KYBERNETIKA — VOLUME 47 (2011), NUMBER 6, PAGES 944–954

ON THE CORE PROPERTY OF THE CYLINDER FUNCTIONS CLASS IN THE CONSTRUCTION OF INTERACTING PARTICLE SYSTEMS

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For general interacting particle systems in the sense of Liggett, it is proven that the class of cylinder functions forms a core for the associated Markov generator. It is argued that this result cannot be concluded by straightforwardly generalizing the standard proof technique that is applied when constructing interacting particle systems from their Markov pregenerators.

Keywords: Markov pregenerator, Markov generator, cylinder function, local function

Classification: 60K35, 82C22

1. INTRODUCTION

Interacting particle systems (IPS) in the sense of Liggett [5] are Feller–Markov processes on a space $\mathbb{X} = W^S$, where W is a compact metric space and S is a finite or countably infinite set. IPS are specified by defining a family of transition rate functions which are sufficiently local in the sense of Definition 2.2 below. Given the transition rates, a Markov pregenerator is defined on some dense subspace $D(\mathbb{X})$ of the space $C(\mathbb{X})$ of continuous functions on $\mathbb{X}$. This operator is closable in $C(\mathbb{X})$ with respect to the supremum norm and the closure is a Markov generator which is associated with a unique Feller–Markov process on $\mathbb{X}$. Thus $D(\mathbb{X})$ is a core for the Markov generator of the IPS constructed in this way.

Definition 1.1. Let A be a closed linear operator on $C(\mathbb{X})$ with domain $\vartheta(A)$. A linear subspace $D \subset \vartheta(A)$ is a *core* for A, if A is the closure of its restriction to D, that is $\overline{A|_D} = A$.

The class $D(\mathbb{X})$ consists of continuous functions that are localized in a well described sense, see (3) below. The class of cylinder functions, that are continuous functions that depend only on finitely many coordinates, see (4) below, forms a proper though dense subset of the localized functions. So it is natural to pose the question wether the smaller class of cylinder functions is also a core. It is widely assumed that this is true. See for instance [1, 2, 4]. However, an explicit proof of this statement has not been provided in the literature yet.

In this note, it is proven that the set of cylinder functions is rich enough to form a core for the Markov generator of an IPS. The core property of the cylinder functions class is derived from the core property of $D(\mathbb{X})$ by approximation arguments. Additional conditions on the transition rates are not necessary. It is argued that a straightforward generalization of the proof technique applied by Liggett in [5, Thm. I.3.9] when showing that $D(\mathbb{X})$ is a core does not work without additional conditions on the transition rates.

Clearly, the issue addressed here is nowhere near as interesting as the original problem of existence and uniqueness of IPS. It is, nevertheless, a component towards rounding off the theory of IPS. Cylinder functions are easier to work with than the less localized functions within $D(\mathbb{X})$. For instance, for finite range IPS, the Markov generator applied to cylinder functions reduces to a finite sum instead of an infinite one. The finding that the class of cylinder functions is actually a core allows to consider only these nicer functions when studying problems such as ergodicity, reversibility, duality or approximation of IPS. As IPS are increasingly used in applications, for instance in physics, finance or theoretical biology, it is expected that the number of studies dealing with those issues is growing as well, thus giving any contribution towards easier handling of the model additional weight.

2. NOTATION

Let S be a countable set of sites and take the symbol $\mathcal{T}$ for the set of all non-empty finite subsets of S. For singletons in $\mathcal{T}$, we usually write x instead of $\{x\}$. Suppose that (W, d) is a compact metric space and denote by $\mathcal{W}$ the Borel σ-field on W. Define the *configuration space* $\mathbb{X} := W^S$. For each $T \subset S$, the set $S \backslash T$ shall be denoted by T^c, $\mathbb{X}_T = W^T$ will represent the configuration space over T and

$$\pi_T : \mathbb{X} \to \mathbb{X}_T : \pi_T(\eta) := (\eta(x))_{x \in T} =: \eta_T$$

denotes the *projection* from $\mathbb{X}$ onto $\mathbb{X}_T$. Let be $\mathcal{F}_T$ the σ-field given by $\mathcal{F}_T := \pi_T^{-1}\left(\bigotimes_{x \in T} \mathcal{W}\right)$. For $u \in \mathbb{X}_T$, $\eta \in \mathbb{X}$, let $\tau_T(\eta, u)$ be the configuration where η_T is replaced by u, that is

$$\tau_T(\eta, u)(z) = \begin{cases} \eta(z), & z \in T^c \\ u(z), & z \in T. \end{cases} \tag{1}$$

