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THE GAMMA-UNIFORM DISTRIBUTION AND ITS APPLICATIONS

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Up to present for modelling and analyzing of random phenomenons, some statistical distributions are proposed. This paper considers a new general class of distributions, generated from the logit of the gamma random variable. A special case of this family is the *Gamma-Uniform* distribution. We derive expressions for the four moments, variance, skewness, kurtosis, Shannon and Rényi entropy of this distribution. We also discuss the asymptotic distribution of the extreme order statistics, simulation issues, estimation by method of maximum likelihood and the expected information matrix. We show that the Gamma-Uniform distribution provides great flexibility in modelling for negatively and positively skewed, convex-concave shape and reverse 'J' shaped distributions. The usefulness of the new distribution is illustrated through two real data sets by showing that it is more flexible in analysing of the data than of the Beta Generalized-Exponential, Beta-Exponential, Beta-Pareto, Generalized Exponential, Exponential Poisson, Beta Generalized Half-Normal and Generalized Half-Normal distributions.

Keywords: Bathtub shaped hazard rate function, convex-concave shaped, ExpIntegralE function, regularized incomplete gamma function, reverse 'J' shaped, Shannon and Rényi entropy

Classification: 93E12, 62A10

1. INTRODUCTION

Recently, attempts have been made to define new families of probability distributions that provide great flexibility in modelling skewed data in practice. One such example is a broad family of univariate distributions generated from the beta distribution, proposed by Jones [10], which extends the original beta family of distributions with the incorporation of two additional parameters. These parameters control the skewness and the tail weight. Earlier, with a similar goal in mind, Eugene et al. [7] defined the family of beta-normal distributions and discussed its properties.

The class of "Beta-generated distributions" is defined as follows:
Consider a continuous cumulative distribution function (cdf) G with probability density function (pdf) g. Then, the cdf of the univariate family of distributions generated by G, is defined by

$$F(x) = \frac{1}{B(\alpha,\beta)} \int_0^{G(x)} w^{\alpha-1}(1-w)^{\beta-1}\,\mathrm{d}w = I_{G(x)}(\alpha,\beta), \qquad \alpha>0,\ \beta>0, \tag{1}$$

where $I_z(a,b) = \frac{B_z(a,b)}{B(a,b)}$ is the regularized incomplete Beta function, in which

$$B_z(a,b) = \int_0^z t^{a-1}(1-t)^{b-1}\,\mathrm{d}t, \qquad 0 \le z \le 1,$$

is the incomplete Beta function and $B(a,b) = B_1(a,b)$ is the Euler Beta function.

Following the terminology of Arnold in the discussion of Jones' [10] paper, the distribution G will be referred to as the "parent distribution" in what follows. Note that the supports of random variables associated with $F(\cdot)$ and $G(\cdot)$ are equal. The pdf corresponding to (1) can be written as

$$f(x) = \frac{1}{B(\alpha,\beta)} g(x) \left\{G(x)\right\}^{\alpha-1} \left\{1 - G(x)\right\}^{\beta-1}, \qquad \alpha > 0,\ \beta > 0. \tag{2}$$

This class of generalized distributions has been receiving considerable attention over the last years, in particular after the works of Eugene et al. [7] and Jones [10].

Eugene et al. [7] introduced what is known as the Beta-Normal (BN) distribution by taking G to be cdf of the Normal distribution. Nadarajah and Kotz [13] introduced what is known as the Beta-Gumbel (BG) distribution by taking G to be cdf of Gumbel distribution. Nadarajah and Gupta [15] introduced the Beta-Fréchet (BF) distribution by taking G to be the Fréchet distribution. Lee et al. [12] defined the Beta-Weibull (BW) distribution by taking G to be cdf of Weibull distribution. Further, Nadarajah and Kotz [14] examined the Beta-Exponential (BE) distribution by taking G to be cdf of Exponential distribution. Alfred Akinsete et al. [1] defined the Beta-Pareto (BP) distribution by taking G to be cdf of Pareto distribution. Fredy Barreto et al. [3] introduced the Beta Generalized Exponential (BGE) distribution by taking G to be cdf of Generalized Exponential distribution. Pescim et al. [17] proposed the Beta-Generalized Half-Normal (BGHN) distribution by taking G to be cdf of Generalized Half-Normal distribution. Paranaíba et al. [16] introduced the Beta Burr XII distribution (BBXII) by taking G to be cdf of Burr XII distribution. Silva et al. [19] introduced the Beta Modified Weibull distribution (BMW) by taking G to be cdf of Beta Modified Weibull distribution. Cordeiro and Lemonte [5] introduced the β Birnbaum–Saunders distribution (β-BS) by taking G to be cdf of Birnbaum–Saunders distribution. Cordeiro and Lemonte [6] introduced the Beta-Laplace distribution (BL) by taking G to be cdf of Laplace distribution. Mahmoudi [18] introduced the Beta-Generalized Pareto distribution (BGP) by taking G to be cdf of Generalized Pareto distribution.

In this paper we will introduce a new class of "Gamma-generated distributions". Consider (similar to the previous class) a continuous cdf G with pdf g. We define the cdf of the univariate family of distributions generated by G as follows:

$$F(x) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \int_0^{G(x)/\overline{G}(x)} e^{-\frac{w}{\beta}} w^{\alpha-1}\,\mathrm{d}w = 1 - Q\left(\alpha, \frac{G(x)}{\beta\overline{G}(x)}\right), \quad \alpha>0,\ \beta>0, \tag{3}$$

where $\overline{G}(x) = 1 - G(x)$ and $Q(a,z) = \frac{\Gamma(a,z)}{\Gamma(a)}$ is the regularized incomplete gamma function where

$$\Gamma(a,z) = \int_z^{\infty} t^{a-1} e^{-t}\,\mathrm{d}t, \qquad z \ge 0,$$

is the incomplete gamma function and $\Gamma(a)$ is Euler gamma function.

