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ON THE ARGMIN–SETS OF STOCHASTIC PROCESSES AND THEIR DISTRIBUTIONAL CONVERGENCE IN FELL–TYPE–TOPOLOGIES

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Let $\epsilon - \mathrm{Argmin}(Z)$ be the collection of all ϵ-optimal solutions for a stochastic process Z with locally bounded trajectories defined on a topological space. For sequences (Z_n) of such stochastic processes and (ϵ_n) of nonnegative random variables we give sufficient conditions for the (closed) random sets $\epsilon_n - \mathrm{Argmin}(Z_n)$ to converge in distribution with respect to the Fell-topology and to the coarser Missing-topology.

Keywords: $\epsilon-$argmin of stochastic process, random closed sets, weak convergence of Hoffmann–Jørgensen, Fell-topology, Missing-topology

Classification: 49J53, 60B10, 60F05, 90C15

1. INTRODUCTION AND MAIN RESULTS

Let T be a locally compact and second countable Hausdorff-space (lcscH). We consider the set $l^\infty(T)$ of all functions $z : T \to \mathbb{R}$ that are bounded on every compact $K \subseteq T$ but not necessarily on T. There will be no notational distinction between $z : T \to \mathbb{R}$ and its (bounded) restriction $z : K \to \mathbb{R}$ on K. Along with $z \in l^\infty(T)$ there is the pertaining collection of all minimizing points

$$\mathrm{Argmin}(z) := \left\{ t \in T : z(t) = \inf_{s \in T} z(s) \right\}$$

and the collection of all ϵ-optimal solutions

$$\epsilon - \mathrm{Argmin}(z) := \left\{ t \in T : z(t) \leq \inf_{s \in T} z(s) + \epsilon \right\}, \quad \epsilon \geq 0.$$

Obviously, $\epsilon - \mathrm{Argmin}(z) \supseteq \mathrm{Argmin}(z)$ with equality, if $\epsilon = 0$. For the stochastic framework we need a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Finally, we consider arbitrary (possibly non-measurable) maps

$$Z_n : \Omega \to l^\infty(T), \quad n \in \mathbb{N},$$

and

$$Z : \Omega \to l^\infty(T).$$

Assume that for every $n \in \mathbb{N}$ there exists a nonnegative random variable ϵ_n defined on $(\Omega, \mathcal{A}, \mathbb{P})$ such that we find a nonempty and closed subset φ_n of $\epsilon_n - \text{Argmin}(Z_n) \neq \emptyset$. Take the singleton $\varphi_n = \{\tau_n\}$ for $\tau_n \in \epsilon_n - \text{Argmin}(Z_n)$ as the most simple example. Or assume that Z_n has lower semicontinuous (lsc) trajectories. Then the whole set $\epsilon_n - \text{Argmin}(Z_n)$ is closed. In any case we obtain maps $\varphi_n : \Omega \to \mathcal{F}$, where $\mathcal{F}$ denotes the family of all closed subsets in T. The *main problem* in this paper is as follows:

Let the stochastic processes Z_n converge in some distributional sense to a limit Z, then what can be said about distributional convergence of the φ_n to some φ and how is φ related to $\text{Argmin}(Z)$?

This kind of question typically arises in stochastic optimization, when the stability of approximative stochastic programs is to be investigated. Many other variations of the main problem stem from asymptotic statistics. Here, procedures are defined by minimizing or maximizing a certain criterion function, for example Maximum-Likelihood-estimators, minimum distance estimators, least squares estimators or Bayes estimators.

Now, to make our problem more precise we first define

$$Z_n \rightsquigarrow Z \text{ in } l^\infty(T)$$

as weak convergence in the sense of Hoffmann–Jørgensen, confer, e. g., van der Vaart and Wellner [14]. According to their Theorem 1.6.1 it is equivalent with

$$Z_n \rightsquigarrow Z \text{ in } l^\infty(K) \quad \text{for all compact } K \subseteq T.$$

Next, the space $\mathcal{F}$ is endowed with the *Fell-topology* τ_{Fell} and the corresponding Borel-σ-algebra $\mathcal{B}_{Fell}$. For the definition of τ_{Fell} we introduce the classes $\mathcal{G}$ and $\mathcal{K}$ of all open and compact subsets in T, respectively. Then the Fell-topology is generated from a subbase which contains all *missing sets* $\mathcal{M}(K) := \{F \in \mathcal{F} : F \cap K = \emptyset\}, K \in \mathcal{K}$ and all *hitting sets* $\mathcal{H}(G) := \{F \in \mathcal{F} : F \cap G \neq \emptyset\}, G \in \mathcal{G}$. It is well-known that $\mathcal{F}$ is compact, second countable and Hausdorff, confer Theorem A2.5 in Kallenberg [5]. Furthermore, the *Kuratowski-metric* induces τ_{Fell}, whence $\mathcal{F}$ is a (nice) metric space. There are several simple families, which generate $\mathcal{B}_{Fell}$, confer Salinetti and Wets [13], p. 386 (Note the arguments there are also true for us, since T admits a separable metrization.) As a consequence many criteria for measurability of a map $\varphi : (\Omega, \mathcal{A}) \to (\mathcal{F}, \mathcal{B}_{Fell})$ are available, e. g.,

$$\{\varphi \cap G \neq \emptyset\} \in \mathcal{A} \text{ for all } G \in \mathcal{G}$$

or

$$\{\varphi \cap K = \emptyset\} \in \mathcal{A} \text{ for all } K \in \mathcal{K}.$$