The set of continuous real functions on $\mathbb{X}$ equipped with the sup-norm $\|\cdot\|$ is denoted by $C(\mathbb{X})$. The closure of a set $B \subset C(\mathbb{X})$ with respect to the sup-norm is $\overline{B}$. For $f \in C(\mathbb{X})$, define

$$\begin{aligned} \Delta_T(f) &:= \sup\{|f(\eta) - f(\zeta)| : \eta, \zeta \in \mathbb{X}, \eta_{T^c} = \zeta_{T^c}\}, \quad T \subset S; \qquad (2)\\ \operatorname{tm}(f) &:= \{x \in S : \Delta_x(f) > 0\}; \\ |||f||| &:= \sum_{x \in S} \Delta_x(f). \end{aligned}$$

Note that the latter sum can take values in $[0, \infty]$. Define further

$$D(\mathbb{X}) := \{f \in C(\mathbb{X}) : |||f||| < \infty\} \tag{3}$$

$$T(\mathbb{X}) := \{f \in C(\mathbb{X}) : \operatorname{tm}(f) \in \mathcal{T} \cup \{\emptyset\}\}. \tag{4}$$

The elements of $T(\mathbb{X})$ are called *local functions* or *cylinder functions*.

One easily finds that

$$T(\mathbb{X}) \subset D(\mathbb{X}) \subset C(\mathbb{X}).$$

In addition, $C(\mathbb{X})$ is the uniform closure of $T(\mathbb{X})$. Indeed, $T(\mathbb{X})$ is a linear subspace of $C(\mathbb{X})$ containing the constant functions. If $f, g \in T(\mathbb{X})$, then $f \cdot g \in T(\mathbb{X})$. In addition, $T(\mathbb{X})$ separates points by the following consideration. Given $\eta, \zeta \in \mathbb{X}, \eta \neq \zeta$, there exists $x \in S$ with $\eta(x) \neq \zeta(x)$. For $f := \mathbb{1}_{\pi_x^{-1}(\eta(x))} \in T(\mathbb{X})$ one finds that $f(\eta) \neq f(\zeta)$. Thus, according to the Stone-Weierstrass Theorem, $T(\mathbb{X})$ is dense in $C(\mathbb{X})$.

Definition 2.1. Suppose that we are given a family $c = (c_T)_{T\in\mathcal{T}}$ of non-negative functions $c_T : \mathbb{X} \times \mathcal{F}_T \to [0, \infty)$. Then c is a *family of transition rate functions*, if the following conditions are satisfied.

(i) For each $T \in \mathcal{T}, \eta \in \mathbb{X}$, the map $c_T(\eta, \cdot)$ is a finite positive measure on $(\mathbb{X}_T, \mathcal{F}_T)$.

(ii) For each $T \in \mathcal{T}$, the function $\eta \mapsto c_T(\eta, \cdot)$ is a continuous map from $\mathbb{X}$ to the space $\mathcal{M}(\mathbb{X}_T, \mathcal{F}_T)$ of finite measures on $(\mathbb{X}_T, \mathcal{F}_T)$ with respect to the topology of weak convergence.

Given a family $c = (c_T(\cdot, \cdot))_{T\in\mathcal{T}}$ of transition rate functions, define

$$c_T(x) := \sup\left\{\|c_T(\eta, \cdot) - c_T(\zeta, \cdot)\|_{\text{tv}} : \eta, \zeta \in \mathbb{X}, \eta_{x^c} = \zeta_{x^c}\right\}, \quad x \in S, T \in \mathcal{T}, \tag{5}$$

where $\|\cdot\|_{\text{tv}}$ is the total-variation norm on $\mathcal{M}(\mathbb{X}_T, \mathcal{F}_T)$. Let be

$$c_T := \sup_{\eta\in\mathbb{X}} c_T\left(\eta, \mathbb{X}_T\right), \quad T \in \mathcal{T},$$

and

$$\gamma(x, z) := \sum_{\substack{T\in\mathcal{T},\\ T\ni x}} c_T(z), \quad x, z \in S, x \neq z, \quad \gamma(x, x) = 0, \quad x \in S. \tag{6}$$