Note that the support of random variables associated with $F(\cdot)$ and $G(\cdot)$ are equal. The pdf corresponding to (3) can be written as

$$f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \frac{g(x)}{\overline{G}^2(x)} \exp\left\{-\frac{G(x)}{\beta\overline{G}(x)}\right\} \left\{\frac{G(x)}{\overline{G}(x)}\right\}^{\alpha-1}, \qquad \alpha > 0,\ \beta > 0. \tag{4}$$

If X has a density of the form (4), then the random variable $W = \frac{G(X)}{\overline{G}(X)}$ has a Gamma distribution Gamma(α,β). The opposite is also true, that is, if W has a Gamma(α,β), then the random variable $X = G^{\leftarrow}(\frac{W}{1+W})$ has a Gamma-generated distribution with density (4), where

$$G^{\leftarrow}(u) = \inf\{x : G(x) \geq u\}, \qquad 0 < u < 1,$$

is left continuous inverse of G. The function $G^{\leftarrow}(u)$ exists for any cdf G and it agrees with $G^{-1}(u)$ if G is a strictly increasing function.

The pdf $f(x)$ in (4) will be most tractable when both functions $G(x)$ and $g(x)$ have simple analytic expressions, also one of the simplest distribution in Statistics is the Uniform distribution. Then, we study the case when $G(x)$ is the cdf of the Uniform distribution in (a, b), i.e, U(a, b). In this case, the random variable X is said to be have the Gamma-Uniform distribution and denoted by GU(α, β, a, b).

2. THE GAMMA-UNIFORM DISTRIBUTION

We are motivated to introduce the Gamma-Uniform (GU) distribution by taking G in (3) to be cdf of a U(a,b). The cdf of the GU distribution becomes

$$F(x) = \int_0^{\frac{x-a}{b-x}} \frac{e^{-\frac{w}{\beta}} w^{\alpha-1}}{\Gamma(\alpha)\beta^\alpha}\, \mathrm{d}w = 1 - Q\left(\alpha, \frac{x-a}{\beta(b-x)}\right), \qquad a < x < b. \tag{5}$$

Although the range of the GU distribution relates on a and b, but it can be used for analyzing and modelling any data set with changing these parameters.

The corresponding pdf and the hazard rate function associated with (5) are:

$$f(x) = \frac{(b-a)e^{-\frac{x-a}{\beta(b-x)}}(\frac{x-a}{b-x})^{\alpha-1}}{(b-x)^2\beta^\alpha\Gamma(\alpha)}, \qquad a < x < b, \tag{6}$$

and

$$h(x) = \frac{(b-a)e^{-\frac{x-a}{\beta(b-x)}}(\frac{x-a}{b-x})^{\alpha-1}}{(b-x)^2\beta^\alpha\Gamma(\alpha, \frac{x-a}{\beta(b-x)})}, \qquad a < x < b, \tag{7}$$

respectively.

Figure 1 illustrates some of the possible shapes of pdf, cdf and hazard rate function of the GU distributions for various values of α, β and $a = 0$, $b = 5$. The graphs (a), (b), (c) and (d) show that the pdf of the GU distributions is positively skewed, negatively

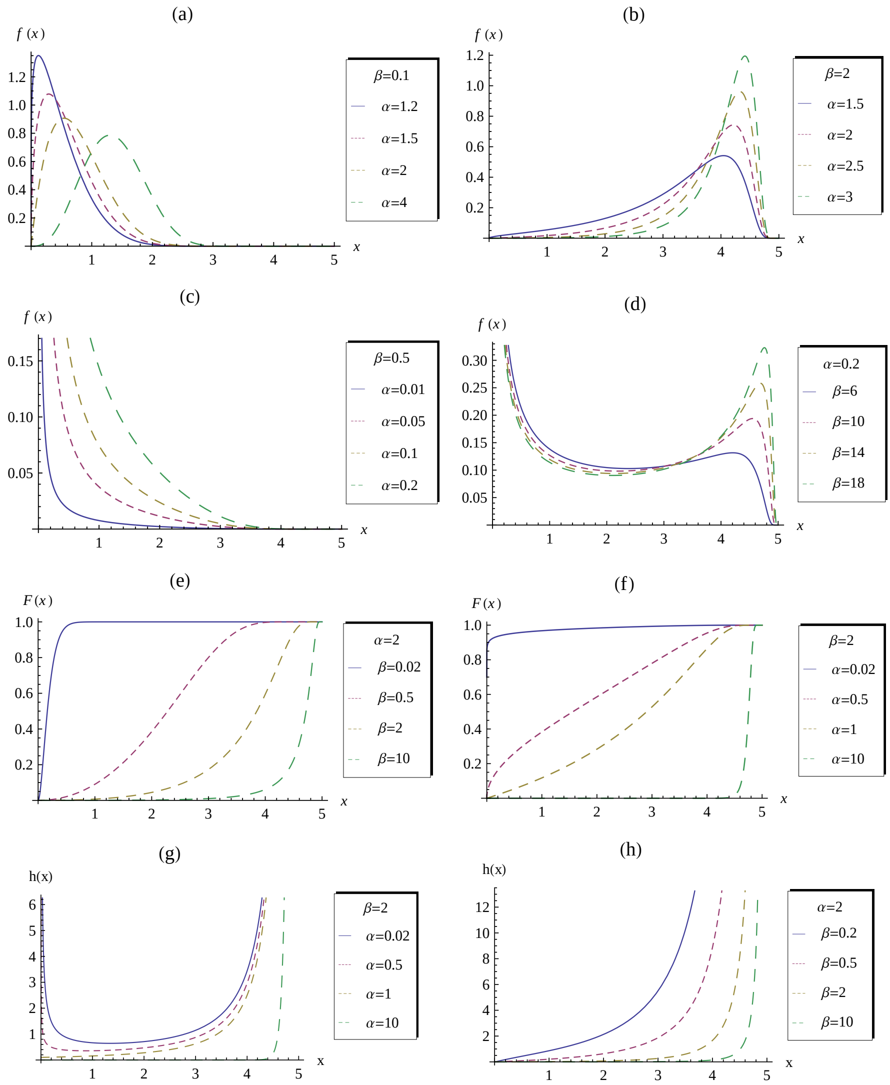

Fig. 1. Plots of the pdf, cdf and hazard rate function of the GU distribution for various values of α, β and $a = 0, b = 5$.

