

## A CONSUMPTION-INVESTMENT PROBLEM MODELLED AS A DISCOUNTED MARKOV DECISION PROCESS

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In this paper a problem of consumption and investment is presented as a model of a discounted Markov decision process with discrete-time. In this problem, it is assumed that the wealth is affected by a production function. This assumption gives the investor a chance to increase his wealth before the investment. For the solution of the problem there is established a suitable version of the Euler Equation (EE) which characterizes its optimal policy completely, that is, there are provided conditions which guarantee that a policy is optimal for the problem if and only if it satisfies the EE. The problem is exemplified in two particular cases: for a logarithmic utility and for a Cobb-Douglas utility. In both cases explicit formulas for the optimal policy and the optimal value function are supplied.

*Keywords:* discounted Markov decision processes, differentiable value function, differentiable optimal policy, stochastic Euler equation, consumption and investment problems

*Classification:* 93E12, 62A10

### 1. INTRODUCTION

This paper concerns a problem of consumption and investment (see [9, 10, 11, 19, 21] and [22]). It was inspired by [9] and [19]. In these references there is considered an investor, who wants to increase his wealth at each time ( $t = 0, 1, 2, \dots$ ); in this case, it is assumed that the investor wishes to allocate his wealth between investment and consumption. Then the objective is to maximize his utility consumption.

In this paper a generalization of the model presented in [9, 10] and [19] is given. In those references, it was assumed that the utility is obtained by a logarithmic or exponential function; now there is considered a general utility function. Also, it is assumed that an investor's wealth is affected by a production function, with the usual properties (see [18]). In economic growth models the production function has been used both in deterministic and stochastic cases (see [5], [11, 18, 20] and [22]). In those cases, the production function was affected by a exogenous variable; in this article, however, it is assumed that the investment is perturbed by a random noise, which represents an interest rate risk. Those classes of model are important in portfolio selection (see [17]) and optimal savings problems (see, [2] and [3]), where the risky investment is prevalent.

In order to find the optimal solution to the problem of consumption and investment, it is presented as a discounted discrete-time Markov Decision Process, with real state and action spaces (see, [5, 9, 11, 18, 19, 20] and [22]).

To do so, firstly, there are studied conditions to guarantee the differentiability of the optimal solution (i. e. the optimal value function and the optimal policy) for the discounted Markov Decision Processes (MDPs) on real spaces. Besides a functional equation to characterize the optimal policy is established; this equation is known as Euler Equation (EE). In particular, a version of the EE in terms of the value iteration functions is obtained.

The EE has been applied in various economic branches. For example, in asset prices (see [10] and [12]) and economic growth (see [5, 11, 18, 20] and [22]). The methodology to solve the EE, in some papers (see [13] and [19]), involves characterizing the optimal policy and proposing a particular class of policies to determine the solution. One of the contributions of this work consists in obtaining a new stochastic version of the EE, in terms of the value iteration functions, and the other contribution is doing it in terms of the optimal policy. This is an extension of the version presented in [7]: in this work a version of the EE is obtained for deterministic system.

Secondly, for the problem of consumption and investment modelled as a discounted Markov decision process there is established a version of the EE and, with an additional condition, it is obtained that, if the optimal policy satisfies the EE, then it is optimal. This part of the article is exemplified in two particular cases, considering a logarithmic utility and a Cobb-Douglas utility. In both cases explicit formulas for the optimal policy and the optimal value function are given.

The paper is organized as follows. In Section 2 there is introduced the basic theory referring to MDPs. After that, in Section 3, there are presented some preview results about the differentiability and Euler equation in the context of discounted MDPs. In Section 4, the problem of consumption and investment is posed and some results about this model are given. Finally, the theory is exemplified with two classical examples; for these instances all the assumptions considered in the paper are verified.

## 2. MARKOV DECISION PROCESSES: BASICS

A discrete-time Markov Decision Model (MDM) (see [4, 14] and [15]) consists of five elements:

$$(X, A, \{A(x) | x \in X\}, Q, r),$$

where  $X$  and  $A$  are Borel spaces of  $\mathbb{R}$ , both with non-empty interior.  $X$  and  $A$  are called the state space and the action space, respectively.  $\{A(x) | x \in X\}$  is a family of non-empty and measurable subsets  $A(x)$  of  $A$ , where  $A(x)$  denotes the set of feasible actions when the system is in state  $x \in X$ . Let  $\mathbb{K}$  be the set of state-action pairs:

$$\mathbb{K} := \{(x, a) | x \in X, a \in A(x)\}.$$

$Q(\cdot | \cdot)$ , called the transition law, is a stochastic kernel on  $X$  given  $\mathbb{K}$ , and  $r : \mathbb{K} \rightarrow \mathbb{R}$  is a measurable function called the one-step reward function.

The dynamics described in this stochastic system works as follows: if the system is observed at a time  $t$ , the state considered is  $x_t = x \in X$  and action  $a_t = a \in A(x)$  is applied, then two things happen:

- a) a reward  $r(x, a)$  is obtained and
- b) the system moves to the next state  $x_{t+1}$  by means of the transition law  $Q$ .

Throughout this paper it will be assumed that  $X$ ,  $A$  and  $A(x)$ ,  $x \in X$ , are convex sets. It is also assumed that the transition law  $Q$  is given by a difference equation of the type:

$$x_{t+1} = L(F(x_t, a_t), \xi_t), \tag{1}$$

$t = 0, 1, \dots$ , with a given initial state  $x_0 = x$ , where  $\{\xi_t\}$  is a sequence of independent and identically distributed (i.i.d.) random variables, independent of  $x$  and taking values in a Borel space  $S \subset \mathbb{R}$ . Let  $\xi$  be a generic element of the sequence  $\{\xi_t\}$ . The density of  $\xi$  is designated by  $\Delta$  and its probability distribution, by  $\mu$ ;  $L : X' \times S \rightarrow X$  is a measurable function, with  $X' \subset \mathbb{R}$ , and  $F : \mathbb{K} \rightarrow X'$ , is a measurable function too.