Definition 2.2. A family $c = (c_T)_{T\in\mathcal{T}}$ of transition rate functions is *admissible*, if

$$K := \sup_{x\in S} \sum_{T\ni x} c_T < \infty, \tag{7}$$

and

$$M := \sup_{x\in S} \sum_{z\in S} \gamma(x, z) < \infty. \tag{8}$$

Remark 2.3. The two conditions (7) and (8) for transition rate functions to be admissible are exactly those stated in [5, I(3.3) and I(3.8)]. They are standard assumptions in the construction of IPS.

Suppose a family $c = (c_T(., .))_{T\in\mathcal{T}}$ of admissible transition rates is given. Define an operator $A : D(\mathbb{X}) \to C(\mathbb{X})$ by

$$Af(\eta) = \sum_{T\in\mathcal{T}} \int_{\mathbb{X}_T} c_T(\eta, \mathrm{d}v)[f(\tau_T(\eta, v)) - f(\eta)], \quad \eta \in \mathbb{X}, f \in D(\mathbb{X}). \tag{9}$$

According to [5, Prop. I.3.2], A is well-defined if c is admissible. The operator A is called *associated* to the family c. By [5, Thm. I.3.9], the closure $\overline{A}$ of A is a Markov generator which generates a Markov semigroup $(T_t)_{t\geq 0}$ on $C(\mathbb{X})$. The corresponding Markov process with rcll-trajectories is called *stochastic interacting particle system (IPS)* generated by A.

3. MAIN RESULTS

Theorem 3.1. Suppose that $c = (c_T(\cdot,\cdot))_{T\in\mathcal{T}}$ is an admissible family of transition rate functions with associated generator $\overline{A}$. Then $T(\mathbb{X})$ is a core for $\overline{A}$.

Corollary 3.2. Suppose that $c = (c_T(\cdot,\cdot))_{T\in\mathcal{T}}$ is an admissible family of transition rate functions. Then the operator $A_0 : T(\mathbb{X}) \to C(\mathbb{X})$ given by

$$A_0 f(\eta) = \sum_{T\in\mathcal{T}} \int_{\mathbb{X}_T} c_T(\eta, \mathrm{d}v)[f(\tau_T(\eta, v)) - f(\eta)], \quad \eta \in \mathbb{X}, f \in T(\mathbb{X}), \tag{10}$$

is a Markov pregenerator whose closure is a Markov generator.

Proof of Theorem 3.1. Define $A_0 := A|_{T(\mathbb{X})}$. The operator is a Markov pregenerator in the sense of Liggett [5, Def. I.2.1], since

1. $\mathbb{1} \in T(\mathbb{X})$ and $A_0\mathbb{1} = 0$, where $\mathbb{1}(\eta) = 1,\ \eta \in \mathbb{X}$;
2. $T(\mathbb{X})$ is dense in $C(\mathbb{X})$;
3. If $f \in T(\mathbb{X})$ and $f(\eta) = \min_{\zeta\in\mathbb{X}} f(\zeta)$, then

$$A_0 f(\eta) = \sum_{T\in\mathcal{T}} \int_{\mathbb{X}_T} c_T(\eta, \mathrm{d}v) \underbrace{[f(\tau_T(\eta, v)) - f(\eta)]}_{\geq 0} \geq 0.$$