For a lot of other equivalent characterizations of measurability confer Theorem 14.3 in Rockafellar and Wets [11]. The usual notation for a random element in $(\mathcal{F}, \mathcal{B}_{Fell})$ is *random closed set* or *closed valued measurable multifunction.* So, once we have that φ and $\varphi_n, n \in \mathbb{N}$, are random closed sets, then distributional convergence is well-defined and denoted by $\varphi_n \xrightarrow{\mathcal{D}} \varphi$ in $\mathcal{F}$. However, it is worthwhile to equip $\mathcal{F}$ with another (coarser) topology τ_{Miss} that is generated by the smaller family consisting only of all missing sets $\mathcal{M}(K), K \in \mathcal{K}$. Especially, weak convergence in the space $(\mathcal{F}, \mathcal{B}_{Miss})$ is still well-defined, confer Gänssler and Stute [3], chapter 8.4 for the definition of weak convergence in general topological spaces. It is denoted by $\varphi_n \xrightarrow{i-\mathcal{D}} \varphi$ in $\mathcal{F}$. The investigation of this alternative concept of weak convergence in $\mathcal{F}$ goes back to Vogel [15,16], who calls it *inner approximation in distribution* or *semi-convergence in distribution*. She gives equivalent characterizations, which go beyond the Portmanteau-Theorem and from which one can infer that the notion of *asymptotic dominance* of Pflug [9] coincides with inner approximation in distribution. Of course, since τ_{Miss} is contained in τ_{Fell} the Portmanteau-Theorem yields that $\mathcal{D}$-convergence entails semi-convergence in distribution. Vogel [15,16] also gives an equivalent characterization of τ_{Miss}-convergence in terms of the Kuratowsi-limessuperior. Moreover, the topology τ_{Miss} is shown to be *quasi-pseudo-metrizable*. However, a metrization is impossible, confer Gersch [4], which is a good reference for further properties of the topological space $(\mathcal{F}, \tau_{Miss})$.

Now, having fixed the underlying topologies we now can give several solutions of our main problem. Our primary result is

Theorem 1.1. For every $n \in \mathbb{N}$ let ϵ_n be a nonnegative random variable and $\varphi_n \subseteq \epsilon_n - \text{Argmin}(Z_n)$ be a random closed set. Moreover, assume Z is lower semicontinuous with random closed set $\text{Argmin}(Z) \neq \emptyset$.

Then, if

$$Z_n \rightsquigarrow Z \quad \text{in } l^\infty(T), \tag{1.1}$$

and

$$\epsilon_n = o_{\mathbb{P}}(1), \tag{1.2}$$

it follows that

$$\varphi_n \xrightarrow{i-\mathcal{D}} \text{Argmin}(Z) \quad \text{in } \mathcal{F}. \tag{1.3}$$

Notice that the special case in which the sets $\epsilon_n - \text{Argmin}(Z_n), n \in \mathbb{N}$, of ϵ_n-optimal solutions are replaced by the sets $\text{Argmin}(Z_n), n \in \mathbb{N}$, of all minimizing points corresponds to $\epsilon_n = 0$ for all $n \in \mathbb{N}$, whence assumption (1.2) is trivially fulfilled. We obtain

Corollary 1.2. Let Z be lower semicontinuous and assume the argmin-sets $\text{Argmin}(Z_n)$, $n \in \mathbb{N}$, and $\text{Argmin}(Z) \neq \emptyset$ are random closed sets. Then from

$$Z_n \rightsquigarrow Z \quad \text{in } l^\infty(T) \tag{1.4}$$

it follows that

$$\text{Argmin}(Z_n) \xrightarrow{i-\mathcal{D}} \text{Argmin}(Z) \quad \text{in } \mathcal{F}.$$

The assertion of Theorem 1.1 can significantly be sharpened if the sequence $(\varphi_n)_n$ is *stochastically bounded* in the following sense:

$$\lim_{i\to\infty} \limsup_{n\to\infty} \mathbb{P}(\varphi_n \not\subseteq T_i) = 0, \tag{1.5}$$

for a sequence

$$T_i, i \in \mathbb{N}, \text{ of compact sets with } T_i^\circ \uparrow T, \tag{1.6}$$

where C° denotes the interior of any set $C \subseteq T$.

Remark 1.3. [i] Since T is locally compact and Hausdorff a sequence with property (1.6) exists as a consequence of Theorem 21 on p. 203 in Royden [12].

[ii] Notice that for every fixed $n \in \mathbb{N}$ the sequence $\mathbb{P}(\varphi_n \not\subseteq T_i)$ is decreasing as $i \to \infty$. Therefore (1.5) is equivalent with:

For every $\epsilon > 0$ there exist natural numbers $i_0 = i_0(\epsilon)$ and $n_0 = n_0(\epsilon)$ such that

$$\mathbb{P}(\varphi_n \not\subseteq T_{i_0}) \le \epsilon \quad \forall\, n \ge n_0. \tag{1.7}$$

[iii] Let $K_i, i \in \mathbb{N}$, be any other sequence of compact sets with property (1.6). Then for T_{i_0} in (1.7) by compactness we find a natural number $m(i_0(\epsilon)) =: m_0(\epsilon) =: m_0$ such that $T_{i_0} \subseteq K_{m_0}$, whence $\mathbb{P}(\varphi_n \not\subseteq K_{m_0}) \le \epsilon \; \forall\, n \ge n_0$.

[iv] By [i] the concept of stochastic boundedness is well-defined for sequences of arbitrary random closed sets. Moreover, by [iii] we know that once (1.5) is fulfilled for a *single* sequence it actually holds for *every* sequence (T_i) with (1.6).