skewed, reverse 'J' shaped and convex-concave shape, respectively, for various values of α and β.

The graph (g) illustrates that the hazard rate function is bathtub shaped. However, the GU distribution provide a reasonable parametric fit for modelling phenomenon with non-monotone failure rate such as the bathtub shaped, which are common in reliability and biological studies. Also, the graph (h) illustrates an increasing failure rates (IFR) function. Hence it can be used in some lifetime data analysis.

3. MOMENTS

In general, exact moments of the GU distribution cannot be calculated. However we derive some closed form expressions for the first four moments, variance, skewness and kurtosis.

In the subsequent, we will use the function $E[n, z]$ which is defined by

$$E[n, z] = \text{ExpIntegralE}[n, z] = \int_1^\infty e^{-zt/t^n}\,\mathrm{d}t.$$

$$\begin{aligned}
E(X) &= \frac{1}{\beta}\Big\{e^{\frac{1}{\beta}}\Big(aE[\alpha, \frac{1}{\beta}] + b\alpha\beta E[1+\alpha, \frac{1}{\beta}]\Big)\Big\},\\
E(X^2) &= \frac{1}{\beta^2}\Big\{\beta\big((a-b)^2 + b^2\beta\big) - (a-b)e^{\frac{1}{\beta}}E[\alpha, \frac{1}{\beta}]\Big(a + a\beta(\alpha-1) - b(1+\beta+\alpha\beta)\Big)\Big\},\\
E(X^3) &= \frac{1}{2\beta^2}\Big\{\beta\Big(\alpha(a-b)^3 + \beta\big(2a^3 - 3\alpha(a-b)^2(a+b) + \alpha^2(a-b)^3\big)\Big)\\
&\quad -(a-b)\alpha e^{\frac{1}{\beta}}E[1+\alpha, \frac{1}{\beta}]\Big(- 2ab\big(1+\beta+2\alpha\beta+\beta^2(\alpha^2-1)\big)\\
&\quad + a^2\big(1+\beta(\alpha-1)(\beta(\alpha-2)+2)\big) + b^2\big(1+\beta(\alpha+2)(\beta+\alpha\beta+2)\big)\Big)\Big\},\\
E(X^4) &= \frac{1}{6\beta^4}\Big\{\beta\Big((a-b)^4 + (a-b)^3\beta\big(a(2\alpha-3) - b(2\alpha+9)\big) + (a-b)^2\beta^2\\
&\quad \times\big(a^2(\alpha-3)(\alpha-2) - 2ab(-6+\alpha+\alpha^2) + b^2(18+\alpha(\alpha+7))\big) + 6b^4\beta^3\Big)\\
&\quad +(b-a)e^{\frac{1}{\beta}}E[\alpha, \frac{1}{\beta}]\Big(- 3a^2b\big(1+\beta+3\alpha\beta+\beta^2(\alpha-1)(2+3\alpha)\\
&\quad +\beta^3(\alpha-2)(\alpha-1)(\alpha+1)\big) + 3ab^2(1+\beta+\alpha\beta)\big(1+\beta(\alpha+2)(2+\beta\\
&\quad \times(\alpha-1))\big) + a^3\big(1+(\alpha-1)\beta(3+\beta(\alpha-2)(3+\beta(\alpha-3)))\big)\\
&\quad +b^3\big(- 1 - \beta(\alpha+3)(3+(2+\alpha)\beta(3+\beta+\alpha\beta))\big)\Big)\Big\}.
\end{aligned}$$

Suppose X is a random variable with the pdf given by (6). If $a = 0$ and $b = 1$ then

$$E(X^n) = (\alpha)_n U\Big(n, 1-\alpha, \frac{1}{\beta}\Big),$$

where $(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$ is Pochhammer symbol and

$$\mathrm{U}(a,b,z) = \frac{1}{\Gamma(a)}\int_0^\infty e^{-zt}t^{a-1}(1+t)^{b-a-1}\,\mathrm{d}t$$

is confluent hyper geometric function.

Further calculations show that first three important central moments, skewness and kurtosis of X can be given by

$$Var(X) = \frac{(a-b)^2}{\beta^2}\Big\{\beta - e^{\frac{1}{\beta}}E[\alpha,\frac{1}{\beta}]\big(1+(\alpha-1)\beta + e^{\frac{1}{\beta}}E[\alpha,\frac{1}{\beta}]\big)\Big\} =: \frac{(a-b)^2 s_1}{\beta^2}$$