By [5, Prop. I.2.5], the operator A_0 has a closure $\overline{A_0}$. Note that $\overline{A_0}$ is again a Markov pregenerator. To show that $\overline{A_0}$ coincides with the closure $\overline{A}$ of A, it is sufficient to verify that $D(\mathbb{X})$ belongs to the domain $\vartheta(\overline{A_0})$ of $\overline{A_0}$. To check this, one has to prove that for given $g \in D(\mathbb{X})$, there are cylinder functions g_n such that $g_n \to g$ and $Ag_n \to Ag$. This will be done in the following. Fix $g \in D(\mathbb{X})$. Let S_n be finite sets that increase to S. Choose a configuration $\xi_0 \in \mathbb{X}$ and denote, for each $\eta \in \mathbb{X}$, by $\eta_n := \tau_{S_n}(\xi_0, \eta)$ the configuration that agrees with η on S_n and is equal to ξ_0 off S_n. Denote by g_n the cylinder approximation to g which is given by $g_n(\eta) = g(\eta_n)$. It holds that

$$|g_n(\eta) - g(\eta)| = |g(\eta_n) - g(\eta)| \leq \Delta_{S_n^c}(g), \quad \eta \in \mathbb{X},$$

where $\Delta_T(g), T \in \mathcal{T}$, is defined in (2). One finds that

$$\Delta_T(g) \leq \sum_{x\in T} \Delta_x(g), \quad T \subset S. \tag{11}$$

Indeed, suppose that $T = \{x_1, x_2, \dots\}$ and consider $\eta, \zeta \in \mathbb{X}$ with $\eta_{T^c} = \zeta_{T^c}$. Define $\eta^0 := \eta, \eta^k := \tau_{x_k}(\eta^{k-1}, \zeta(x_k)), k = 1, 2, \dots$ Then $|g(\eta^{k-1}) - g(\eta^k)| \le \Delta_{x_k}(g)$, hence

$$\begin{aligned} |g(\eta) - g(\zeta)| &\le \sum_{k=1}^{n} |g(\eta^{k-1}) - g(\eta^k)| + |g(\eta^n) - g(\zeta)| \\ &\le \sum_{k=1}^{n} \Delta_{x_k} g + |g(\eta^n) - g(\zeta)|, \quad n \in \mathbb{N}. \end{aligned}$$

Since $\lim_{n\to\infty} \eta^n = \zeta$ in the product topology on $\mathbb{X}$ and g is continuous, one obtains (11) by letting n converge to infinity.

It follows from (11) that

$$||g_n - g|| \le \Delta_{S_n^c} g \le \sum_{x \notin S_n} \Delta_x(g) \underset{n}{\longrightarrow} 0. \tag{12}$$

For any $\eta \in \mathbb{X}$ and any finite $V \subset S$, one finds

$$\begin{aligned} &|Ag_n(\eta) - Ag(\eta)| \\ &\quad \le \sum_{T: T\cap V \neq \emptyset} \int_{\mathbb{X}_T} c_T(\eta, \mathrm{d}v) \left[|g_n(\tau_T(\eta, v)) - g(\tau_T(\eta, v))| + |g_n(\eta) - g(\eta)| \right] \\ &\qquad + \sum_{T: T\cap V = \emptyset} \int_{\mathbb{X}_T} c_T(\eta, \mathrm{d}v) \left[|g_n(\tau_T(\eta, v)) - g_n(\eta)| + |g(\tau_T(\eta, v)) - g(\eta)| \right] \\ &\quad \le 2||g_n - g|| \sum_{T: T\cap V \neq \emptyset} c_T + \sum_{T: T\cap V = \emptyset} c_T(\Delta_T(g_n) + \Delta_T(g)), \end{aligned}$$

where c_T is defined in (5). By (7) and considering that $\Delta_x(g_n) \le \Delta_x(g)$, the latter sum can be estimated as follows

$$\begin{aligned} \sum_{T: T\cap V = \emptyset} c_T(\Delta_T(g_n) + \Delta_T(g)) &\le \sum_{T: T\cap V = \emptyset} c_T \sum_{x\in T} (\Delta_x(g_n) + \Delta_x(g)) \\ &\le 2 \sum_{x \notin V} \Delta_x(g) \sum_{T \ni x, T\cap V = \emptyset} c_T \\ &\le 2 \sum_{x \notin V} \Delta_x(g) \sum_{T \ni x} c_T \\ &\le 2K \sum_{x \notin V} \Delta_x(g). \end{aligned}$$

Note that by (7)

$$\sum_{T: T\cap V \neq \emptyset} c_T \le \sum_{x \in V} \sum_{T \ni x} c_T \le K|V|,$$

hence

$$||Ag_n - Ag|| \le 2K \left[|V| \, ||g_n - g|| + \sum_{x \notin V} \Delta_x(g) \right].$$

