except the origin) [28,29]. Because the variable of chaotic system is bounded, we can let $M_2 = \max\{|(1-29a) - \hat{x}_3| + |\eta|\}$, $M_3 = \max\{|\hat{x}_2 + \eta| + |\eta|\}$, and have

Theorem 3.1. The equilibrium points $O_\pm$ of the incommensurate fractional system (10) are asymptotically stable if $k_1 > 1$, $k_2 > 29a + M_2$, $k_3 > -\frac{(a+5)}{3} + M_3 > 0$, $0 < k_{4,5,6} < 1$.

P r o o f. By Lemma 2.2, the eigenvalue of $A(X)$ lies in the circles

$$|\lambda_1 + (25a + 10 + k_1)| \le 25a + 11, |\lambda_2 + (1 - 29a + k_2)| \le |(1 - 29a) - \hat{x}_3| + |\eta| + 1,$$

$$\left|\lambda_3 + (\frac{(a+8)}{3} + k_3)\right| \le |\hat{x}_2 + \eta| + |\eta| + 1, |\lambda_4 + 1| \le k_4, |\lambda_5 + 1| \le k_5, |\lambda_6 + 1| \le k_6.$$

According to the condition, when $k_1 > 1$, $k_2 > 29a + M_2$, $k_3 > -\frac{(a+5)}{3} + M_3 > 0$, $0 < k_{4,5,6} < 1$, we can see that all the circles lie in the left of imaginary axis, and the real of all the eigenvalue of $A(X)$ less than zero, i. e., all λ satisfy $|\arg(\lambda)| > \frac{\pi}{2} > \frac{\Lambda\pi}{2}$ for $0 < \Lambda = \max\{\alpha, \beta, \gamma, p, q, r\} < 1$. By Lemma 2.1, we can know that the zero solution of the control system (12) is asymptotically stable, i. e. the equilibrium points $O_\pm$ of the incommensurate fractional system (10) is asymptotically stable if $k_1 > 1$, $k_2 > 29a+M_2$, $k_3 > -\frac{(a+5)}{3} + M_3 > 0$, $0 < k_{4,5,6} < 1$. □

Remarks: For the equilibrium point O, the coefficient matrix of the control system is

$$A(X) = \begin{pmatrix} -(25a+10+k_1) & 25a+10 & 0 & 1 & 0 & 0 \\ (28-35a) - x_3 & -(1-29a+k_2) & 0 & 0 & 1 & 0 \\ x_2 & 0 & -\left(\frac{(a+8)}{3} + k_3\right) & 0 & 0 & 1 \\ -k_4 & & & -1 & & \\ & -k_5 & & & -1 & \\ & & -k_6 & & & -1 \end{pmatrix}$$

and the Theorem 3.1 will become

Theorem 3.2. The equilibrium point O of the incommensurate fractional system (10) is asymptotically stable if $k_1 > 1$, $k_2 > 29a + M_2$, $k_3 > -\frac{(a+5)}{3} + M_3 > 0$, $0 < k_{4,5,6} < 1$, where $M_2 = \max\{|(28 - 35a) - x_3|\}$, $M_3 = \max\{|x_2|\}$.

3.2. GS with observer

Rewrite the fractional unified chaotic system as

$$\begin{cases} \frac{\mathrm{d}^\sigma X}{\mathrm{d}t^\sigma} = AX + BF(X) \\ \sigma = (\alpha, \beta, \gamma)^{\mathrm{T}} \end{cases} \tag{13}$$

where, $A = \begin{pmatrix} -25a-10 & 25a+10 & 0 \\ 28-35a & 29a-1 & 0 \\ 0 & 0 & -\frac{a+8}{3} \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 1 \end{pmatrix}$, $F(X) = \begin{pmatrix} x_1x_3 \\ x_1x_2 \end{pmatrix}$, and suppose the system (13) has the output $F(X)$, based on the design idea of nonlinear observer, we configure the transmitted synchronizing signal as

$$G(X) = F(X) + KX \tag{14}$$

where $K \in \mathbb{R}^{2\times 3}$ is a feedback gain matrix to be decided. Construct the following fractional order observer

$$\frac{\mathrm{d}^\sigma Y}{\mathrm{d}t^\sigma} = P^{-1}APY + P^{-1}B(G(X) - KPY) \tag{15}$$

where, $Rank(P) = 3$.