In general, a policy is a sequence  $\pi = \{\pi_t\}$  of stochastic kernels, defined on  $A$  given the history of the process.  $\Pi$  is used to denote the set of policies. A particular class of these policies are the stationary ones:

$$\mathbb{F} := \{ f : X \rightarrow A \mid f \text{ is a measurable function and } f(x) \in A(x) \text{ for all } x \in X \}$$

(in this case  $\pi$  is denoted by  $f$ ).

For each  $\pi \in \Pi$  and  $x \in X$ , it will be defined that

$$v(\pi, x) := E_x^\pi \left[ \sum_{t=0}^{\infty} \alpha^t r(x_t, a_t) \right].$$

$v(\pi, x)$  is called total discounted reward, where  $\alpha \in (0, 1)$  is the discount factor.

**Definition 2.1.** A policy  $\pi^* \in \Pi$  is optimal, if for each  $x \in X$ ,

$$v(\pi^*, x) = \sup_{\pi \in \Pi} v(\pi, x).$$

The function defined by

$$V(x) := \sup_{\pi \in \Pi} v(\pi, x),$$

$x \in X$ , will be called the *optimal value function*.

The *optimal control problem* consists in determining an optimal policy.

**Definition 2.2.** A measurable function  $\lambda : X \rightarrow \mathbb{R}$  is said to be a solution to the optimality equation (OE) if it satisfies

$$\lambda(x) = \sup_{a \in A(x)} \{ r(x, a) + \alpha E [\lambda(L(F(x, a), \xi))] \}, \tag{2}$$

$x \in X$ .

**Definition 2.3.** The value iteration functions are defined as

$$v_n(x) = \max_{a \in A(x)} \{r(x, a) + \alpha E[v_{n-1}(L(F(x, a), \xi))]\}, \tag{3}$$

for all  $x \in X$  and  $n = 1, 2, \dots$ , with  $v_0(x) = 0$ .

**Remark 2.4.** Under certain assumptions (see [14] and [15]), it is possible to demonstrate that for each  $n = 1, 2, \dots$ , there exists a stationary policy  $f_n \in \mathbb{F}$  such that the maximum in (3) is attained, i. e.,

$$v_n(x) = r(x, f_n(x)) + \alpha E[v_{n-1}(L(F(x, f_n(x)), \xi))],$$

$x \in X$ .

The Assumption 2.5 named, for short, Basic Assumption (BA) is fulfilled in a wide variety of cases. (See, for example, [14], p.46, Assumptions 4.2.1 and 4.2.2, and Theorem 4.2.3, and also see Lemma 4.2.8, p. 49.)

**Assumption 2.5.**

- a) The optimal value function  $V$  is a solution to the OE (see (2)).
- b) For every  $x \in X$ ,  $v_n(x) \rightarrow V(x)$ , as  $n \rightarrow \infty$ .

### 3. MARKOV DECISION PROCESSES: PRELIMINARY RESULTS

Throughout this section it is assumed that BA holds. Let  $X$  and  $Y$  be Euclidian spaces and consider the following notation:  $C^2(X, Y)$  denotes the set of functions  $l : X \rightarrow Y$  with a continuous second derivative (when  $X = Y$ ,  $C^2(X, Y)$  will be denoted by  $C^2(X)$  and in some cases it will be written only as  $C^2$ ).

Let  $\Gamma : X \times Y \rightarrow \mathbb{R}$  be a measurable function such that  $\Gamma \in C^2(X \times Y, \mathbb{R})$ .  $\Gamma_x, \Gamma_y$  and  $\Gamma_{yy}$  denote the partial derivative of  $\Gamma$  with respect to  $x$  and  $y$ , and the second partial derivative with respect to  $y$ . The interior of the set  $X$  is denoted by  $int(X)$ .

Let  $\Lambda : \mathbb{K} \rightarrow \mathbb{R}$  be a measurable function and

$$\lambda(x) := \sup_{a \in A(x)} \Lambda(x, a),$$

$x \in X$ .

The proof of the following theorem, can be consulted in [8], Theorem 1.

**Theorem 3.1.** Suppose that

- a)  $\Lambda \in C^2(int(\mathbb{K}); \mathbb{R})$ . Furthermore,  $\Lambda_{aa}(x, \cdot) < 0$ , for every  $x \in X$ .
- b) There is a function  $f : X \rightarrow A$  such that  $f(x) \in int(A(x))$  and  $\lambda(x) = \Lambda(x, f(x))$ , for every  $x \in X$ .

Then  $f \in C^1(int(X); A)$  and  $\lambda \in C^2(int(X); \mathbb{R})$ .

Define  $G : \mathbb{K} \rightarrow \mathbb{R}$  by

$$G(x, a) := r(x, a) + \alpha H(x, a), \tag{4}$$

where  $H(x, a) = E[V(L(F(x, a), \xi))]$ ,  $(x, a) \in \mathbb{K}$ .

The following result is a direct consequence of the previous theorem.

**Theorem 3.2.** Suppose that  $r, H \in C^2(\text{int}(\mathbb{K}); \mathbb{R})$ ,  $r_{aa}(x, \cdot) < 0$  and  $H_{aa}(x, \cdot) \leq 0$  and  $f(x) \in \text{int}(A(x))$ , for every  $x \in X$ , where  $f$  is the optimal policy. Then  $f \in C^1(\text{int}(X); A)$  and  $V \in C^2(\text{int}(X); \mathbb{R})$ .

**Theorem 3.3.** Suppose that

- a)  $r, H \in C^2(\text{int}(\mathbb{K}); \mathbb{R})$ ,  $r_{aa}(x, \cdot) < 0$ ,  $H_{aa}(x, \cdot) \leq 0$  and  $F_a(x, a)$  is invertible;
- b)  $f(x) \in \text{int}(A(x))$  for every  $x \in X$ .