Given $\epsilon > 0$, there exists a finite set $V_0 \subset S$ such that

$$\sum_{x \notin V_0} \Delta_x(g) < \epsilon/2,$$

since $g \in D(\mathbb{X})$. By (12), it holds for each sufficiently large n that

$$2K|V_0|||g_n - g|| \leq \epsilon/2.$$

Hence $||Ag_n - Ag|| < \epsilon$ for each sufficiently large n and therefore $Ag_n \to Ag$ in supremum norm. □

Proof of Corollary 3.2. As is argued in the proof of Theorem 3.1, the closure $\overline{A_0}$ of the operator A_0 exists and is a Markov pregenerator in the sense of Liggett [5, Def. I.2.1]. In addition, $\overline{A_0}$ agrees with the closure $\overline{A}$ of the operator A that is defined in (9). By [5, Thm. I.3.9], $\overline{A} = \overline{A_0}$ is a Markov generator of a Markov semigroup on $C(\mathbb{X})$. □

Remark 3.3.

1. If condition (7) is replaced by the slightly weaker condition

$$\sum_{T \ni x} c_T < \infty \quad \text{for each } x \in S, \tag{13}$$

then the operator A_0 given by (10) is still well-defined. Indeed, adjusting the arguments in the proof of [5, Thm. I.3.2] slightly, one finds

$$\begin{aligned} |A_0 f(\eta)| &\leq \sum_{T \in \mathcal{T}} c_T \Delta_T(f) \leq \sum_{T \in \mathcal{T}} c_T \sum_{x \in \mathrm{tm}(f) \cap T} \Delta_x(f) \\ &\leq \sum_{x \in \mathrm{tm}(f)} \Delta_x(f) \sum_{T \ni x} c_T, \quad \eta \in \mathbb{X}, f \in T(\mathbb{X}). \end{aligned}$$

The last sum is finite since it consists of a finite number of finite summands, therefore the series defining $A_0 f$ converges uniformly and defines a continuous function. This fact has been stated already in [5, § I.6], but without detailed argument.

It is also clear from the proof of Theorem 3.1 that A_0 is a Markov pregenerator if only (13) is assumed for the transition rates. In [5, § I.6], it was was shown that a solution to the the martingale problem for A_0 exists under (13). If this solution to the martingale problem is unique, then the corresponding Markov process is a Feller process whose Markov generator is an extension of A_0 [5, Th. I.6.8]. In general, particularly without condition (8), it remains open wether $T(\mathbb{X})$ is a core in this case.

2. Condition (8) is required to prove that, under (7), the closure of A is a Markov generator, see next section. It was argued in [5, § I.7] that (7) alone is not sufficient for this.

3. It becomes clear in the proof of Theorem 3.1 that condition (7), which is stronger than condition (13) in that it requires a uniform bound on the total rate at which a given coordinate can change, ensures that $D(\mathbb{X}) \subset \vartheta(\overline{A_0})$ and $\overline{A_0}|_{D(\mathbb{X})} = A$. If one drops condition (8) but additionally assumes that the martingale problem for A_0 has a unique solution, then $\overline{A_0}$ can be uniquely extended to a Markov generator, see [5, Thm. 6.8]. If moreover $D(\mathbb{X})$ is a core of that Markov generator, then the proof of Theorem 3.1 shows that $T(\mathbb{X})$ is a core, as well.

4. DISCUSSION: THE STRAIGHTFORWARD GENERALIZATION OF THE STANDARD PROOF TECHNIQUE FAILS

An alternate way to show that the closure of A_0 is a Markov generator could be to prove that there exists a $\lambda > 0$ such that $\mathcal{R}(\lambda I - A_0)$ is dense in $C(\mathbb{X})$, see [3, Ch. 1,Prop. 3.1]. This idea was pursued in the construction of IPS from the Markov pregenerator A specified on $D(\mathbb{X})$ [5, Thm. I.3.9]. If this attempt is successful, then the domain $T(\mathbb{X})$ of the pregenerator A_0 is directly a core for the Markov generator $\overline{A_0}$.