$$\begin{aligned}
E[\{X-E(X)\}^3] &= \frac{(a-b)^3}{2\beta^3}\Bigg\{-\beta(1+(\alpha-2)\beta) + e^{\frac{1}{\beta}}E[\alpha,\frac{1}{\beta}] \\
&\quad \times\Big(1+2\beta(\alpha-4)+\beta^2(\alpha-2)(\alpha-1)+2e^{\frac{1}{\beta}}E[\alpha,\frac{1}{\beta}] \\
&\quad \times\big(3+3\beta(\alpha-1)+2e^{\frac{1}{\beta}}E[\alpha,\frac{1}{\beta}]\big)\Big)\Bigg\} =: \frac{(a-b)^3 s_2}{2\beta^3} \\
E[\{X-E(X)\}^4] &= \frac{(a-b)^4}{6\beta^4}\Bigg\{\beta\Big(1+\beta\big(-3+2\alpha+(6-5\alpha+\alpha^2)\beta\big)\Big) \\
&\quad -e^{\frac{1}{\beta}}E[\alpha,\frac{1}{\beta}]\Bigg(1+3\beta(\alpha-5)+3\beta^2(\alpha-5)(\alpha-2) \\
&\quad +\beta^3(\alpha-3)(\alpha-2)(\alpha-1)+6e^{\frac{1}{\beta}}E[\alpha,\frac{1}{\beta}] \\
&\quad \times\Big(2\big(1+\beta(-5+2\alpha+(2-3\alpha+\alpha^2)\beta)\big)+3e^{\frac{1}{\beta}} \\
&\quad \times E[\alpha,\frac{1}{\beta}]\big(2+2\beta(\alpha-1)+e^{\frac{1}{\beta}}E[\alpha,\frac{1}{\beta}]\big)\Big)\Bigg)\Bigg\} =: \frac{(a-b)^4 s_3}{6\beta^4}
\end{aligned}$$

and

$$\text{Skewness}(X) = \frac{-s_2}{2\sqrt{s_1}}, \qquad \text{Kurtosis}(X) = \frac{s_3}{6s_1^2} - 3,$$

respectively. Note that the skewness and kurtosis measures depend only on α and β.

These measures are calculated and summarized in Table 1 for some parameters values. The required numerical evaluations were implemented using the Mathematica (version 7) software through the some commands like `ExpIntegralE`.

From Table 1, it is concluded that for a constant α, the skewness measurement is a decreasing function of β.

4. PERCENTILES

The pth percentile x_p, is defined by $F(x_p)=p$. From (5), we have $1-Q\big(\alpha, \frac{x_p-a}{\beta(b-x_p)}\big)=p$. If $z_{1-p} = \frac{x_p-a}{\beta(b-x_p)}$, then $z_{1-p} = Q^{-1}(\alpha, 1-p)$, where Q^{-1} is the inverse of regularized incomplete gamma function. Thus $x_p = \frac{a+\beta b z_{1-p}}{1+\beta z_{1-p}}$.

α	β	E(X)	Var(X)	Skewness(X)	Kurtosis(X)
0.02	0.2	-4.9660	0.0446	10.2131	133.3170
	2	-4.8178	0.7369	6.0870	40.3935
	10	-4.6067	2.2505	4.3676	18.8672
0.5	0.2	-4.2079	0.8622	1.6905	2.9297
	2	-1.5568	7.0084	0.3140	-1.1780
	10	0.9443	9.7743	-0.5957	-1.0360
2	0.2	-2.3945	1.6547	0.3393	-0.4545
	2	2.3073	2.3584	-1.3031	1.7293
	10	4.2015	0.5784	-3.3861	17.4109
10	0.2	1.5189	0.5329	-0.5709	0.4121
	2	4.4769	0.0294	-1.3776	3.7337
	10	4.8903	0.0015	-1.5605	5.0870

Tab. 1. Mean, variance, skewness and kurtosis of GU$(\alpha, \beta, -5, 5)$ for various values of α, β.

Example 4.1. If $X \sim GU(2, 4, 0, 5)$ and $p = 0.5$, then $z_{0.5} = Q^{-1}(2, 0.5) = 1.67835$. Therefore the median of the distribution is $x_{0.5} = 4.35178$.

5. ASYMPTOTIC PROPERTIES

Let $X_1, \ldots, X_n$ be a random sample of (3). Sometimes one would be interested in the asymptotic of extreme values $X_{n:n} = \max(X_1, \ldots, X_n)$ and $X_{1:n} = \min(X_1, \ldots, X_n)$. The limiting distribution of $Y_n = \frac{X_{n:n} - b_n}{a_n}$ is Type I (Exponential type), since $\lim_{n\to\infty} n[1 - F(a_n y + b_n)] = e^{-y}$ in which a_n and b_n are the solution of the system $F(a_n + b_n) = 1 - (ne)^{-1}$ and $F(b_n) = \frac{n-1}{n}$. Hence, it follows of Theorems 7.8.3 and 7.8.5 from [2] that

$$P(Y_n \leq y) \approx G^{(1)}(y) = \exp(-e^{-y}), \qquad -\infty < y < \infty.$$

and the exact distribution of Y_n is

$$G_n(y) = [F_X(a_n y + b_n)]^n = 1 - \Big[Q\Big(\alpha, \frac{a_n y + b_n - a}{\beta(b - a_n y - b_n)}\Big)\Big]^n, \quad \frac{a - b_n}{a_n} < y < \frac{b - b_n}{a_n}.$$

The limiting distribution of $W_n = \frac{X_{1:n} + b_n}{an}$ is type III (limiting Type), since $\lim_{y\to 0} \dfrac{F(ky - b_n)}{F(y - b_n)} = k^\alpha$ where $b_n = -\inf\{x|\ F(x) > 0\} = -a$ and $a_n = b_n + s_n$ in which s_n is the solution of $nF(s_n) = 1$. Hence, it follows from Theorem 7.8.6 from [2] that

$$P(W_n \leq w) \approx H^{(3)}(w) = 1 - \exp(-w^\alpha), \qquad w > 0.$$

and the exact distribution of W_n is

$$H_n(w) = 1 - [1 - F_X(a_n w - b_n)]^n = 1 - \Big[Q\Big(\alpha, \frac{a_n w}{\beta(b - a_n w - a)}\Big)\Big]^n, \quad 0 < w < \frac{b - a}{a_n}.$$

Example 5.1. Suppose again that $X \sim GU(2,4,0,5)$, and we are interested to obtain the distribution of the maximum and minimum of a random sample of size $n = 100$. It can be easily shown that b_n and a_n is obtained from the solution of the system $1 - Q\big(2, \frac{b_n}{4(5-b_n)}\big) = \frac{99}{100}$ and $1 - Q\big(2, \frac{a_n+b_n}{4(5-a_n-b_n)}\big) = 1 - (100e)^{-1}$ which gives $a_n = 0.025748$ and $b_n = 4.81853$.