Then  $f$  satisfies the Euler's equation

$$(r_a F_a^{-1})(x, f(x)) = -\alpha E[W(L(F(x, f(x)), \xi), f(L(F(x, f(x)), \xi))) L'(F(x, f(x)), \xi)], \tag{5}$$

for each  $x \in \text{int}(X)$ , where the function  $W$  is defined by

$$W(x, a) := [r_x - r_a F_a^{-1} F_x](x, a), \tag{6}$$

$(x, a) \in \mathbb{K}$ .

*Proof.* Fix  $x \in X$ . Note that the assumptions of Theorem 3.2 hold, then  $f \in C^1(\text{int}(X); A)$  and  $V \in C^2(\text{int}(X); \mathbb{R})$  and therefore  $G \in C^2(\text{int}(\mathbb{K}); \mathbb{R})$  where  $G$  is defined in (4). Thus, it follows that for  $a \in \text{int}(A(x))$ ,

$$G_a(x, a) = r_a(x, a) + \alpha E[V'(L(F(x, a), \xi)) L'(F(x, a), \xi)] F_a(x, a).$$

Then, using the first-order condition and the invertibility of  $F_a$ , it follows that,  $G_a(x, f(x)) = 0$ , i. e.,

$$-r_a F_a^{-1}(x, f(x)) = \alpha E[V'(L(F(x, f(x)), \xi)) L'(F(x, f(x)), \xi)]. \tag{7}$$

On the other hand, since  $V$  satisfies (2) and  $f \in \mathbb{F}$  is the optimal policy, then

$$\begin{aligned} V(x) &= r(x, f(x)) + \alpha E[V(L(F(x, f(x)), \xi))] \\ &= G(x, f(x)). \end{aligned}$$

Thus using that  $G_a(x, f(x)) = 0$ , it is possible to obtain the following envelope formula:

$$\begin{aligned} V'(x) &= G_x(x, f(x)) + G_a(x, f(x))f'(x), \\ &= G_x(x, f(x)) \end{aligned}$$

equivalently,

$$V'(x) = r_x(x, f(x)) + \alpha E[V'(L(F(x, f(x)), \xi))L'(F(x, f(x)), \xi)]F_x(x, f(x)). \quad (8)$$

Substituting (7) in (8), it is obtained that

$$V'(x) = W(x, f(x)), \quad (9)$$

hence

$$V'(L(F(x, f(x)), \xi)) = W(L(F(x, f(x)), \xi), f(L(F(x, f(x)), \xi))). \quad (10)$$

Finally, (5) is obtained by substituting (10) in (7). Since  $x$  is arbitrary, Theorem 3.3 follows.  $\square$

#### 4. CONSUMPTION-INVESTMENT PROBLEM

In this section, there is considered a consumption-investment model defined as follows. Suppose that at each time  $t$ , the current wealth  $x_t$  generates an output  $h(x_t)$ , and a part of it,  $a_t$ , is consumed, and the rest  $i_t = h(x_t) - a_t$  is invested ( $h : [0, \infty) \rightarrow [0, \infty)$  is a production function). It is assumed that borrowing is not allowed, so  $i_t \in [0, h(x_t)]$  and, equivalently,  $a_t \in [0, h(x_t)]$ . This investment will lead to another wealth in the next period  $t+1$ ; in this case, it is assumed that the relation between wealth and consumption is given by

$$x_{t+1} = \xi_t(h(x_t) - a_t),$$

$t = 0, 1, 2, \dots$ , where  $\{\xi_t\}$  is a sequence of i.i.d. random variables taking values in  $S = [0, \infty)$ , independent of  $x_0$ , where  $x_0 = x \in X = [0, \infty)$  is the initial capital stock. Let  $\xi$  be a generic element of the sequence  $\{\xi_t\}$ ; it is assumed that  $\xi$  is a random variable with density  $\Delta \in C^2([0, \infty))$ . In this case,  $\xi$  is the return per invested dollar.

**Assumption 4.1.** It is assumed that the function  $h$  satisfies the following:

- a)  $h \in C^2((0, \infty))$ ,
- b)  $h$  is concave on  $X$ ,
- c)  $h' > 0$  and  $h(0) = 0$ .

The set of feasible actions is given for each  $x_0 = x \in X$  as  $A(x) = [0, h(x)]$ .

The objective is to maximize the investor's consumption utility over all  $\pi \in \Pi$ :

$$v(\pi, x) = E_x^\pi \left[ \sum_{t=0}^{\infty} \alpha^t U(a_t) \right], \quad (11)$$

where  $U : [0, \infty) \rightarrow \mathbb{R}$  is a measurable function, which is called utility function.

**Assumption 4.2.** The utility function  $U$  satisfies the following properties:

- a)  $U \in C^2((0, \infty), \mathbb{R})$ , which is strictly increasing and strictly concave,
- b)  $U'$  is invertible with the inverse  $u$ ,
- c)  $U'(0) = \infty$  and  $\lim_{x \rightarrow \infty} U'(x) = 0$ ,
- d) there exists a function  $\vartheta$  defined on  $S$  such that  $E[\vartheta(\xi)] < \infty$ , and

$$|U'(h(s(h(x) - a)))h'(s(h(x) - a))s\Delta(s)| \leq \vartheta(s), \tag{12}$$

$$s \in S, a \in (0, h(x)).$$

- e) The interchange between derivatives and integrals is valid (see Remark 4.3).

**Remark 4.3.** The Assumption 4.2 d) will be used in the proof of Lemma 5.3 to ensure the interiority of the optimal policy. And the Assumption 4.2 e) will be used in the proof of Lemma 5.1. This assumption in some particular cases can be verified by means of the Dominated Convergence Theorem (see Assumption 5.4, below).

Throughout the paper, the model described is named Consumption-Investment Problem (CIP).