Theorem 1.4. Under the assumptions of Theorem 1.1 assume in addition that the sequence $(\varphi_n)_n$ is stochastically bounded. Then

$$\limsup_{n\to\infty} \mathbb{P}\left(\bigcap_{j=1}^{r} \{\varphi_n \cap F_j \neq \emptyset\}\right) \le \mathbb{P}\left(\bigcap_{j=1}^{r} \{\varphi \cap F_j \neq \emptyset\}\right) \tag{1.8}$$

for all $r \in \mathbb{N}$ and all *closed* $F_1, \ldots, F_r \in \mathcal{F}$.

Moreover, if all φ_n are nonempty and if $\mathrm{Argmin}(Z) = \{\tau\}$ for some mapping $\tau : \Omega \to T$ then actually

$$\varphi_n \xrightarrow{\mathcal{D}} \mathrm{Argmin}(Z) \quad \text{in } \mathcal{F}. \tag{1.9}$$

Corollary 1.5. For every $n \in \mathbb{N}$ let $\varphi_n \subseteq \epsilon_n - \mathrm{Argmin}(Z_n)$ be a nonempty random closed set. Then there exists a sequence (τ_n) of *measurable selections of* φ_n, that is $\tau_n : \Omega \to T$ is Borel-measurable with $\tau_n \in \varphi_n$.

Moreover assume that $\mathrm{Argmin}(Z) = \{\tau\}$ is a random closed set and that

$$\lim_{i\to\infty} \limsup_{n\to\infty} \mathbb{P}(\tau_n \notin T_i) = 0 \tag{1.10}$$

where $T_i, i \in \mathbb{N}$, is a sequence with (1.6).

Then, if (1.1) and (1.2) are true, it follows that

$$\tau_n \xrightarrow{\mathcal{D}} \tau \quad \text{in } T. \tag{1.11}$$

Indeed, (1.11) holds for *every* sequence (τ_n) of measurable selections of φ_n with property (1.10).

Remark 1.6. [i] Remark 1.3 carries over analogously. Especially, (1.10) is valid for all sequences (T_i) with (1.6).

[ii] Note that in our results we require that $\operatorname{Argmin}(Z)$ is a random closed set. This property is not guaranteed a priori by our assumption on Z to be a stochastic process (i. e. $Z(t) \equiv Z(\cdot)(t)$ is a real random variable for every $t \in T$) with lower semicontinuous trajectories. Such processes are also called *lsc integrands* and in general they are not *normal integrands* in the sense of Rockafellar and Wets [11]. For the latter the corresponding collection of all minimizing points actually is a random closed set, confer Theorem 14.37 of Rockafellar and Wets [11]. However, Theorem 14.40 in [11] says that if the above measurability condition is sharpened a bit Z is even a normal integrand and we have that $\operatorname{Argmin}(Z)$ is a random closed set.

[iii] We like to mention that sets of the type as occurring in (1.5), (1.7) or (1.8) are measurable, because of the measurability characterizations given in Theorem 14.3 of Rockafellar and Wets [11]. Consequently, the corresponding $\mathbb{P}$-probabilities are well-defined.

[iv] From (1.8) we can easily deduce Theorem 2 of Ferger [2]. To see this let $T = \mathbb{R}$ and choose $T_i := [-i, i]$. Consider a measurable selection $\tau_n \in \epsilon_n - \operatorname{Argmin}(Z_n)$. Observe that the smallest and largest minimizing point $\sigma = \min \operatorname{Argmin}(Z)$ and $\lambda = \max \operatorname{Argmin}(Z)$ exist, since Z is lower semicontinuous, whence $\operatorname{Argmin}(Z)$ is closed. These quantities are Borel-measurable. This can be derived with analogue arguments as in the proof of Corollary 1 of Ferger [2]. Put $\varphi_n := \{\tau_n\}$ and observe that (1.5) is equivalent with stochastic boundedness (uniform tightness) of the sequence (τ_n). Therefore we may apply (1.8) with $r = 1$ and $F_1 = (-\infty, x], x \in \mathbb{R}$ (which is closed, but not compact!), to conclude:

$$\limsup_{n\to\infty} \mathbb{P}(\tau_n \le x) \le \mathbb{P}(\operatorname{Argmin}(Z) \cap (-\infty, x] \ne \emptyset) \le \mathbb{P}(\sigma \le x) \text{ for all } x \in \mathbb{R},$$

where the second inequality follows from $\{\operatorname{Argmin}(Z) \cap (-\infty, x] \ne \emptyset\} \subseteq \{\sigma \le x\}$. In the same way with $F_1 = [x, \infty)$ and by taking complements we obtain:

$$\liminf_{n\to\infty} \mathbb{P}(\tau_n < x) \ge \mathbb{P}(\operatorname{Argmin}(Z) \cap [x, \infty) = \emptyset) \ge \mathbb{P}(\lambda < x) \text{ for all } x \in \mathbb{R},$$

where we use $\{\operatorname{Argmin}(Z) \cap [x, \infty) = \emptyset\} \supseteq \{\lambda < x\}$. Thus we have shown that Theorem 1.4 includes Theorem 2 of Ferger [2].

[v] Vogel [16] shows that if (1.8) holds only for all *compact* $F_1, \ldots, F_r$ then (1.3) follows and vice versa. Therefore the first assertion of Theorem 1.4 is strictly stronger than inner approximation in distribution.