For minimum of a random sample, we have $b_n = -a = 0$ and s_n is obtained from $1 - Q\big(2, \frac{s_n}{4(5-s_n)}\big) = \frac{1}{100}$ which gives $s_n = 1.86367$, thus $a_n = 1.86367$.

In Figure 2, graphs of $G_n(y)$, $G^{(1)}$, $H_n(w)$ and $H^{(3)}(w)$ for $n = 100$ are illustrated.

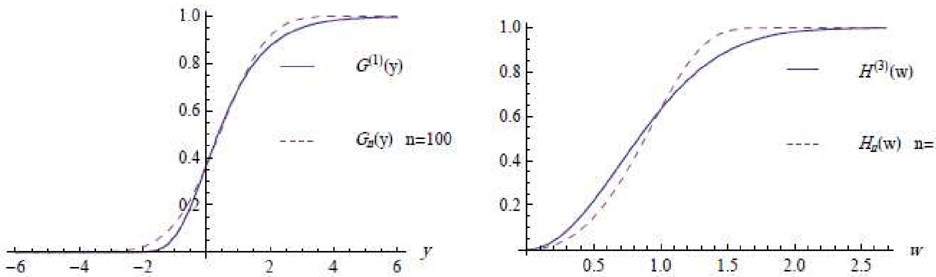

Fig. 2. Comparison of cdf $G_n(y)$ with limiting cdf $G^{(1)}(y)$ and cdf $H_n(w)$ with limiting cdf $H^{(3)}(w)$ for $n = 100$.

According to the Figure 2, it is concluded that for the extremes, the limiting distribution is a good approximation to the original one.

6. SHANNON AND RÉNYI ENTROPY

An entropy of a random variable X with the pdf $f(\cdot)$, is a measure of variation of the uncertainty. Denote by $\mathcal{H}_{Sh}(f)$ the well-known Shannon entropy introduced by Shannon (1948). It is defined by

$$\mathcal{H}_{Sh}(f) = E[-\log f(X)] = -\int_{\mathcal{X}} f(x)\log(f(x))\,\mathrm{d}x. \tag{8}$$

One of the main extensions of this entropy was defined by Rényi (1961). This generalized entropy measure is given by

$$\mathcal{H}_R(\lambda) = \mathcal{H}_R(\lambda, f) = \frac{1}{1-\lambda}\log\int_{\mathcal{X}} f(x)^{\lambda}\,\mathrm{d}x, \quad \lambda > 0 \text{ and } \lambda \neq 1. \tag{9}$$

The additional parameter λ is used to describe complex behaviour in probability models and the associated process under study. Rényi entropy $\mathcal{H}_R(\lambda)$ is monotonically decreasing in λ while Shannon entropy (8) is obtained from $\mathcal{H}_R(\lambda)$ for $\lambda \uparrow 1$; see for more details [20].

The Shannon entropy of the GU distribution is

$$\begin{aligned}\mathcal{H}_{Sh}(f) &= \alpha+\log\beta+\log\Gamma(\alpha)-(\alpha-1)\psi(\alpha)+\beta^{-\alpha}\Big\{(\alpha-1)\times\Big(\beta^{\alpha}\log\beta \\ &\quad -U^{(1,0,0)}(\alpha,\alpha+1,\frac{1}{\beta})-(\alpha+1)U^{(0,1,0)}(\alpha,\alpha+1,\frac{1}{\beta})\Big\}+\log(b-a) \\ &= \mathcal{H}_{Sh}(f_W)+\beta^{-\alpha}\Big\{(\alpha-1)\times\Big(\beta^{\alpha}\log\beta-U^{(1,0,0)}(\alpha,\alpha+1,\frac{1}{\beta}) \\ &\quad -(\alpha+1)U^{(0,1,0)}(\alpha,\alpha+1,\frac{1}{\beta})\Big\}+\mathcal{H}_{Sh}(f_U),\end{aligned}$$

where $U^{(1,0,0)}(a,b,z)=\partial_a U(a,b,z)$, $U^{(0,1,0)}(a,b,z)=\partial_b U(a,b,z)$, $\psi(z)=\mathrm{d}\ln\Gamma(z)/\mathrm{d}z=\Gamma^{'}(z)/\Gamma(z)$ is Poly Gamma function, $\mathcal{H}_{Sh}(f_W)$ is Shannon entropy of $G(\alpha,\beta)$ and $\mathcal{H}_{Sh}(f_U)$ is Shannon entropy of $\mathrm{U}(a,b)$.