#### 4.1. Remarks about the CIP

1. In the MDP's literature there exist conditions to guarantee the validity of BA (see Assumption 2.5) for CIP. For example:

- a) BA holds if there exists a sequence  $\{X_j\}_{j \in \mathbb{N}_0}$  of non-empty Borel subsets of  $X$  such that  $X_j \subset X_{j+1}$  for all  $j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$  and

$$X = \bigcup_{j \geq 0} \text{int}(X_j).$$

Besides, for any  $j \in \mathbb{N}_0$  let  $m_j$  be

$$m_j := \sup_{x \in X_j} \sup_{a \in A(x)} u^+(a), \tag{13}$$

where  $u^+(a) = \max\{u(a), 0\}$ . Assume that  $m_0 > 0$ ,  $m_j < \infty$  for every  $j \in \mathbb{N}_0$  and

$$\alpha \limsup_{j \rightarrow \infty} \frac{m_{j+1}}{m_j} < 1. \tag{14}$$

Finally, suppose that for each  $j \in \mathbb{N}_0$  and  $x \in X_j$ ,  $a \in A(x)$ ,

$$Q(X_{j+1} | x, a) = 1, \tag{15}$$

then by Theorem 1 in [16], BA holds.

- b) If there exists a continuous weight function  $w$  and constants  $c, \beta > 0$  such that  $1 \leq \beta < 1/\alpha$  and, for every  $x \in X$ ,

$$\sup_{a \in [0, h(x)]} |U(a)| \leq cw(x), \tag{16}$$

$$\sup_{a \in [0, h(x)]} E[w(\xi(h(x) - a))] \leq \beta w(x), \tag{17}$$

then by Theorem 4.1 in [15], BA holds.

2. In the economics literature Assumption 4.2 c) is known as Inada’s condition (see [5, 11, 18, 19, 20, 21] and [22]). In some papers a similar assumption for the production function is considered (see [5, 11, 20, 21] and [22]). Now, in this paper the Inada’s condition for the production function is not necessary.
3. For the CIP there exists an optimal policy. Indeed, let  $\theta$  be a bounded-continuous function on  $X$  and define

$$\Theta(x, a) = E[\theta(\xi(h(x) - a))], \tag{18}$$

$(x, a) \in \mathbb{K}$ . Observe that  $\Theta$  is continuous in  $a \in [0, h(x)]$  for each  $x \in X$ . In this case the transition law is weakly continuous (see [14]). Since the utility function is continuous and  $A(x)$  is a compact set, for each  $x \in X$ , by Theorem 3.3.5 in [14] there exists an optimal policy.

4. Observe that the CIP satisfies a similar condition (in terms of rewards) to Condition 1 (C1) of [6], so it is possible to guarantee that  $V$  is strictly increasing, strictly concave and the optimal policy is unique.

### 5. MAIN RESULTS ABOUT THE CIP

Firstly note that the value iteration functions are given by:

$$v_n(x) = \max_{a \in [0, h(x)]} \{U(a) + \alpha E[v_{n-1}(\xi(h(x) - a))]\},$$

$x \in X$ ,  $n \geq 1$ , with  $v_0(x) = 0$ . And the corresponding dynamic programming equation for the CIP is given by

$$V(x) = \max_{a \in [0, h(x)]} \{U(a) + \alpha E[V(\xi(h(x) - a))]\}, \tag{19}$$

$x \in X$ .

For each  $n \geq 1$ , define  $G^n : \widehat{\mathbb{K}} \rightarrow \mathbb{R}$  by

$$G^n(x, a) = U(a) + \alpha H^n(x, a),$$

where  $H^n : \widehat{\mathbb{K}} \rightarrow \mathbb{R}$  is defined as

$$H^n(x, a) := E[v_{n-1}(\xi(h(x) - a))], \tag{20}$$

and

$$\widehat{\mathbb{K}} := \{(x, a) \mid x > 0, a \in (0, h(x))\}. \tag{21}$$



**Lemma 5.1.** For the value iteration functions there exist their maximizers  $\{f_n\}_{n \geq 1}$  and for  $n \geq 2$ ,  $f_n(x) \in (0, h(x))$  with  $x > 0$ . Furthermore,  $v_n \in C^2((0, \infty); \mathbb{R})$  and  $f_n \in C^1((0, \infty))$ .

*Proof.* The proof will be made by induction. Fix  $x > 0$ . Since  $U$  is increasing, then  $v_1(x) = U(h(x))$  and  $f_1(x) = h(x)$ . Take  $n = 2$ . By definition,

$$v_2(x) = \max_{a \in [0, h(x)]} \{U(a) + \alpha H^2(x, a)\},$$

where  $H^2$  is defined as in (20). It is known that  $U \in C^2((0, \infty), \mathbb{R})$  and  $h \in C^2((0, \infty))$ , and that both are concave, hence  $v_1 \in C^2((0, \infty); \mathbb{R})$ ,

$$v_1'(x) = U'(h(x))h'(x)$$

and  $v_1''(x) < 0$ . By Assumption 4.2 e),  $H^2 \in C^2(\widehat{\mathbb{K}}; \mathbb{R})$  and  $H_{aa}^2(x, a) < 0$ . Therefore,  $G^2 \in C^2(\widehat{\mathbb{K}}; \mathbb{R})$ .

Note that  $v_1'(0) = \infty$  and since

$$G_a^2(x, a) = U'(a) - \alpha E[v_1'(\xi(h(x) - a))\xi],$$

then, from Assumption 4.2 c), it follows that  $G_a^2(x, 0) = +\infty$  and  $G_a^2(x, h(x)) = -\infty$ . Now by the Intermediate Value Theorem, there is  $\bar{a} \in (0, h(x))$  such that  $G_a^2(x, \bar{a}) = 0$ . Observe that  $G^2$  is strictly concave, therefore  $\bar{a} := f_2(x)$  is unique.

Now using the fact that  $v_2(x) = G^2(x, f_2(x))$  and applying Theorem 3.1 (in this case, taking  $\Lambda = G^2$  and  $\lambda = v_2$ ), it is possible to obtain that  $v_2 \in C^2((0, \infty); \mathbb{R})$  and  $f_2 \in C^1((0, \infty))$ . Note also that  $v_2'(0) = \infty$ , by Assumption 4.2 c).