In this section, the arguments in the proof of [5, Thm. I.3.9] shall be outlined and adapted to pregenerators $A_0 : T(\mathbb{X}) \to C(\mathbb{X})$. The domain $T(\mathbb{X})$ of these pregenerators is smaller than that of the pregenerators $A : D(\mathbb{X}) \to C(\mathbb{X})$ considered in the original work. It will be discussed what the essential generalization step is and which additional assumptions have to be imposed. By means of an example, it is shown that the standard proof technique fails if this assumption is dropped.

Firstly, A_0 is approximated by bounded Markov generators. As in [5], suppose that $(S_n)_{n\in\mathbb{N}}$ is an increasing sequence of finite sets exhausting S. Fix $n \in \mathbb{N}$ and define

$$c_T^{(n)}(\cdot,\cdot) = \begin{cases} c_T(\cdot,\cdot), & T \subset S_n \\ 0, & T \in \mathcal{T}, T \not\subset S_n. \end{cases} \tag{14}$$

Then $c^{(n)} = \left(c_T^{(n)}(\cdot,\cdot)\right)_{T\in\mathcal{T}}$ is a family of transition rate functions. It satisfies

$$c_T^{(n)} = \sup_{\eta\in\mathbb{X}} c_T^{(n)}(\eta, \mathbb{X}_T) \le \sup_{\eta\in\mathbb{X}} c_T(\eta, \mathbb{X}_T) = c_T < \infty, \tag{15}$$

since $c = (c_T(\cdot,\cdot))$ is admissible. Further, $c_T^{(n)}(x) = 0$ for $x \in S, T \in \mathcal{T}, T \not\subset S_n$ and $c_T^{(n)}(x) = c_T(x)$ for $x \in S, T \subset S_n$, hence

$$\gamma^{(n)}(x,z) = \sum_{\substack{T\subset S_n,\\ T\ni x}} c_T(z) \le \gamma(x,z), \quad x,z \in S. \tag{16}$$

Consequently, $c^{(n)}$ is admissible, because c is admissible. Note that

$$\sum_{T\in\mathcal{T}} c_T^{(n)} = \sum_{\substack{T\in\mathcal{T}\\ T\subset S_n}} c_T < \infty, \tag{17}$$

since the latter sum consists of a finite number of summands. Denote by $A^{(n)}$ the operator which is associated to $c^{(n)}$, that is

$$A^{(n)}f(\eta) := \sum_{T\in\mathcal{T}} \int_{\mathbb{X}_T} (f(\tau_T(\eta,v)) - f(\eta))c_T^{(n)}(\eta, \mathrm{d}v), \quad f \in C(\mathbb{X}), \eta \in \mathbb{X}.$$

By (17), $A^{(n)}$ is a bounded operator, hence

$$\mathcal{R}\left(I - \lambda A^{(n)}\right) = C(\mathbb{X}), \quad \lambda > 0, n \in \mathbb{N}. \tag{18}$$

Secondly, it will be verified that for $n \in \mathbb{N}, g \in T(\mathbb{X}), \lambda > 0$, there exists $f_n \in T(\mathbb{X})$ such that

$$f_n - \lambda A^{(n)} f_n = g. \tag{19}$$

This is the essential generalization step and the point were the following additional assumption is necessary. Note that this assumption is in particular satisfied if $S = \mathbb{Z}^d$ and the transition rates are of *finite range* in the sense of [5, Def. I.4.17].

Assumption 4.1. Suppose that the family $c = (c_T)_{T\in\mathcal{T}}$ of admissible transition rate functions is *local* in the sense that

$$\{x : c_T(x) > 0\} \in \mathcal{T}, \quad T \in \mathcal{T}.$$

In the original work it was shown that for $g \in D(\mathbb{X})$ there exists $f_n \in D(\mathbb{X})$ such that $f_n - \lambda A^{(n)} f_n = g$. However, as it will become apparent in the next step, it is necessary that f_n belongs to the domain of the pregenerator. To prove (19), fix $n \in \mathbb{N}$, $g \in T(\mathbb{X})$ and $\lambda > 0$. By (18), there exists a function $f_n \in C(\mathbb{X})$ such that $f_n - \lambda A^{(n)} f_n = g$. It remains to show that $f_n \in T(\mathbb{X})$. Suppose that $x \in S\backslash S_n$. Then for all $T \in \mathcal{T}$ with $x \in T$ it holds that $T \not\subset S_n$. This implies that

$$c_T^{(n)}(u) = 0, \quad u \in S, T \in \mathcal{T}, T \ni x.$$

Therefore

$$\gamma^{(n)}(x,u) = \sum_{\substack{T\in\mathcal{T}\\ T\ni x}} c_T^{(n)}(u) = 0, \quad u \in S, x \in S\backslash S_n. \tag{20}$$