The Rényi entropy of the GU distribution if $\lambda\alpha-\lambda+1>0$ is

$$\begin{aligned}\mathcal{H}_R(\lambda,f) &= \frac{1}{1-\lambda}\Big[\log\Gamma(\lambda\alpha-\lambda+1)-(\lambda\alpha-\lambda+1)\log\lambda-\lambda\log\Gamma(\alpha)-(\lambda-1)\log\beta \\ &\quad +(\lambda\alpha-\lambda+1)\log\frac{\lambda}{\beta}+\log U(\lambda\alpha-\lambda+1,\lambda+\lambda\alpha,\frac{\lambda}{\beta})\Big]+\log(b-a) \\ &= \mathcal{H}_R(\lambda,f_W)+\frac{1}{1-\lambda}\Big[(\lambda\alpha-\lambda+1)\log\frac{\lambda}{\beta}+\log U(\lambda\alpha-\lambda+1,\lambda+\lambda\alpha,\frac{\lambda}{\beta})\Big] \\ &\quad +\mathcal{H}_R(\lambda,f_U),\end{aligned}$$

where $\mathcal{H}_R(\lambda,f_W)$ is Rényi entropy of $\mathrm{Gamma}(\alpha,\beta)$ and $\mathcal{H}_R(\lambda,f_U)$ is Rényi entropy of $\mathrm{U}(a,b)$.

7. SIMULATION

For simulation of the distribution, note from (5) that if W is a random number from a $\mathrm{Gamma}(\alpha,\beta)$, then

$$G^{-1}\left(\frac{W}{1+W}\right)=\frac{W}{1+W}(b-a)+a=\frac{a+bW}{1+W}$$

will follow the pdf of (5).

8. INFERENCE

In this section, estimating by the method maximum likelihood (ML) are discussed. The log-likelihood function for a random sample $X_1,\ldots,X_n$ from (6) is:

$$\begin{aligned}\ell:=\log L(\alpha,\beta,a,b) &= n\log(b-a)-n\alpha\log\beta-n\log\Gamma(\alpha)-\sum_{i=1}^{n}\left(\frac{x_i-a}{\beta(b-x_i)}\right) \\ &\quad +(\alpha-1)\sum_{i=1}^{n}\log\left(\frac{x_i-a}{b-x_i}\right)-2\sum_{i=1}^{n}\log(b-x_i)\end{aligned}$$

The required numerical evaluations were implemented using the R software through the package (stats4), command `mle` with the L-BFGS-B method, because the boundaries of the range of the distribution are restricted by parameters a and b.

We simulate n =20, 40, 60 and 100 times the GU distribution for $\alpha = 2$, $\beta = 0.4$, $a = 0, -5$ and $b = 5$. For each sample size, we compute the MLE's of α, β, a and b. We repeat this process 1000 times and compute the average estimators (AE), Bias and the mean squared errors (MSE). The results are reported in Table 2.

		$a = 0$			$a = -5$		
n		AE	Bias	MSE	AE	Bias	MSE
20	α	1.644	-0.356	12.847	1.660	-0.340	9.726
	β	1.390	0.990	2.612	1.697	1.297	6.264
	a	0.218	0.218	3.441	-4.608	0.392	16.070
	b	4.419	-0.581	2.302	3.823	-1.177	11.389
40	α	1.852	-0.148	4.893	1.749	-0.251	4.956
	β	0.752	0.352	0.387	0.878	0.478	0.572
	a	0.149	0.149	0.711	-4.643	0.357	3.131
	b	4.734	-0.266	1.259	4.281	-0.719	5.782
60	α	1.842	-0.158	1.463	1.656	-0.344	2.789
	β	0.636	0.236	0.187	0.866	0.466	0.516
	a	0.129	0.129	0.146	-4.637	0.363	1.472
	b	4.841	-0.159	0.979	4.315	-0.685	4.320
100	α	1.887	-0.113	0.898	1.711	-0.289	0.842
	β	0.534	0.134	0.074	0.659	0.259	0.182
	a	0.080	0.080	0.086	-4.735	0.265	0.331
	b	4.874	-0.126	0.549	4.468	-0.532	2.183

Tab. 2. Estimated AE, Bias and MSE based on 1000 simulations of the GU distribution with $\alpha = 2, \beta = 0.4$, $a = 0, -5$ and $b = 5$, with n=20, 40, 60, and 100.

Comparing the performance of all the estimators, it is observed that values the MSE's decrease as the sample size increase.

For fixed a and b, to obtain interval estimation and to test for α and β, one requires the 2× 2 expected information matrix $\mathbf{I}$. One obtains $\mathbf{I} = \begin{bmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{bmatrix}$, where

$$I_{11} = -E\left[\frac{\partial^2 f(X;\boldsymbol{\theta})}{\partial\alpha^2}\right] = \psi^{'}(\alpha), \quad I_{12} = I_{21} = -E\left[\frac{\partial^2 f(X;\boldsymbol{\theta})}{\partial\alpha\partial\beta}\right] = \frac{1}{\beta},$$

$$I_{22} = -E\left[\frac{\partial^2 f(X;\boldsymbol{\theta})}{\partial\beta^2}\right] = \frac{\alpha}{\beta^2},$$

and $\boldsymbol{\theta} = (\theta_1, \theta_2) := (\alpha, \beta)$.

Under conditions that are fulfilled for parameters in the interior of the parameter space but not on the boundary, the asymptotic distribution of $\sqrt{n}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})$ is $N_2(\mathbf{0}, \mathbf{I}^{-1}(\boldsymbol{\theta}))$, where $\hat{\boldsymbol{\theta}}$ is the MLE of θ. An asymptotic confidence interval (ACI) with significance level γ for each parameter θ_i is given by

$$\text{ACI}(\theta_i, 100(1-\gamma)) = \left(\hat{\theta}_i - z_{\gamma/2}\sqrt{I^{-1}(\hat{\boldsymbol{\theta}})_i/n},\ \hat{\theta}_i + z_{\gamma/2}\sqrt{I^{-1}(\hat{\boldsymbol{\theta}})_i/n}\right),\ i = 1, 2,$$

where $I^{-1}(\hat{\boldsymbol{\theta}})_i$ is the ith diagonal element of $\mathbf{I}^{-1}(\boldsymbol{\theta})$ estimated at $\hat{\boldsymbol{\theta}}$. $I^{-1}(\hat{\boldsymbol{\theta}})_1$ and $I^{-1}(\hat{\boldsymbol{\theta}})_2$ can be written as

$$I^{-1}(\hat{\boldsymbol{\theta}})_1 = \frac{I_{22}}{I_{11}I_{22} - I_{12}^2} = \frac{\hat{\alpha}}{\hat{\alpha}\psi'(\hat{\alpha})},$$

and

$$I^{-1}(\hat{\boldsymbol{\theta}})_2 = \frac{I_{11}}{I_{11}I_{22} - I_{12}^2} = \frac{\hat{\beta}^2\psi'(\hat{\alpha})}{\hat{\alpha}\psi'(\hat{\alpha})},$$

respectively.