Suppose that for some  $n > 2$ ,  $v_{n-1} \in C^2((0, \infty); \mathbb{R})$ ,  $v_{n-1}''(x) < 0$  and  $v_{n-1}'(0) = \infty$ . Again, since

$$v_n(x) = \max_{a \in [0, h(x)]} \{U(a) + \alpha H^n(x, a)\},$$

and in a similar way to the case  $n = 2$ , it is possible to conclude that  $H^n \in C^2(\widehat{\mathbb{K}}; \mathbb{R})$  and  $H_{aa}^n(x, a) < 0$ . Then  $f_n(x) \in (0, h(x))$ ,  $G^n \in C^2(\widehat{\mathbb{K}}; \mathbb{R})$  and  $G^n$  is strictly concave. Finally, using Theorem 3.1 (in this case, taking  $\Lambda = G^n$  and  $\lambda = v_n$ ) it is possible to guarantee that  $v_n \in C^2((0, \infty); \mathbb{R})$  and  $f_n \in C^1((0, \infty))$ .  $\square$

Now, there is presented a version of Euler's equation (EE) for the value iteration functions.

**Lemma 5.2.** The value iteration functions satisfy the Euler's equation

$$v_n'(x) = \alpha E \{v_{n-1}'[\xi(h(x) - u(v_n'(x)/h'(x)))]\xi\} h'(x), \quad (22)$$

for all  $x \in (0, \infty)$ .

*Proof.* Let  $x > 0$  and  $n \geq 2$  be fixed. By Lemma 5.1, if  $a \in (0, h(x))$ , it results that

$$G_a^n(x, a) = U'(a) - \alpha E[v_{n-1}'(\xi(h(x) - a))\xi].$$

Using the first-order condition for the optimality of  $G^n$  and the fact that  $f_n(x) \in (0, h(x))$ , it is obtained that  $G_a^n(x, f_n(x)) = 0$ , i. e.,

$$U'(f_n(x)) = \alpha E [v'_{n-1}(\xi(h(x) - f_n(x)))\xi]. \tag{23}$$

Since

$$\begin{aligned} v_n(x) &= U(f_n(x)) + \alpha E [v_{n-1}(\xi(h(x) - f_n(x)))] \\ &= G^n(x, f_n(x)). \end{aligned}$$

Thus using that  $G_a^n(x, f(x)) = 0$ , it is possible to obtain the following envelope formula

$$\begin{aligned} v'_n(x) &= G'_x(x, f(x)) + G'_a(x, f(x))f'(x), \\ &= G'_x(x, f(x)) \end{aligned}$$

equivalently,

$$v'_n(x) = \alpha E [v'_{n-1}(\xi(h(x) - f_n(x)))\xi] h'(x). \tag{24}$$

Substituting (23) in (24), it is obtained that

$$v'_n(x) = U'(f_n(x))h'(x).$$

Therefore, by the invertibility of  $U'$ ,

$$f_n(x) = u \left( \frac{v'_n(x)}{h'(x)} \right). \tag{25}$$

(Recall that  $u$  is the inverse of the function  $U'$ .) Finally, substituting (25) in (23), the result follows.  $\square$

**Lemma 5.3.** For each  $x > 0$ , the optimal policy  $f(x) \in (0, h(x))$ .

*Proof.* Let  $x > 0$  be fixed and suppose, by contradiction, that the optimal policy is the (constant) zero function. Then by Definition 2.1 it follows that

$$V(x) = v(0, x) = \frac{U(0)}{1 - \alpha},$$

where  $v$  is defined in (11).

Since  $U$  and  $h$  are strictly increasing, then

$$V(x) = \frac{U(0)}{1 - \alpha} < U(h(x)) + \frac{\alpha}{1 - \alpha}U(0),$$

but this is impossible since  $U(h(x)) + \frac{\alpha}{1 - \alpha}U(0) = v(h, x)$ , contradicting the definition of  $V$ .

On the other hand, if  $h$  is the optimal policy, then

$$\begin{aligned} V(x) &= v(h, x) \\ &= U(h(x)) + \frac{\alpha}{1-\alpha}U(0). \end{aligned}$$

Let  $g : [0, h(x)] \rightarrow \mathbb{R}$  be defined as

$$g(a) = U(a) + \alpha E[U(h(\xi(h(x) - a)))] + \frac{\alpha^2}{1-\alpha}U(0).$$

Note that  $g$  is continuous and strictly concave, so there exists a unique  $\bar{a} \in [0, h(x)]$  that maximizes to  $g$ . If  $\bar{a} \neq h(x)$ , then

$$V(x) \geq g(\bar{a}) > g(h(x)) = V(x),$$

which is impossible. Therefore,  $\bar{a} = h(x)$ .

On the other hand, by (12) it follows that, for  $a \in (0, h(x))$ ,

$$g'(a) = U'(a) - \alpha E[U'(h(\xi(h(x) - a)))h'(\xi(h(x) - a))\xi],$$

and by Assumption 4.2 c), it is concluded that

$$\lim_{a \rightarrow h(x)} g'(a) = -\infty.$$

In particular, there is  $\tilde{a} \in (0, h(x))$  such that  $g'(\tilde{a}) < 0$ , implying that  $g$  is strictly decreasing in  $[\tilde{a}, h(x)]$ . But then  $h(x)$  cannot be the maximizer of  $g$ , i. e., there is a contradiction.  $\square$

Define  $R$  as  $R(x, a, y) = \Delta\left(\frac{y}{h(x)-a}\right)\frac{1}{h(x)-a}$ ,  $x > 0$ ,  $a \in [0, h(x)]$  and  $y > 0$ . Observe that  $R \in C^2$ . In fact, the following assumption makes possible the derivation procedure under the expectation.

**Assumption 5.4.**  $|R_x(x, a, y)| \leq \Phi_1(a, y)$ ,  $|R_a(x, a, y)| \leq \Phi_2(x, y)$ ,  $|R_{xx}(x, a, y)| \leq \Phi_3(a, y)$ ,  $|R_{xa}(x, a, y)| \leq \Phi_4(a, y)$  and  $|R_{xx}(x, a, y)| \leq \Phi_5(a, y)$ , where  $\Phi_i$  are integrable functions,  $i = 1, \dots, 5$ .