By [5, Lemma I.3.4.(b)],

$$\Delta_u(f_n) \le \Delta_u(g) + \lambda \sum_{x\in S} \gamma^{(n)}(x,u)\Delta_x(f_n), \quad u \in S. \tag{21}$$

Hence, by (20) and (6),

$$\begin{aligned}
\Delta_u(f_n) &= \Delta_u(g) + \lambda \sum_{x\in S_n} \sum_{\substack{T\in\mathcal{T},\\ T\ni x}} c_T^{(n)}(z)\Delta_x(f_n) \\
&= \Delta_u(g) + \lambda \sum_{x\in S_n} \sum_{\substack{T\subset S_n,\\ T\ni x}} c_T(u)\Delta_x(f_n) \\
&= \Delta_u(g) + \lambda \sum_{T\subset S_n} c_T(u) \sum_{x\in T} \Delta_x(f_n) \\
&\le \Delta_u(g) + \lambda \left(\sum_{x\in S_n} \Delta_x(f_n)\right) \sum_{T\subset S_n} c_T(u), \quad u \in S.
\end{aligned}$$

Thus

$$\Delta_u(f_n) = 0, \quad u \notin \mathrm{tm}(g) \cup \bigcup_{T \subset S_n} \{y : c_T(y) > 0\}.$$

Since $c = (c_T(\cdot,\cdot))_{T \in \mathcal{T}}$ is local, $\bigcup_{T \subset S_n} \{y : c_T(y) > 0\} \in \mathcal{T}$. Hence $\mathrm{tm}(f_n) \in \mathcal{T}$ and therefore $f_n \in T(\mathbb{X}) = \vartheta(A_0)$.

In the last step, define

$$g_n := f_n - \lambda A_0 f_n = (I - \lambda A_0) f_n \in \mathcal{R}(I - \lambda A_0).$$

It was shown in the proof of [5, Thm. I.3.9.],[1] that for $g \in T(\mathbb{X})$

$$\lim_{n \to \infty} \|g - g_n\| = 0,$$

if λ is sufficiently small. Since $g_n \in \mathcal{R}(I - \lambda A_0), n \in \mathbb{N}$, it follows that g is in the closure of $\mathcal{R}(I - \lambda A_0)$. Hence

$$T(\mathbb{X}) \subset \overline{\mathcal{R}(I - \lambda A_0)}$$

for sufficiently small $\lambda > 0$. Since $T(\mathbb{X})$ is dense in $C(\mathbb{X})$ it follows that

$$\overline{\mathcal{R}(I - \lambda A_0)} = C(\mathbb{X}),$$

which proves the theorem. □

If the Assumption 4.1 is dropped, the standard proof technique fails. This can be seen with the help of the following example. It is conceivable that with a more elaborate cut-off in the construction of $A^{(n)}$ the scheduled proof technique could still work. However, since Theorem 3.1 covers the issue in its generality, the problem of alternate cut-off mechanisms is not further followed up here.

Let be $S = \mathbb{Z}^d$ with $d = 1, 2, \ldots$, and $W = \{0, 1\}$. Suppose that $N : \mathbb{N} \to S$ is a bijection. Define

$$f(\eta) := \sum_{n=1}^{\infty} \eta(N(n)) 2^{-n}, \quad \eta \in \mathbb{X}.$$

One finds easily that

$$\Delta_x(f) = 2^{-N^{-1}(x)} > 0, \quad x \in S.$$

Hence $f \in D(\mathbb{X}) \backslash T(\mathbb{X})$. The family $c = (c_T(\cdot,\cdot))_{T \in \mathcal{T}}$ of transition rate functions shall be given by

$$c_T(\eta, u) := \begin{cases} c(x, \eta), & T = \{x\}, u = 1 - \eta(x) \\ 0, & \text{otherwise}, \end{cases}$$

where

$$c(x, \eta) = f(\theta_x \eta), \quad x \in S, \eta \in \mathbb{X}.$$

[1] In [5, Thm. I.3.9.], functions $g \in D(\mathbb{X})$ are considered. Recall that $T(\mathbb{X}) \subset D(\mathbb{X})$. It is for this property that condition (8) has to be imposed on the transition rates.