Corollary 1.5 is in accordance with the Argmin-Theorem of van der Vaart and Wellner [14]. Here and there the assumption (1.10) is *necessary* for (1.11). Indeed, every distributional convergent sequence (τ_n) is *stochastically bounded* in the sense of (1.10). This is as a simple consequence of the Portmanteau-Theorem taking into account that we can find compact T_i with (1.6) such that every T_i is a continuity-set of τ. In contrast, distributional convergence of random closed sets (φ_n) does not entail stochastic boundedness (1.5). To see this consider (non-random) φ_n equal to the straight line through the origin in $\mathbb{R}^2$, which as n tends to infinity rotate to the ordinate $\varphi = \{0\} \times \mathbb{R}$. Then (φ_n) converges to φ in the Fell-Topology, confer Rockafellar and Wets [11], p. 118, and especially $\varphi_n \xrightarrow{\mathcal{D}} \varphi$, but clearly (φ_n) is not stochastically bounded.

Note that we have not imposed any continuity requirements on the trajectories of Z_n, so far. Indeed, if we make concessions in this regard, then in case of $T = \mathbb{R}^d$ it is possible to sharpen Corollary 1.2 by replacing (1.4) through a weaker type of distributional convergenc . To this end recall that $\mathrm{Argmin}(Z_n)$ is closed if Z_n has lower semicontinuous (lsc) realizations. So, let us assume that

$$Z_n : \Omega \to SC(\mathbb{R}^d), \quad n \in \mathbb{N},$$

and

$$Z : \Omega \to SC(\mathbb{R}^d),$$

are maps from Ω into the set $SC(\mathbb{R}^d)$ of all lsc functions $z : \mathbb{R}^d \to \overline{\mathbb{R}}$ with $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$. This is the traditional function space used in stochastic optimization from which we like to repeat very shortly some few basic definitions. To each $z \in SC(\mathbb{R}^d)$ there belongs the *epigraph of* z defined by

$$\mathrm{epi}(z) := \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : z(x) \le y\}.$$

A map $Z : \Omega \to SC(\mathbb{R}^d)$ is called *random lower semicontinuous function* or *normal integrand* if $\mathrm{epi}(Z)$ is a random closed set. Thus $\mathrm{epi}(Z)$ induces a probability measure on $(\mathcal{F}, \mathcal{B}_{Fell})$ and one defines: A sequence (Z_n) of random lsc functions *epi-converges in distribution* to a limit Z if $\mathrm{epi}(Z_n) \xrightarrow{\mathcal{D}} \mathrm{epi}(Z)$ in $\mathcal{F}$. In symbol: $Z_n \stackrel{\mathrm{epi}-\mathcal{D}}{\longrightarrow} Z$. The following Theorem follows directly from Theorem 1.3 of Pflug [9] and Lemma 2.1 of Vogel [16].

Theorem (Pflug–Vogel). Let $T = \mathbb{R}^d$. If $Z_n \stackrel{\mathrm{epi}-\mathcal{D}}{\longrightarrow} Z$ then

$$\mathrm{Argmin}(Z_n) \stackrel{i-\mathcal{D}}{\longrightarrow} \mathrm{Argmin}(Z) \quad \text{in } \mathcal{F}.$$

The above result in fact holds a further refinement of Corollary 1.2 under the supplemental requirement of lower semicontinuity of the $Z_n, n \in \mathbb{N}$. The reason for this is that weak convergence (1.4) entails epi-convergence in distribution, whereas the converse does not hold, confer Proposition 1 of Pflug [10]. (The proof given there carries over, since the Portmanteau-Theorem of van der Vaart and Wellner

[14] can be applied in the same manner.) Thus the additional assumption of lower-semicontinuity yields that the strictly weaker concept of epi-convergence in distribution actually is sufficient for distributional semi-convergence of the involved argmin-sets. Indeed, as Pflug [10] points out, the space $SC(\mathbb{R}^d)$ of all lower semicontinuous functions endowed with the epigraph-topology is in some sense the minimal setting to allow for the implication in the last theorem. However note that by Theorem 2.25 of Gersch [4] epi-convergence in distribution is equivalent with convergence of the finite dimensional distributions plus so-called *stochastic equi-lower semicontinuity* of the sequence (Z_n), where the latter is the counterpart of *asymptotic tightness* in case of weak convergence (1.4). So, in a specific situation it is to decide which of these two conditions is easier to verify.

Remark 1.7. In empirical process theory it is more convenient to work with the space $D(\mathbb{R})$ of all cad-lag functions (continuous from the right with limits from the left) or the multivariate generalization $D(\mathbb{R}^d)$ endowed with a suitable so-called Skorokhod-topology. One very nice feature of that topology is that every cad-lag stochastic process is Borel-measurable when viewed as map into $D(\mathbb{R}^d)$. Moreover, according to Lindvall [7] convergence in distribution in $D(\mathbb{R})$ is equivalent with convergence in distribution in $D[-a,a]$ for (almost) every positive a, where $D[-a,a]$ is the traditional Skorokhod-space, confer, e. g., Billingsley [1]. (The extension to $D(\mathbb{R}^d)$ is proved by Lagodowski and Rychlik [6].)

Now, Vogel's [15] Lemma 8.6(ii) says that traditional weak convergence of Z_n to Z when viewed as maps into the function space $D(\mathbb{R})$ entails distributional convergence of epi$(\overline{Z}_n)$ to epi$(\overline{Z})$ in $\mathcal{F}$, where $\overline{z}$ denotes the lsc regularization of a function z. Therefore by the above theorem of Vogel and Pflug we obtain:

$$Z_n \stackrel{\mathcal{D}}{\longrightarrow} Z \text{ in } D(\mathbb{R}) \text{ implies } \operatorname{Argmin}(\overline{Z}_n) \stackrel{i-\mathcal{D}}{\longrightarrow} \operatorname{Argmin}(\overline{Z}) \text{ in } \mathcal{F}.$$

From this Vogel [15] deduces an improved version of Ferger's [2] Theorem 3 and in addition extends it to ϵ_n-optimal solutions, confer Corollary 8.9 in [15].