**Lemma 5.5.**  $V \in C^2((0, \infty), \mathbb{R})$  and the optimal policy  $f \in C^1((0, \infty))$ .

*Proof.* Fix  $x > 0$ . Let  $\psi$  be defined as

$$\psi(a) := E[V(\xi(h(x) - a))],$$

$a \in (0, h(x))$ . Using the Change of Variable Theorem (see [1]), it can be obtained that

$$\psi(a) := \int V(y)\Delta\left(\frac{y}{h(x)-a}\right)\frac{dy}{h(x)-a}$$

and by Assumption 5.4, it follows that  $\psi \in C^2((0, h(x)))$ .

On the other hand, since  $V$  is strictly concave (see (19) and by Subsection 4.1, Remark 4),  $\psi$  is concave. Therefore,  $\psi \in C^2$  and is concave, implying that  $\psi'' < 0$ .

Since  $U'' < 0$  (see Assumption 4.2 a)) and  $\psi'' < 0$ , then  $G \in C^2(\widehat{\mathbb{K}}, \mathbb{R})$  and  $G_{aa} < 0$ , where

$$G(x, a) := U(a) + \alpha\psi(a)$$

and  $\widehat{\mathbb{K}}$  is defined as (21).

Finally, by Lemma 5.3 and by Theorem 3.1, the result follows. □

The following result allows to characterize the optimal policy for CIP with its corresponding Euler equation.

**Lemma 5.6.** The optimal policy  $f$  satisfies the following Euler equation for each  $x \in (0, \infty)$ :

$$U'(f(x)) = \alpha E[h'(\xi(h(x) - f(x)))U'(f(\xi(h(x) - f(x))))\xi]. \tag{26}$$

Conversely, if  $f \in \mathbb{F}$  is a policy which satisfies (26) for each  $x \in (0, \infty)$  and if

$$\lim_{t \rightarrow \infty} \alpha^t E_x^f [h'(x_t)U'(f(x_t))x_t] = 0, \tag{27}$$

then  $f$  is optimal.

*Proof.* Let  $f$  be the optimal policy. Note that assumptions of Theorem 3.3 hold. Firstly, observe that by Lemma 5.5  $V \in C^2((0, \infty), \mathbb{R})$  and by Subsection 4.1, Remark 4  $V$  is strictly concave (see (19)). Furthermore, by Assumption 4.2. e) and the fact that  $h \in C^2((0, \infty))$ , it is obtained that  $H \in C^2(\widehat{\mathbb{K}}, \mathbb{R})$  and  $H_{aa}(x, a) \leq 0$ , where  $H(x, a) := E[V(\xi(h(x) - a))]$  (see (4)). On the other hand, taking  $r = U$  and, as  $U$  is strictly concave and  $U \in C^2((0, \infty), \mathbb{R})$ , then  $U'' < 0$ . Finally, note that  $F_a^{-1} = -1$ . Therefore Assumption a) of Theorem 3.3 holds. Now, Assumption b) of Theorem 3.3 holds, due to Lemma 5.3.

In this case,

$$W(x, a) = h'(x)U'(a), \tag{28}$$

$(x, a) \in \widehat{\mathbb{K}}$  (see (6)) then substituting (28) in (5), (26) is obtained.

On the other hand, let  $f$  be a function that satisfies (26) and (27) and  $x > 0$  be fixed. Let  $\widehat{f} \in \mathbb{F}$  be another function and for  $t = 0, 1, \dots$ , the trajectories of the policies  $f$  and  $\widehat{f}$  be denoted by  $x_t$  and  $\bar{x}_t$ , respectively, and  $a_t = f(x_t)$  and  $\bar{a}_t = \widehat{f}(\bar{x}_t)$  denote their actions, respectively, where  $x_0 = \bar{x}_0 = x$  for both.

Since  $U$  is strictly concave and  $U \in C^2$ , applying Theorem 2.17 in [11], p. 258, it can be obtained that

$$E \left[ \sum_{t=0}^{T-1} \alpha^t (U(a_t) - U(\bar{a}_t)) \right] \geq E \left[ \sum_{t=0}^{T-1} \alpha^t U'(a_t)(a_t - \bar{a}_t) \right],$$

for  $T > 1$ , which is positive integer. Since  $x_{t+1} = \xi_t(h(x_t) - a_t)$ ,  $\bar{x}_{t+1} = \xi_t(h(\bar{x}_t) - \bar{a}_t)$ , and  $\xi_t > 0$  almost everywhere (a. e.), then

$$\frac{x_t - \bar{x}_t}{\xi_{t-1}} = h(x_{t-1}) - h(\bar{x}_{t-1}) - (a_{t-1} - \bar{a}_{t-1}),$$

or equivalently,

$$a_{t-1} - \bar{a}_{t-1} = h(x_{t-1}) - h(\bar{x}_{t-1}) - \frac{x_t - \bar{x}_t}{\xi_{t-1}}.$$

Since  $h$  is concave and  $h \in C^2$ , again by the Theorem 2.17 in [11], it results that

$$a_{t-1} - \bar{a}_{t-1} \geq h'(x_{t-1})(x_{t-1} - \bar{x}_{t-1}) - \frac{x_t - \bar{x}_t}{\xi_{t-1}}.$$

Straightforward computations allow to obtain that

$$\begin{aligned} \sum_{t=0}^{T-1} \alpha^t U'(a_t)(a_t - \bar{a}_t) &\geq \sum_{t=0}^{T-1} \alpha^t U'(a_t) \left( h'(x_t)(x_t - \bar{x}_t) - \frac{x_{t+1} - \bar{x}_{t+1}}{\xi_t} \right) \\ &\geq \sum_{t=1}^{T-1} \left[ \alpha^{t-1} \frac{(x_t - \bar{x}_t)}{\xi_{t-1}} [\alpha \xi_{t-1} h'(x_t) U'(a_t) - U'(a_{t-1})] \right] \\ &\quad - \frac{\alpha^{T-1}}{\xi_{T-1}} U'(a_{T-1}) x_T. \end{aligned}$$

Note that the last inequality is due to the fact that  $U'(x)$ ,  $x_T$ ,  $\alpha$  and  $\xi_t$  are strictly positive.