We simulate n =20, 40, 60 and 100 times the GU distribution for $\alpha = 2$, $\beta = 0.4$ and fixed $a = 0, -5$ and $b = 5$. For each sample size, we compute the MLE's of α and β. We also compute the asymptotic confidence interval in each replications and repeat this process 1000 times and compute the AE, Bias, MSE, coverage probabilities (CP) and ACI. The results are reported in Table 3.

	n		AE	Bias	MSE	CP	ACI
a=0	20	α	2.290	0.290	0.596	0.96	(1.286, 3.293)
		β	0.380	1.890	0.018	0.87	(0.214, 0.547)
	40	α	2.182	0.182	0.282	0.96	(1.506, 2.859)
		β	0.385	1.782	0.009	0.88	(0.265, 0.504)
	60	α	2.094	0.094	0.154	0.95	(1.564, 2.624)
		β	0.396	1.694	0.007	0.92	(0.295, 0.496)
	100	α	2.053	0.053	0.079	0.96	(1.651, 2.456)
		β	0.396	1.653	0.003	0.93	(0.319, 0.474)
a=-5	20	α	2.339	0.339	0.682	0.97	(1.314, 3.364)
		β	0.375	1.939	0.016	0.86	(0.210, 0.539)
	40	α	2.148	0.148	0.258	0.95	(1.482, 2.814)
		β	0.389	1.748	0.009	0.90	(0.269, 0.510)
	60	α	2.084	0.084	0.139	0.96	(1.557, 2.611)
		β	0.394	1.684	0.006	0.93	(0.294, 0.493)
	100	α	2.041	0.041	0.072	0.96	(1.641, 2.441)
		β	0.399	1.641	0.004	0.93	(0.321, 0.477)

Tab. 3. Estimated AE, Bias, MSE, CP and ACI based on 1000 simulations of the GU distribution with fixed $a = 0, -5$, $b = 5$ and $\alpha = 2, \beta = 0.4$ with n= 20, 40, 60 and 100.

It is observed that for all the parametric values the MSE's and the biases decrease as the sample size increase. It is also interesting to observe that the asymptotic confidence interval maintains the nominal coverage probabilities. Therefore, the MLE's and their the asymptotic results can be used for estimating and constructing confidence intervals even for small sample sizes.

9. APPLICATIONS

In this section, we fit the GU model to two real data sets and show that the GU distribution is more flexible in analyzing of the data than of the Beta Generalized-Exponential (BGE) [3], Beta-Exponential (BE) [14], Beta-Pareto (BP)[1], Generalized Exponential(GE) [9], Exponential Poisson (EP) [11], Beta Generalized Half-Normal (BGHN) [17] and Generalized Half-Normal (GHN)[4] distributions.

The data set is given in Feigl and Zelen [8] for two groups of patients who died of acute myelogenous leukemia. The patients were classified into the two groups according to the presence or absence of a morphologic characteristic of white cells. The patients termed AG positive were identified by the presence of Auer rods and/or significant granulative of the leukemic cells in the bone marrow at diagnosis. For the AG negative patients these factors were absent.

In order to compare the models, we used following four criterions: Akaike Information Criterion(AIC), Bayesian Information Criterion (BIC), Kolmogorov–Smirnov (K-S) statistic and the P-value from the chi-square goodness of fit test, where the lower values of AIC, BIC and K-S statistic and the upper value of P-value for models indicate that these models could be chosen as the best model to fit the data. The K-S statistic and the corresponding P-value evaluations were implemented using the R software through the command `ks.test`. The results of the K-S test are just illustrative from this reason, unknown parameters of distribution were replaced by their ML estimates.

9.1. The myelogenous leukemia data for AG positive

In this subsection we fit a GU model to one real data set. The convex-concave data is obtained from Feigl and Zelen [8] represent observed survival times (weeks) for AG positive. The data set is: 65, 156, 100, 134, 16, 108, 121, 4, 39, 143, 56, 26, 22, 1, 1, 5, 65.

Table 4 lists the MLEs of the parameters from the fitted GU, BGHN, GHN, GE, EP and BP models and the values of the following statistics: AIC, BIC and K-S statistic. The computations were performed with the R software. These results indicate that the GU model has the lowest values for the AIC and BIC criteria among the fitted models, and therefore it could be chosen as the best model.

Model	Parameters	AIC	BIC	K-S	P-value
GU	$\hat{\alpha} = 0.297$, $\hat{\beta}$=6.657, $\hat{a}$=0.99, $\hat{b}$=166.39	162.4	165.8	0.105	0.992
BGHN	$\hat{a} = 0.09$, $\hat{b}$=0.40, $\hat{\alpha}$=5.99, $\hat{\theta}$=132.49	174.9	178.2	0.095	0.998
GHN	$\hat{\alpha} = 0.76$, $\hat{\theta}$= 73.62	176.6	178.3	0.145	0.865
GE	$\hat{\alpha} = 0.757$, $\hat{\lambda}$=0.013	177.6	179.3	0.148	0.853
EP	$\hat{\lambda} = 0.01$, $\hat{\beta}$=0.016	178.6	180.3	0.158	0.789
BP	$\hat{a} = 20.35$, $\hat{b}$= 32.71, $\hat{\theta}$=0.01, $\hat{k}$= 0.06	189.6	192.9	0.198	0.515

Tab. 4. MLEs of the model parameters for the myelogenous leukemia data (AG positive) and the measures AIC, BIC and K-S statistics.