Remark 1.8. Let the space $l^\infty(\mathbb{R})$ be endowed with the metric of uniform convergence on compacta and the pertaining Borel-σ-algebra. Then similarly as in the above remark Vogel's [15] Lemma 8.6(iii) results in:

$$Z_n \stackrel{\mathcal{D}}{\longrightarrow} Z \text{ in } l^\infty(\mathbb{R}) \text{ implies } \operatorname{Argmin}(\overline{Z}_n) \stackrel{i-\mathcal{D}}{\longrightarrow} \operatorname{Argmin}(\overline{Z}) \text{ in } \mathcal{F}.$$

Observe that the conclusion essentially is the same as in our Corollary 1.2 for $T = \mathbb{R}$, but there is a main difference between these two results. Namely, the assumption $Z_n \stackrel{\mathcal{D}}{\longrightarrow} Z$ in $l^\infty(\mathbb{R})$ is much more restrictive than $Z_n \rightsquigarrow Z$ in $l^\infty(\mathbb{R})$ because in case of traditional weak convergence $\stackrel{\mathcal{D}}{\longrightarrow}$ Borel-measurability of Z and $Z_n, n \in \mathbb{N}$, is required. In contrast to the situation of Remark 1.7 this assumption can and does fail for locally bounded stochastic processes as pointed out by van der Vaart and Wellner [14], p. 3. Indeed, e. g., the empirical distribution function or the classical empirical process do not meet measurability. Actually, this failure was the main reason for establishing a theory of weak convergence $\rightsquigarrow$ for arbitrary (possibly non-measurable) maps.

We like to end this section with a short discussion on the relations between distributional convergence and semi-convergence in distribution in the framework of general random closed sets. Our starting point is the following

Proposition 1.9. Let φ and $\varphi_n, n \in \mathbb{N}$, be random closed sets such that

$$\varphi_n \xrightarrow{i-\mathcal{D}} \varphi \quad \text{in } \mathcal{F}. \tag{1.12}$$

and (φ_n) is stochastically bounded. Then

$$\limsup_{n\to\infty} \mathbb{P}\left(\bigcap_{j=1}^{r}\{\varphi_n \cap F_j \neq \emptyset\}\right) \leq \mathbb{P}\left(\bigcap_{j=1}^{r}\{\varphi \cap F_j \neq \emptyset\}\right) \tag{1.13}$$

for all $r \in \mathbb{N}$ and all $F_1, \ldots, F_r \in \mathcal{F}$.

If in addition $\varphi_n, n \in \mathbb{N}$, are nonempty and $\varphi = \{\tau\}$ for some mapping $\tau : \Omega \to T$ then

$$\varphi_n \xrightarrow{\mathcal{D}} \varphi \quad \text{in } \mathcal{F}. \tag{1.14}$$

So, semi-convergence in distribution to a singleton plus stochastic boundedness ensures convergence in distribution, for which in turn however stochastic boundedness in general is not necessary as we have pointed out above by a counter-example. Therefore, one could expect that actually a sharper result holds which comes along without the boundedness requirement. Thus we ask whether the following conclusion is true:

$$\varphi_n \xrightarrow{i-\mathcal{D}} \varphi = \{\tau\} \quad \text{in } \mathcal{F} \quad \Rightarrow \quad \varphi_n \xrightarrow{\mathcal{D}} \varphi = \{\tau\} \quad \text{in } \mathcal{F}\ ? \tag{1.15}$$

The answer is negative as we show by the following

Counter-example. Let us assume that the implication (1.15) is true. For the sake of convenience we only consider $T = \mathbb{R}^d$, but our arguments can easily be extended to general T lcscH. To this end introduce the non-random sets $\varphi_n := \{t_n\}$, where $t_n := 0$, if n is even, and $|t_n| > n$, if n is odd, where $|\cdot|$ denotes the Eulidean norm. Then the Kuratowski-limessuperior is equal to

$$K - \limsup_{n\to\infty} \varphi_n = \{0\},$$

whence by Lemma 2.1 of Vogel [16] we have that $\varphi_n \to \varphi := \{0\}$ in τ_{Miss}. Now, for every open $O \in \tau_{Miss}$ that contains φ it follows that

$$\liminf_{n\to\infty} \mathbb{P}(\varphi_n \in O) = \liminf_{n\to\infty} 1_{\{\varphi_n \in O\}} = 1 = \mathbb{P}(\varphi \in O),$$

whereas for $O \in \tau_{Miss}$ not containing φ one has that

$$\liminf_{n\to\infty} \mathbb{P}(\varphi_n \in O) \geq 0 = \mathbb{P}(\varphi \in O).$$

Thus by the Portmanteau-Theorem of Gänssler and Stute [3], Proposition 8.4.9, it follows that $\varphi_n \xrightarrow{i-\mathcal{D}} \varphi$, so that by our assumption $\varphi_n \xrightarrow{\mathcal{D}} \varphi$. Another application of the Portmanteau-Theorem yields that $g(\varphi_n) \to g(\varphi)$ for all functions $g : \mathcal{F} \to \mathbb{R}$ which are bounded and continuous with respect to τ_{Fell}. As a consequence $\varphi_n \to \varphi$ in τ_{Fell}, because otherwise there is an open set $O \in \tau_{Fell}$ that contains φ and a subsequence $(\varphi_{n_k})_{k\in\mathbb{N}}$ such that $\varphi_{n_k} \notin O$ for all $k \in \mathbb{N}$. According to Uryson's Lemma there exists a continuous function g such that $g(\varphi) = 0$ and $g = 1$ on the complement $\overline{O} := \mathcal{F} \setminus O$ as well as $0 \leq g \leq 1$. Therefore, $g(\varphi_{n_k}) = 1 \to 1 \neq 0 = g(\varphi)$, whence $g(\varphi_n) \nrightarrow g(\varphi)$, a contradiction. This shows that $\varphi_n \to \varphi$ in τ_{Fell}, which in turn is a contradiction to the fact that the Kuratowski-limesinferior $K - \liminf_{n\to\infty} \varphi_n$ is empty. Hence the implication in (1.15) is not true.