Here, the shift operators $\theta_x : \mathbb{X} \to \mathbb{X}$ are defined for $x \in S$ by $\theta_x \eta(z) = \eta(z+x), z \in \mathbb{Z}^d$.

Fix $x \in S, x \neq 0$. For $\eta, \zeta \in \mathbb{X}$ with $\eta_{x^c} = \zeta_{x^c}$, it holds that

$$\begin{aligned} \|c_0(\eta, \cdot) - c_0(\zeta, \cdot)\|_{\text{tv}} &= |c_0(\eta, 0) - c_0(\zeta, 0)| + |c_0(\eta, 1) - c_0(\zeta, 1)| \\ &= |c(0, \eta) - c(0, \zeta)| \\ &= |f(\eta) - f(\zeta)|. \end{aligned}$$

Hence $c_0(x) = \Delta_x(f),\ x \in S, x \neq 0$. Since c is translation-invariant, it follows that

$$c_y(x) = \Delta_{x-y}(f), \quad x, y \in S, x \neq y.$$

Obviously,

$$c_T(x) = 0, \quad T \in \mathcal{T}, |T| \neq 1.$$

Further,

$$\gamma(x, z) = \sum_{\substack{T \in \mathcal{T} \\ T \ni x}} c_T(z) = c_x(z) = \Delta_{z-x}(f), \quad x, z \in S, x \neq z.$$

In particular,

$$\begin{aligned} \sup_{x \in S} \sum_{T \ni x} c_T &= \sup_{x \in S} \sum_{T \ni x} \left(\sup_{\eta \in \mathbb{X}} c_T(\eta, \mathbb{X}_T) \right) \\ &= \sup_{\eta \in \mathbb{X}} c_0(\eta, W) \\ &= \sup_{\eta \in \mathbb{X}} f(\eta) \\ &= 1, \end{aligned}$$

and

$$\begin{aligned} \sup_{x \in S} \sum_{z \in S} \gamma(x, z) &= \sup_{x \in S} \sum_{z \in S} \Delta_{z-x}(f) \\ &= \sup_{x \in S} \sum_{z \in S} 2^{-N^{-1}(z-x)} \\ &= \sum_{n \in \mathbb{N}} 2^{-n} \\ &= 1, \end{aligned}$$

therefore $c = (c_T(\cdot, \cdot))_{T \in \mathcal{T}}$ is admissible but non-local. Concerning the proof of Theorem 3.1, one finds that for sufficiently large $n \in \mathbb{N}$ the function f_n which solves the equation

$$f_n - \lambda A^{(n)} f_n = g,$$

with respect to some $g \in T(\mathbb{X}), \lambda > 0$, is not a cylinder function. Indeed, choose n such that $S_n \supset \text{tm}(g)$. By (21) and the subsequent considerations, it holds that

$$\begin{aligned} \Delta_u(f_n) &\leq \Delta_u(g) + \lambda \sum_{T \subset S_n} c_T(u) \sum_{x \in T} \Delta_x(f_n) \\ &= \Delta_u(g) + \lambda \sum_{z \in S_n} c_z(u) \Delta_z(f_n) \\ &= \Delta_u(g) + \lambda \sum_{z \in S_n} \Delta_{u-z}(f) \Delta_z(f_n), \quad u \in S. \end{aligned}$$

If g is non-constant, then the case $\Delta_x(f_n) = 0, x \in S_n$ is excluded. Therefore we can assume that $\Delta_x(f_n) > 0$ for some $x \in S_n$. Then

$$\lambda \sum_{z \in S_n} \Delta_{u-z}(f)\Delta_z(f_n) > 0, \quad u \in S,$$

hence $\Delta_u(f_n) > 0, u \in S$, and thus $f_n \notin T(\mathbb{X})$.

(Received October 29, 2010)

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