Theorem 3.3. If (A, B) is controllable, and $|\arg(\lambda_i(A - BK))| > \frac{\Lambda\pi}{2}$, $i = 1, 2, 3$, $\Lambda = \max\{\alpha, \beta, \gamma\}$, then $\lim_{t\to\infty} \|PY - X\| = 0$, i. e. system (13) and (15) will approach GS with the observe (15).

Proof. Let $e = PY - X$, then

$$\begin{aligned} \tfrac{\mathrm{d}^\sigma e}{\mathrm{d}t^\sigma} &= P\tfrac{\mathrm{d}^\sigma Y}{\mathrm{d}t^\sigma} - \tfrac{\mathrm{d}^\sigma X}{\mathrm{d}t^\sigma} \\ &= APY + B(G(X) - KPY) - AX - BF(X) \\ &= A(PY - X) - BK(PY - X) = (A - BK)e. \end{aligned} \tag{16}$$

By the Lemma 2.1, the system (16) is globally asymptotically stable under the conditions of Theorem 3.3, i. e. $\lim_{t\to\infty} \|e\| = 0$. □

Remarks: When $P = I$, where I is an identical matrix, the system (13) and (15) are in CS. When $P = -I$, the two systems are anti-synchronized. When $P = kI$ and $k \neq \pm 1$ are constant, the two systems are in GS.

4. SC TECHNIQUES BASED ON CS

The use of CS for secure information transmission implies the presence of at least two unidirectionally coupled identical chaotic oscillators. There are many methods based

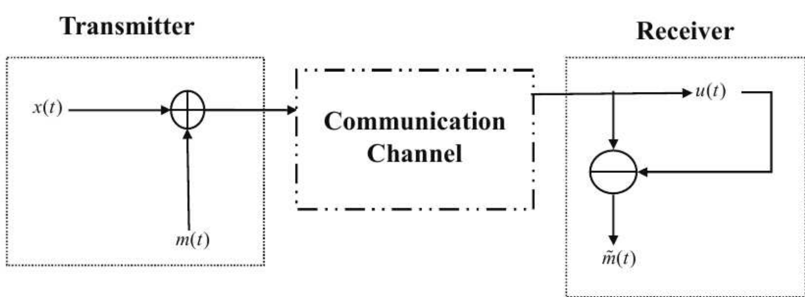

Fig. 2. Basic structure of a typical chaotic masking system.

on this principle, i. e., chaotic masking, chaotic regime switching, the nonlinear mixing of an information signal with a chaotic one, the modulation of control parameters of a transmitting oscillator with a valid information signal, etc. [2, 7, 12, 15, 33]. These methods constitute the basis of many SC techniques, hence, we consider the chaotic masking by using the fractional chaotic system in this section.

Chaotic masking is one of the first and simplest techniques for transmitting information in a secure fashion. A schematic diagram of this method is shown in Figure 2. The information signal $m(t)$ is combined in the summator with a carrier generated by the chaotic system $x(t)$ for transmission through the communication channel. The received signal causes complete chaotic synchronization of the chaotic oscillator $u(t)$ in the receiver, as a result, the dynamics of the receiving oscillator become identical to that of the transmitting one. The detected signal $\tilde{m}(t)$ is produced after passing through the subtractor as the difference between the received signal and the synchronous response of the chaos oscillator in the receiver.

5. NUMERICAL EXPERIMENTS

Simulation 1: Chaos control using fractional controller

Let $a = 0.7$, the fractional unified chaotic system (10) represents Lorenz fractional chaotic system. Take $(\alpha, \beta, \gamma) = (0.85, 0.9, 0.95)$, $(p, q, r) = (0.8, 0.9, 0.8)$, $k_1 = 2$, $k_2 = 100$, $k_3 = 80$, $k_4, 5, 6 = 0.5$, by theorem 3.1, the zero solution of the incommensurate fractional system (12) is asymptotically stable (see Figure 3).