To sum up, convergence in distribution $\varphi_n \xrightarrow{\mathcal{D}} \varphi$ always entails semi-convergence in distribution $\varphi_n \xrightarrow{i-\mathcal{D}} \varphi$, whereas the converse is not true, even if the limit is a singleton. However in that special case the additional requirement of stochastic boundedness makes the reverse conclusion to be valid.

2. PROOFS

P r o o f. (Theorem 1.1) According to Vogel [16] for the derivation of (1.3) we have to show that

$$\limsup_{n\to\infty} \mathbb{P}\left(\bigcap_{j=1}^{r} \{\varphi_n \cap K_j \neq \emptyset\}\right) \leq \mathbb{P}\left(\bigcap_{j=1}^{r} \{\varphi \cap K_j \neq \emptyset\}\right) \tag{2.1}$$

for all $r \in \mathbb{N}$ and all compact $K_1, \ldots, K_r \in \mathcal{K}$.

As before let $\overline{C} := T \setminus C$ denote the complement of a set $C \subseteq T$. We start with

$$\begin{aligned} E_n := &\bigcap_{j=1}^{r} \{\varphi_n \cap K_j \neq \emptyset\} \\ &\subseteq \bigcap_{j=1}^{r} \left\{ \inf_{t\in K_j} Z_n(t) \leq \inf_{t\in\overline{K_j}} Z_n(t) + \epsilon_n \right\} &(2.2) \\ &\subseteq \bigcap_{j=1}^{r} \left\{ \inf_{t\in K_j} Z_n(t) \leq \inf_{t\in\overline{K_j}\cap T_i} Z_n(t) + \epsilon_n \right\} \quad \forall\, i \in \mathbb{N}. &(2.3) \end{aligned}$$

To see the inclusion (2.2) let us assume that the event E_n occurs, but that

$$\inf_{t\in K_j} Z_n(t) > \inf_{t\in\overline{K_j}} Z_n(t) + \epsilon_n \ \text{ for at least one } 1 \leq j \leq r\,. \tag{2.4}$$

Since $\varphi_n \cap K_j \neq \emptyset$ we find a point $\tau_n \in \varphi_n \cap K_j$. Conclude that

$$\begin{aligned}
Z_n(\tau_n) &\geq \inf_{t\in K_j} Z_n(t) && \text{because } \tau_n \in K_j \\
&> \inf_{t\in \overline{K_j}} Z_n(t) + \epsilon_n && \text{by (2.4)} \\
&\geq \inf_{t\in T} Z_n(t) + \epsilon_n && \text{because } \overline{K_j} \subseteq T \\
&\geq Z_n(\tau_n) && \text{because } \tau_n \in \varphi_n \subseteq \epsilon_n - \text{Argmin}(Z_n).
\end{aligned}$$

Thus assumption (2.4) yields a contradiction and we have shown (2.2).

The second inclusion (2.3) simply follows from the fact that $\overline{K_j} \cap T_i \subseteq \overline{K_j} \;\; \forall\, i \in \mathbb{N}$.

Next, introduce the compact set $C_i := \bigcup_{j=1}^r K_j \cup T_i$ and note that for every fixed $M \subseteq C_i$ the mapping

$$z \mapsto \inf_{t\in M} z(t)\ , \quad z \in \ell^\infty(C_i)\ ,$$

is continuous on its domain $\ell^\infty(C_i)$ endowed with the sup-norm. It follows that for every $i \in \mathbb{N}$

$$H_i : z \mapsto \left(\inf_{t\in K_j} z(t) - \inf_{t\in \overline{K_j}\cap T_i} z(t) \right)_{1\leq j\leq r}$$

is a continuous map from $l^\infty(C_i)$ to $\mathbb{R}^r$. Therefore assumption (1.1) and Theorem 1.3.6 (Continuous Mapping Theorem CMT) of van der Vaart and Wellner [14] yield that

$$H_i(Z_n) \rightsquigarrow H_i(Z) \;\text{ in } \mathbb{R}^r \;\text{ as }\; n \to \infty \quad \forall\, i \in \mathbb{N}.$$

By assumption (1.2) the vector $\delta_n := (\epsilon_n, \ldots, \epsilon_n) \in \mathbb{R}^r$ converges to zero in $\mathbb{P}$-probability. Therefore by Theorem 1.10.2 of van der Vaart and Wellner [14] we obtain

$$H_i(Z_n) - \delta_n \rightsquigarrow H_i(Z) \;\text{ in } \mathbb{R}^r \;\text{ as }\; n \to \infty \quad \forall\, i \in \mathbb{N}.$$