Now, since  $a_t = f(x_t)$  and  $f$  satisfies (26), it follows that

$$U'(a_{T-1}) = \alpha E_x^f [h'(x_T) U'(a_T) \xi_{T-1} | x_{T-1}]. \tag{29}$$

Using (29) it is possible to conclude that

$$E_x^f \left[ \alpha^{T-1} \frac{x_T}{\xi_{T-1}} U'(a_{T-1}) \right] = E \left[ \alpha^{T-1} \frac{x_T}{\xi_{T-1}} \alpha E_x^f [h'(x_T) U'(a_T) \xi_{T-1} | x_{T-1}] \right].$$

Now, note that  $\frac{x_T}{\xi_{T-1}} = h(x_{T-1}) - f(x_{T-1})$ , i. e.,  $\frac{x_T}{\xi_{T-1}}$  depends on  $x_{T-1}$ , then by properties of the conditional expectation, it can be obtained that

$$E_x^f \left[ \alpha^{T-1} \frac{x_T}{\xi_{T-1}} U'(a_{T-1}) \right] = E_x^f [\alpha^T h'(x_T) U'(f(x_T)) x_T],$$

i. e., (27) follows.

Similarly,

$$\begin{aligned} &E_x^f \left[ \frac{(x_t - \bar{x}_t)}{\xi_{t-1}} [\alpha \xi_{t-1} h'(x_t) U'(a_t) - U'(a_{t-1})] \right] \\ &= E_x^f \left[ \frac{(x_t - \bar{x}_t)}{\xi_{t-1}} E[\alpha \xi_{t-1} h'(x_t) U'(a_t) - U'(a_{t-1}) | x_{t-1}] \right]. \end{aligned}$$

Then, since  $f$  satisfies (26), it follows that

$$U'(a_{t-1}) = E_x^f [\alpha \xi_{t-1} h'(x_t) U'(a_t) | x_{t-1}],$$

implying that

$$E_x^f [\alpha \xi_{t-1} h'(x_t) U'(a_t) - U'(a_{t-1}) | x_{t-1}] = 0,$$

consequently

$$E_x^f \left[ \frac{(x_t - \bar{x}_t)}{\xi_{t-1}} [\alpha \xi_{t-1} h'(x_t) U'(a_t) - U'(a_{t-1})] \right] = 0.$$

Hence

$$E_x^f \left[ \sum_{t=0}^{T-1} \alpha^t (U(a_t) - U(\bar{a}_t)) \right] \geq -E_x^f [\alpha^T h'(x_T) U'(f(x_T)) x_T].$$

Then letting  $T \rightarrow \infty$ , it follows that

$$E_x^f \left[ \sum_{t=0}^{\infty} \alpha^t U(a_t) \right] \geq E_x^f \left[ \sum_{t=0}^{\infty} \alpha^t U(\bar{a}_t) \right].$$

Therefore  $f$  is optimal. □

## 6. EXAMPLES

### 6.1. Example: logarithmic utility

This example is posed in [8] and [22], but with the difference that this is a stochastic version.

Suppose that

$$U(a_t) = \ln(a_t),$$

and the production function  $h(x) = x^\gamma$ , with  $\gamma \in (0, 1)$ ; and

$$x_{t+1} = \xi_t (x_t^\gamma - a_t),$$

$a_t \in [0, x_t^\gamma]$  and  $x_0 = x \in X := [0, \infty)$ , where  $\{\xi_t\}$  is a sequence of i.i.d. random variables independent of  $x$ , taking values in  $S := (0, 1)$ . Then let  $\xi$  be a generic element of the sequence  $\{\xi_t\}$ . Suppose that  $\mu_\gamma := E[\ln(\xi^\gamma)] < \infty$ . The density of  $\xi$  is designated by  $\Delta$ . It will be considered that  $\Delta \in C^2((0, \infty))$ .

**Lemma 6.1.** The logarithmic utility example satisfies BA.

*Proof.* Notice that the transition law is weakly continuous by Section 4.1 (see Remark 3).

Fix a number  $\epsilon > 1$  and put  $X_j := [0, j + \epsilon]$ , for  $j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ . Note that  $X_j \subset X_{j+1}$  and

$$X = \bigcup_{j \geq 0} \text{int}(X_j).$$

On the other hand, for each  $j \in \mathbb{N}_0$ ,  $m_j$  defined in (13) is as follows:

$$m_j = \gamma \ln(j + \epsilon),$$

and it is easy to see that (14) holds. Moreover, for  $t = 0, 1, 2, \dots$ , if  $x_t = x \in X$  and  $a_t \in A(x)$ , there exists  $j \in \mathbb{N}_0$  such that  $x \in X_j$  and, since  $\xi \in (0, 1)$ , it follows that  $x_{t+1} \in X_{j+1}$ , i. e., (15) holds, by Subsection 4.1, Remark 1a). □

**Lemma 6.2.** For each  $n = 1, 2, \dots$ ,

$$v_n(x) = \gamma k_n \ln(x) + c_n, \tag{30}$$

where  $x \in X$ ,  $c_n \in \mathbb{R}$  and  $k_n = \sum_{t=0}^{n-1} (\alpha\gamma)^t$ ,  $n = 2, 3, \dots$