Note that regarding to Table 4, the P-value of the GU distribution (0.992) is slightly less than the P-value of the BGHN distribution (0.998) but based on the AIC, the GU distribution is the best fitted distribution.

9.2. The myelogenous leukemia data for AG negative

In this subsection we fit a GU model to one real data set. The reverse 'J' shaped data is obtained from Feigl and Zelen [8] represent observed survival times (weeks) for AG negative. The data set is: 56, 65, 17, 17, 16, 22, 3, 4, 2, 3, 8, 4, 3, 30, 4, 43.

Table 5 lists the MLEs of the parameters from the fitted GU, EP, BP, GHN, BGHN, BGE, BE and GE models and the values of the following statistics: AIC , BIC and K-S. These results indicate that the GU model has the lowest values for the AIC, BIC and K-S statistics among the fitted models, and thus the GU distribution is more suitable than other distributions.

Model	Parameters	AIC	BIC	K-S	P-value
GU	$\hat{\alpha}$ = 0.46, $\hat{\beta}$=0.30, $\hat{a}$=1.99, $\hat{b}$=165.39	123	126.1	0.18	0.682
EP	$\hat{\lambda}$ =1.01, $\hat{\beta}$=0.04	129.1	130.6	0.211	0.476
BP	$\hat{a}$ =1.53, $\hat{b}$=9.88, $\hat{\theta}$=1.86, $\hat{k}$=0.09	129.7	132.8	0.22	0.445
GHN	$\hat{\alpha}$ =0.74 ,$\hat{\theta}$=22.79	130.2	131.8	0.22	0.42
BGHN	$\hat{a}$ = 148.23, $\hat{b}$=94.77, $\hat{\alpha}$=0.06, $\hat{\theta}$=136.5	131.9	134.98	0.23	0.355
BGE	$\hat{a}$ = 37.95, $\hat{b}$=3.33, $\hat{\lambda}$=0.013, $\hat{\alpha}$=0.04	132.9	135.9	0.23	0.352
BE	$\hat{a}$ = 0.96, $\hat{b}$=2.998, $\hat{\lambda}$=0.017	131.5	133.8	0.24	0.336
GE	$\hat{\alpha}$ = 0.097, $\hat{\lambda}$=0.053	129.5	131	0.24	0.330

Tab. 5. MLEs of the model parameters for myelogenous leukemia data (AG negative) and the measures AIC, BIC and K-S statistics.

The plots of the estimated densities of distributions fitted to the data set (top plots) and empirical and four fitted cdf (bottom plots) for the myelogenous leukemiain data in Figure 3 shows that the GU distribution gives a better fit than the other models.

10. CONCLUSIONS

In this paper, a new flexible class of distributions is considered. Then for a special case of this class, i.e, the Gamma-Uniform distribution, some moments, Shannon and Rényi entropy are derived. Finally, the asymptotic distribution of the extreme order statistics and simulation issues, estimation by method of maximum likelihood and the expected information matrix are discussed.

An application of the GU distribution to two real data sets are given to demonstrate that this distribution can be used quite effectively to provide better fit than other available models.

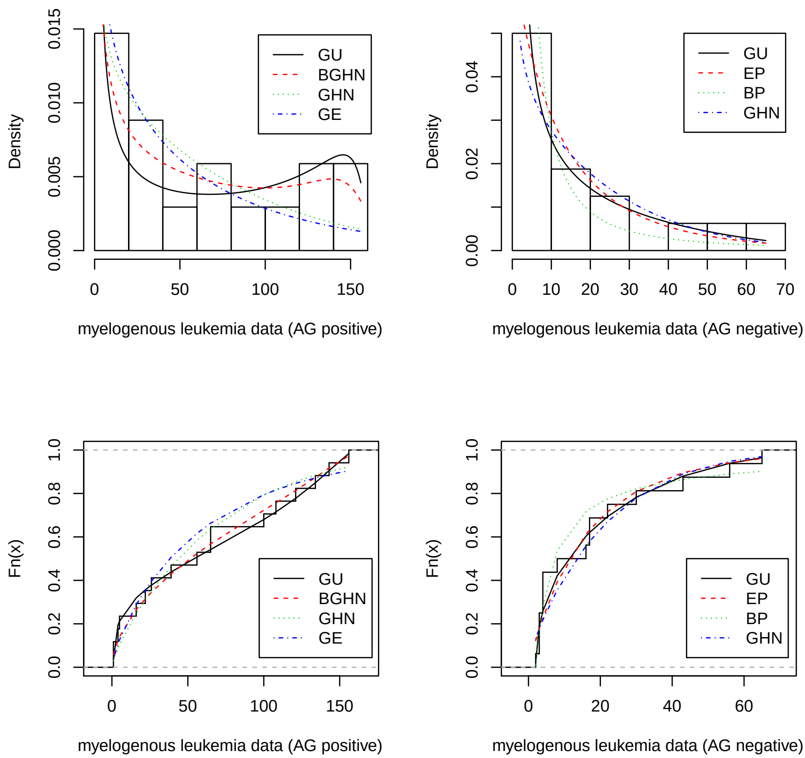

Fig. 3. Estimated densities of distributions (top plots), empirical and four fitted cumulative distribution functions (bottom plots) for the myelogenous leukemia data.

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