Since the half-space $(-\infty, 0]^r$ is closed in $\mathbb{R}^r$ the Portmanteau Theorem of van der Vaart and Wellner [14], p. 18, ensures that for all $i \in \mathbb{N}$:

$$\limsup_{n\to\infty} \mathbb{P}(E_n) \leq \limsup_{n\to\infty} \mathbb{P}\left(H_i(Z_n) - \delta_n \in (-\infty, 0]^r\right) \leq \mathbb{P}(H_i(Z) \in (-\infty, 0]^r) = \mathbb{P}(E_i) \tag{2.5}$$

where

$$E_i := \bigcap_{j=1}^{r} \left\{ \inf_{t\in K_j} Z(t) \leq \inf_{t\in \overline{K_j}\cap T_i} Z(t) \right\} \quad \forall\, i \in \mathbb{N}. \tag{2.6}$$

Since the sequence (E_i) is monotone decreasing we can conclude that

$$\begin{aligned}
&\lim_{i\to\infty} \mathbb{P}(E_i) \\
&= \mathbb{P}\left(\bigcap_{i=1}^{\infty} E_i\right) \\
&= \mathbb{P}\left(\bigcap_{j=1}^{r}\bigcap_{i=1}^{\infty}\left\{\inf_{t\in K_j} Z(t) \leq \inf_{t\in \overline{K_j}\cap T_i} Z(t)\right\}\right) \\
&= \mathbb{P}\left(\bigcap_{j=1}^{r}\left\{\inf_{t\in K_j} Z(t) \leq \inf_{t\in \overline{K_j}} Z(t)\right\}\right) && (2.7) \\
&\leq \mathbb{P}\left(\bigcap_{j=1}^{r}\{\mathrm{Argmin}(Z)\cap K_j \neq \emptyset\}\right). && (2.8)
\end{aligned}$$

To see (2.7) note that for each fixed $1 \leq j \leq r$ we have that

$$\inf_{t\in K_j} Z(t) \leq \inf_{t\in \overline{K_j}\cap T_i} Z(t) \quad \forall\, i \in \mathbb{N} \iff \inf_{t\in K_j} Z(t) \leq \inf_{i\in\mathbb{N}} \inf_{t\in \overline{K_j}\cap T_i} Z(t).$$

Since

$$\inf_{i\in\mathbb{N}} \inf_{t\in \overline{K_j}\cap T_i} Z(t) = \inf\{Z(t) : t \in \cup_{i=1}^{\infty} \overline{K_j}\cap T_i\} = \inf_{t\in \overline{K_j}} Z(t),$$

we arrive at the desired equality (2.7). As to the inequality (2.8) observe that on the event $E := \bigcap_{j=1}^{r}\{\inf_{t\in K_j} Z(t) \leq \inf_{t\in \overline{K_j}} Z(t)\}$ one has that for every $1 \leq j \leq r$

$$\inf_{t\in T} Z(t) = \inf_{t\in K_j} Z(t) = Z(\tau_j)$$

for some $\tau_j \in K_j$, since K_j is compact and Z is lower semicontinuous. Conclude that $\tau_j \in \mathrm{Argmin}(Z) \cap K_j$ for all $1 \leq j \leq r$ which shows

$$E \subseteq \bigcap_{j=1}^{r}\{\mathrm{Argmin}(Z)\cap K_j \neq \emptyset\},$$

whence (2.8) follows. Taking the limit $i \to \infty$ we obtain (2.1) from (2.5) and (2.8). This finishes our proof. □

P r o o f. (Corollary 1.2) Recall that $\mathrm{Argmin}(Z_n) = \epsilon_n - \mathrm{Argmin}(Z_n)$ with $\epsilon_n = 0$, whence (1.2) is satisfied. An application of Theorem 1.1 yields the result. □

P r o o f. (Theorem 1.4) From Theorem 1.1 we know that $\varphi_n \xrightarrow{i-\mathcal{D}} \varphi$ in $\mathcal{F}$ with $\varphi := \mathrm{Argmin}(Z)$. Thus the assertion follows from Proposition 1.9. □

Proof. (Corollary 1.5) The existence of measurable selections is guaranteed by Corollary 14.6 in Rockafellar and Wets [11]. Here, note that φ_n is nonempty by assumption, whence the domain dom $\varphi_n := \{\omega \in \Omega : \varphi_n(\omega) \neq \emptyset\}$ is equal to Ω.

To prove the second part of the corollary let (τ_n) be *any* sequence of measurable selections of φ_n with property (1.10). Put $\tilde{\varphi}_n := \{\tau_n\}, n \in \mathbb{N}$. We check the conditions of Theorem 1.4. First, note that $\{\tilde{\varphi}_n \cap G \neq \emptyset\} = \{\tau_n \in G\} \in \mathcal{A}$ for all open G, whence each $\tilde{\varphi}_n$ is a random closed set. Further, since $\{\tau_n \notin T_i\} = \{\tilde{\varphi}_n \nsubseteq T_i\}$ assumption (1.10) guarantees the validity of (1.5) for the sequence $(\tilde{\varphi}_n)$. Thus for every closed set F it follows from (1.8) with $r = 1$ that

$$\limsup_{n\to\infty} \mathbb{P}(\tau_n \in F) = \limsup_{n\to\infty} \mathbb{P}(\{\tau_n\} \cap F \neq \emptyset) \leq \mathbb{P}(\{\tau\} \cap F \neq \emptyset) = \mathbb{P}(\tau \in F),$$

from which the assertion (1.11) follows by the Portmanteau-Theorem, confer Proposition 8.4.9 of Gänssler and Stute [3]. □

The proof of Proposition 1.9 relies on an a sufficient (and necessary) condition for convergence in distribution of random closed sets given in Theorem 2.1 of Norberg [8], confer also Theorem 14.27 in Kallenberg [5].