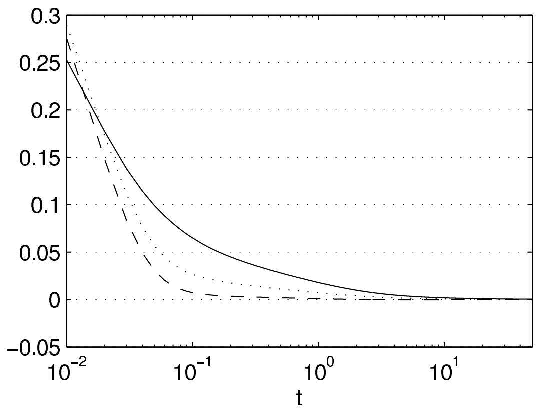

Fig. 3. State diagram of system (12) ($\hat{x}_1$: real line, $\hat{x}_2$: dash line, $\hat{x}_3$: dot line).

Simulation 2: GS of the incommensurate fractional unified chaotic system using observe

Let the closed-loop poles at $[-2, -1, -2]$, by the pole assignment algorithm of multi-input system, we can obtain a feedback gain matrix $K = \begin{pmatrix} -28.0727 & 5.2 & 0 \\ 0 & 0 & -0.9 \end{pmatrix}$. Denote

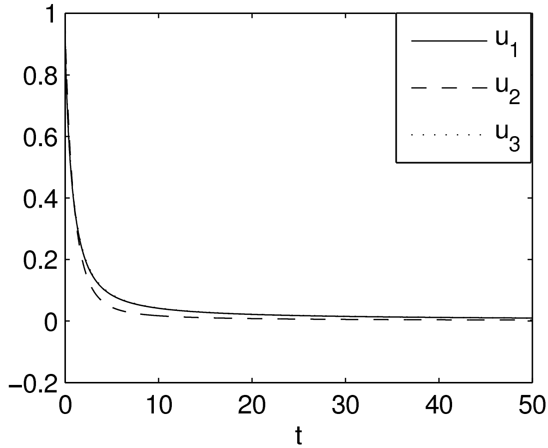

Fig. 4. Fractional controller u.

$P = 0.3I$, and I is an identity matrix, from Figures 6 to 8, we can see that the state X equals PY.

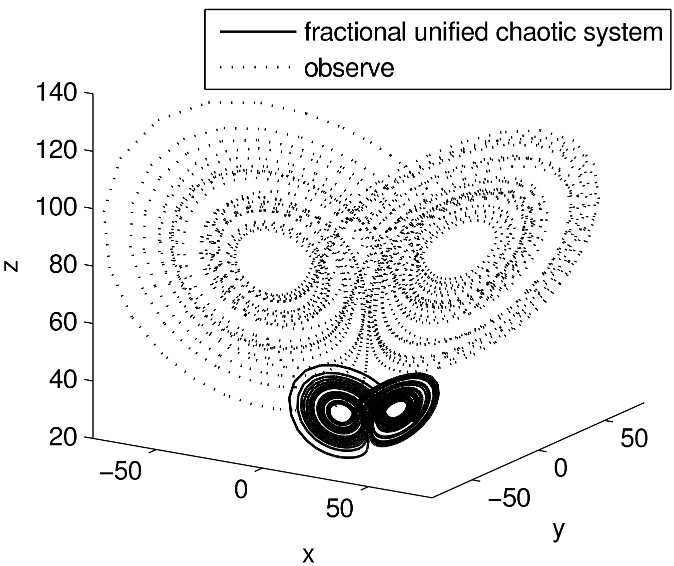

Fig. 5. GS of system (13) and the observe (15).