To formulate it let $\mathcal{C}$ denote the Borel-σ-algebra on T and define the family of sets $\hat{\mathcal{C}} := \{C \in \mathcal{C} : C \text{ is relatively compact}\}$. Moreover, for any random closed set φ let

$$\hat{\mathcal{C}}_\varphi := \{C \in \hat{\mathcal{C}} : \mathbb{P}(\varphi \cap C^\circ \neq \emptyset) = \mathbb{P}(\varphi \cap C^c \neq \emptyset)\},$$

where C° and C^c denotes the interior and the closure, respectively, of (any) set C. Here, the probabilities occuring in $\hat{\mathcal{C}}_\varphi$ are well-defined by definition of $\mathcal{B}_{Fell}$, since C° is open and C^c is compact.

Theorem (Norberg). Let $\varphi, \varphi_1, \varphi_2, \ldots$ be random closed in some lcscH space T. Then $\varphi_n \xrightarrow{\mathcal{D}} \varphi$ if and only if

$$\mathbb{P}(\varphi_n \cap C \neq \emptyset) \to \mathbb{P}(\varphi \cap C \neq \emptyset) \quad \forall\, C \in \hat{\mathcal{C}}_\varphi. \tag{2.9}$$

Note that the probabilities in (2.9) are well-defined, since $\{F \in \mathcal{F} : F \cap C \neq \emptyset : C \in \hat{\mathcal{C}}\}$ generates the Borel-σ-algebra $\mathcal{B}_{Fell}$ as is shown by Norberg [8], p. 727.

Proof. (Proposition 1.9) For the proof of (1.13) observe that for all $i \in \mathbb{N}$:

$$\bigcap_{j=1}^{r} \{\varphi_n \cap F_j \neq \emptyset\} \subseteq \bigcap_{j=1}^{r} \{\varphi_n \cap (F_j \cap T_i) \neq \emptyset\} \cup \{\varphi_n \nsubseteq T_i\}.$$

To see the inclusion assume that there exists some $1 \leq j \leq r$ such that $\varphi_n \cap (F_j \cap T_i) = \emptyset$ and that $\varphi_n \subseteq T_i$. Then $\varphi_n \cap F_j = (\varphi_n \cap F_j \cap T_i) \cup (\varphi_n \cap F_j \cap \overline{T}_i) = \varphi_n \cap F_j \cap \overline{T}_i \subseteq \varphi_n \cap \overline{T}_i = \emptyset$, because $\varphi_n \subseteq T_i$, whence $\varphi_n \cap F_j = \emptyset$. Next, note that $F_j \cap T_i, 1 \leq j \leq r$ are compact. Thus by (1.12) and Lemma 2.1 of Vogel [16] we obtain:

$$\limsup_{n\to\infty} \mathbb{P}\left(\bigcap_{j=1}^{r} \{\varphi_n \cap F_j \neq \emptyset\}\right) \leq \mathbb{P}\left(\bigcap_{j=1}^{r} \{\varphi \cap (F_j \cap T_i) \neq \emptyset\}\right) + \limsup_{n\to\infty} \mathbb{P}(\varphi_n \nsubseteq T_i).$$

Taking the limit $i \to \infty$ yields the assertion (1.13) upon noticing (1.5) and that

$$\bigcap_{j=1}^{r} \{\varphi \cap (F_j \cap T_i) \neq \emptyset\} \uparrow \bigcap_{j=1}^{r} \{\varphi \cap F_j \neq \emptyset\} \text{ as } i \to \infty.$$

For the proof of (1.14) we have to deduce (2.9) of Norberg's Theorem. The key step for this is to show that

$$\liminf_{n\to\infty} \mathbb{P}(\varphi_n \cap G \neq \emptyset) \geq \mathbb{P}(\varphi \cap G \neq \emptyset) \; \forall \, G \in \mathcal{G}. \tag{2.10}$$

Indeed, then it follows

$$\begin{aligned}
\mathbb{P}(\varphi \cap C^\circ \neq \emptyset) &\leq \liminf_{n\to\infty} \mathbb{P}(\varphi_n \cap C^\circ \neq \emptyset) && \text{by (2.10))} \\
&\leq \liminf_{n\to\infty} \mathbb{P}(\varphi_n \cap C \neq \emptyset) \\
&\leq \limsup_{n\to\infty} \mathbb{P}(\varphi_n \cap C \neq \emptyset) \\
&\leq \limsup_{n\to\infty} \mathbb{P}(\varphi_n \cap C^c \neq \emptyset) \\
&\leq \mathbb{P}(\varphi \cap C^c \neq \emptyset) && \text{by (1.13)} \\
&= \mathbb{P}(\varphi \cap C^\circ \neq \emptyset) && \text{because } C \in \hat{\mathcal{C}}_\varphi \\
&= \mathbb{P}(\varphi \cap C \neq \emptyset) && \text{because } C^\circ \subseteq C \subseteq C^c.
\end{aligned}$$

Thus is remains to show (2.10). But to see this note that

$$\begin{aligned}
\mathbb{P}(\varphi \cap G \neq \emptyset) &= \mathbb{P}(\tau \in G) \\
&= 1 - \mathbb{P}(\tau \in \overline{G}) \\
&= 1 - \mathbb{P}(\varphi \cap \overline{G} \neq \emptyset) \\
&\leq 1 - \limsup_{n\to\infty} \mathbb{P}(\varphi_n \cap \overline{G} \neq \emptyset) && \text{by (1.13), because } \overline{G} \text{ is closed} \\
&= \liminf_{n\to\infty} \mathbb{P}(\varphi_n \cap \overline{G} = \emptyset) \\
&\leq \liminf_{n\to\infty} \mathbb{P}(\varphi_n \cap G \neq \emptyset) && \text{because } \varphi_n \neq \emptyset.
\end{aligned}$$

This completes the proof. □

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