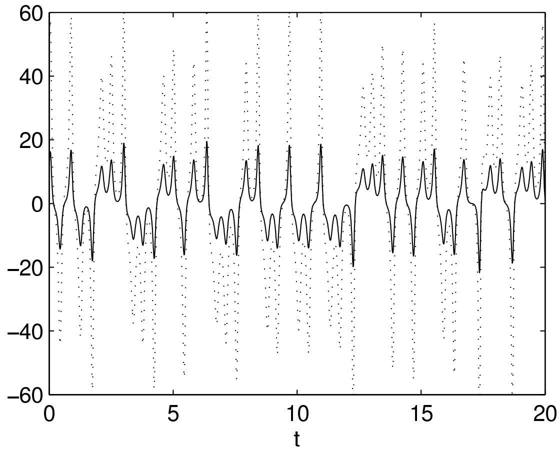

Fig. 6. State diagram of system (13) and (15), $X_1 - Y_1$ (X_1: real line, Y_1: dot line).

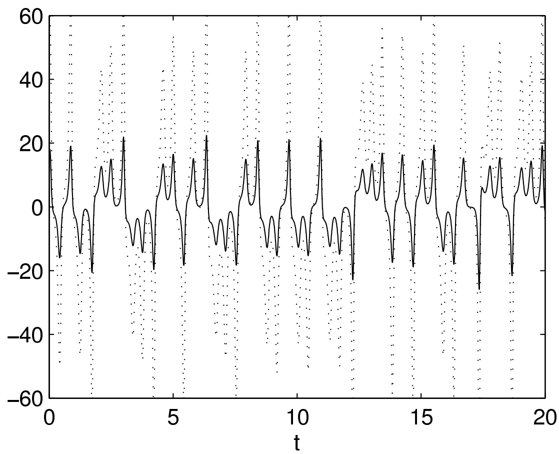

Fig. 7. State diagram of system (13) and (15), $X_2 - Y_2$ (X_2: real line, Y_2: dot line).

Simulation 3: Applications of GS in SC

Let the information signal $m(t) = \frac{1}{2}\cos t + \frac{1}{5}\sin\frac{t}{5}$, The synchronization of transmitted signal and mixed signal see Figure 13.

6. CONCLUSION

This paper deals with a fractional calculus based control strategy for chaos suppression in the FDEs systems. The application of fractional calculus and Gershgorin circle theorem

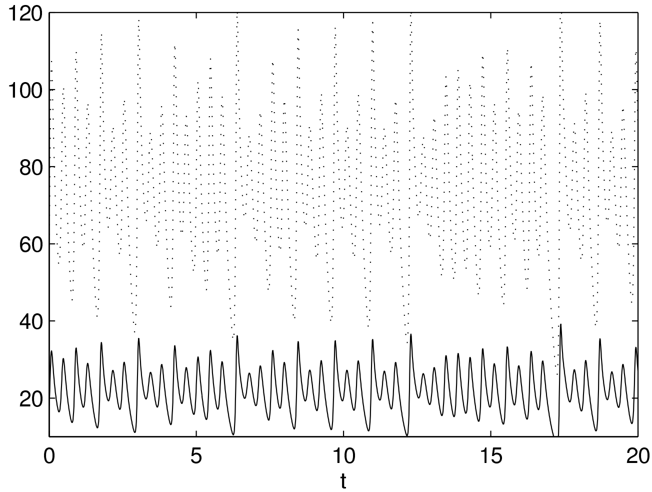

Fig. 8. State diagram of system (13) and (15), $X_3 - Y_3$ (X_3: real line, Y_3: dot line).

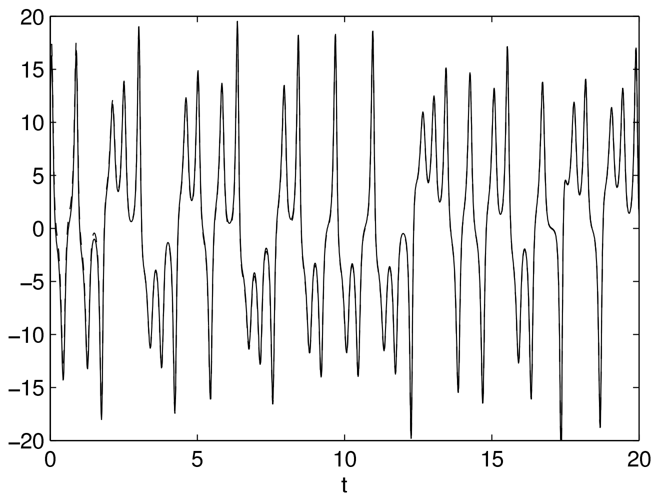

Fig. 9. State diagram of system (13) and (15), $X_1 - (PY)_1$ (X_1: real line, $(PY)_1$: dash line).

in controlling of the incommensurate fractional unified chaotic system is presented. We propose a controller structure based on the fractional differentiator which can stabilize some fixed points in chaotic system. Also offer observer for GS of the fractional unified chaotic system and apply this synchronization technology in SC. Numerical simulations confirm the efficiency of the proposed controller in suppression of chaotic oscillations

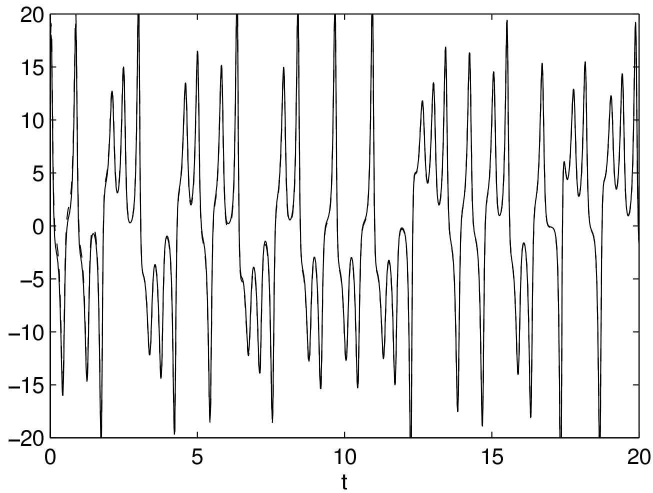

Fig. 10. State diagram of system (13) and (15), $X_2 - (PY)_2$ (X_2: real line, $(PY)_2$: dash line).

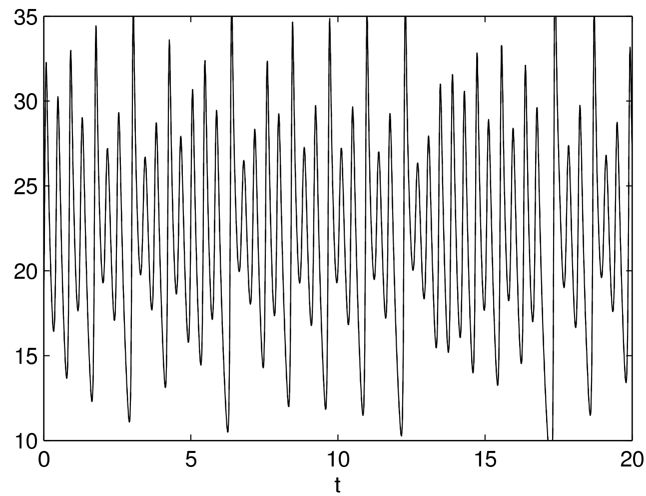

Fig. 11. State diagram of system (13) and (15), $X_3 - (PY)_3$ (X_3: real line, $(PY)_3$: dash line).

and the proposed chaotic masking scheme's success in the communication application. Another topics, such as the influences of the noise in the parameters and structure of the controlled system can be investigated for future research. Moreover, in future works, we also can use fractional controller practically in control of chaotic systems, such as multi-scroll chaotic systems which have been advanced by some effective design

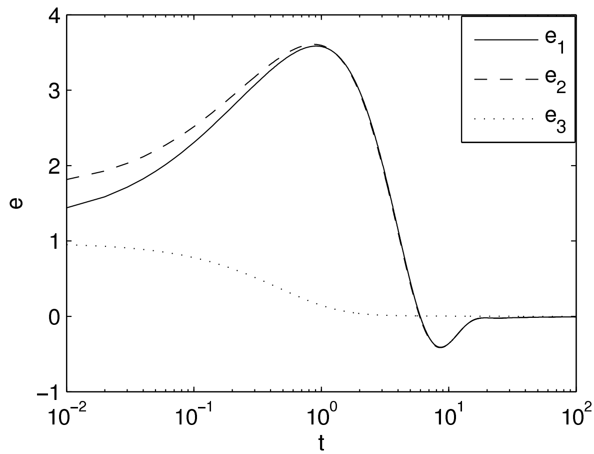

Fig. 12. State diagram of error system (16).

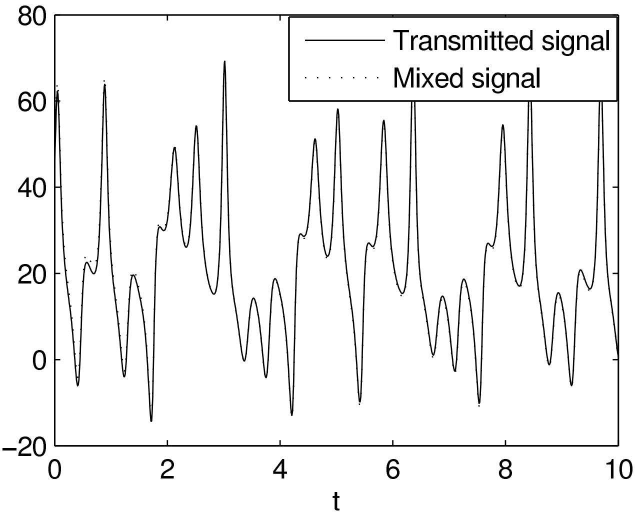

Fig. 13. The transmitted signal and mixed signal of chaotic masking system.

methods using piecewise-linear functions, cellular neural networks, nonlinear modulating functions, circuit component design, switching manifolds, etc. [16, 17, 19, 20]. Take some special multi-scroll chaotic systems which designed by sine function [38-40] as an example, since $\sin x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{(2n-1)!} = x\left(1 + \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!}\right)$ for $x \in \mathbb{R}$, obviously, when this multi-scroll chaotic system transformed into the form of (8), the coefficient matrix includes $1+\sum_{n=1}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!}$. Because the variable of chaotic system

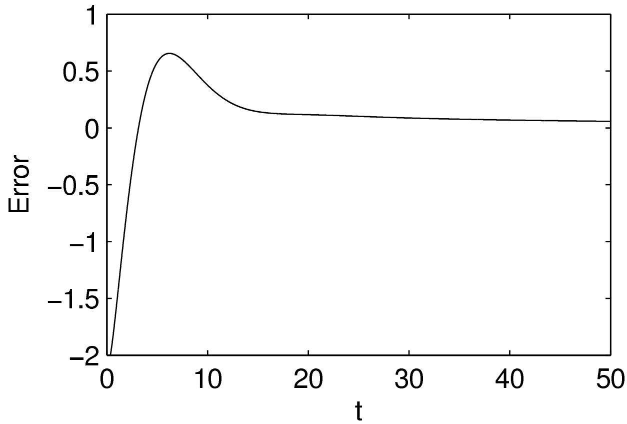

Fig. 14. Error of the information signal $m(t)$ and the detected signal $\tilde{m}(t)$.

is bounded, we can let $|x| \leq M$, then $\left|1 + \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!}\right| \leq 1 + \sum_{n=1}^{\infty} \frac{M^{2n}}{(2n+1)!}$. And we can see that the series $\sum_{n=1}^{\infty} \frac{M^{2n}}{(2n+1)!}$ is convergence, like the theorem 3.1, we can obtain that the equilibrium points will be asymptotically stable under the conditions of the appropriate feedback control parameters k using Lemma 2.1 and Lemma 2.2. We leave these for future work, and will discuss the practicality in more detail later.

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