*Proof.* The proof will be made by induction. Fix  $x > 0$ . Take  $n = 1$ . Then, directly,  $v_1(x) = \gamma \ln(x)$ . In this case,  $H^2$  (see (20)) is given by

$$H^2(x, a) = \gamma \ln(x^\gamma - a) + \mu_\gamma,$$

$a \in A(x)$ . Note that  $H^2 \in C^2(\widehat{\mathbb{K}}; \mathbb{R})$  and  $H_{aa}^2(x, a) < 0$ , then using Lemma 5.2,

$$\begin{aligned} v_2'(x) &= \alpha E \left[ v_1' \left( \xi \left( x^\gamma - \frac{\gamma x^{\gamma-1}}{v_2'(x)} \right) \right) \xi \right] \gamma x^{\gamma-1}, \\ &= \alpha E \left[ \gamma \xi / \xi \left( x^\gamma - \frac{\gamma x^{\gamma-1}}{v_2'(x)} \right) \right] \gamma x^{\gamma-1}, \end{aligned}$$

so it follows that  $v_2'(x) = \frac{\gamma}{x}(1 + \alpha\gamma)$ , hence

$$v_2(x) = \gamma \ln(x)(1 + \alpha\gamma) + c_2.$$

Now suppose that for some  $n > 2$ ,  $v_{n-1}$  satisfies (30). Then  $H^n$  (see (20)) is as follows:

$$H^n(x, a) = \gamma k_{n-1} \ln(x^\gamma - a) + \gamma k_{n-1} \mu_\gamma + c_{n-1}.$$

Note that  $H^n \in C^2(\widehat{\mathbb{K}}; \mathbb{R})$  and  $H_{aa}^n(x, a) < 0$ , then using Lemma 5.2,

$$\begin{aligned} v_n'(x) &= \alpha E \left[ v_{n-1}' \left( \xi \left( x^\gamma - \frac{\gamma x^{\gamma-1}}{v_{n-1}'(x)} \right) \right) \xi \right] \gamma x^{\gamma-1}, \\ &= \alpha E \left[ \gamma \xi / \xi \left( x^\gamma - \frac{\gamma x^{\gamma-1}}{v_{n-1}'(x)} \right) \right] \gamma x^{\gamma-1}, \end{aligned}$$

and a straightforward computation allows to obtain that

$$v_n'(x) = \frac{\gamma}{x} \sum_{t=0}^{n-1} (\alpha\gamma)^t.$$

Finally, (30) is obtained integrating the last equality. □

**Corollary 6.3.** For each  $n = 1, 2, \dots$ ,

$$f_n(x) = \frac{x^\gamma}{k_{n+1}},$$

$x \in X$ , with  $k_{n+1} = \sum_{t=0}^n (\alpha\gamma)^t$ ,  $n = 0, 1, 2, \dots$ , where

$$f_n(x) = \arg \max_{a \in A(x)} \{ \ln(a) + \alpha H^n(x, a) \}.$$

*Proof.* Fix  $x > 0$ . By Lemma 6.1 it follows that

$$v_n(x) = \max_{a \in [0, x^\gamma]} \{ \ln(a) + \alpha E [\gamma \ln(\xi(x^\gamma - a))k_{n-1} + c_{n-1}] \}.$$

Let  $\bar{g}$  be defined as

$$\bar{g}(a) := \ln(a) + \alpha\gamma k_n \ln((x^\gamma - a) + \alpha\gamma k_n \mu_\gamma + c_n),$$

$a \in (0, x^\gamma)$ . Using the first order condition for the optimality of  $\bar{g}$  and the fact that  $f_n(x) \in (0, h(x))$ , it is possible to obtain that

$$f_n(x) = \frac{x^\gamma}{k_{n+1}}.$$

□

Observe that for each  $x \geq 0$ ,  $f_n(x) \rightarrow \tilde{f}(x)$ , where

$$\tilde{f}(x) := x^\gamma(1 - \alpha\gamma), \tag{31}$$

and  $\tilde{f}(x) \in [0, x^\gamma]$ , i.e.  $\tilde{f}$  is an admissible deterministic stationary policy. Furthermore, evaluating  $\tilde{f}$  in (11) permits to obtain that  $v(\tilde{f}, x) = K \ln(x) + C > -\infty$ , where  $K, C \in \mathbb{R}$ .

**Corollary 6.4.** For the logarithmic utility example,

$$V(x) = \frac{\gamma}{1 - \alpha\gamma} \ln(x) + C,$$

$x > 0$ , where

$$C = \frac{1}{1 - \alpha} \left[ \ln(1 - \alpha\gamma) + \frac{\alpha\gamma}{1 - \alpha\gamma} (\mu_\gamma + \ln(\alpha\gamma)) \right],$$

and  $\tilde{f}$  is the optimal policy, where  $\tilde{f}$  is defined in (31).

*Proof.* Fix  $x > 0$ . Since  $v_n(x) \rightarrow V(x)$ , and  $k_n \rightarrow 1/(1 - \alpha\gamma)$ ,  $n \rightarrow \infty$ , (notice that  $0 < \alpha\gamma < 1$ ), then from (30) it follows that  $\{c_n\}$  is convergent. Let  $C := \lim_{n \rightarrow \infty} c_n$ . So

$$V(x) = \frac{\gamma}{1 - \alpha\gamma} \ln(x) + C.$$

On the other hand, since  $U'(a) = 1/a$  and  $h'(x) = \gamma x^{\gamma-1}$ , then

$$\begin{aligned} & \alpha E [h' \left( \xi \left( h(x) - \tilde{f}(x) \right) \right) U' \left( \tilde{f} \left( \xi \left( h(x) - \tilde{f}(x) \right) \right) \right) \xi] \\ &= \alpha E \left[ \frac{\gamma (\xi(x^\gamma - x^\gamma(1 - \alpha\gamma)))^{\gamma-1}}{(\xi(x^\gamma - x^\gamma(1 - \alpha\gamma)))^\gamma (1 - \alpha\gamma)} \xi \right] \end{aligned}$$

and a straightforward computation yields that (26) holds. Now, observe that

$$\begin{aligned} \alpha^t E \left[ h'(x_t) U'(\tilde{f}(x_t)) x_t \right] &= \alpha^t E \left[ \frac{\gamma x_t^{\gamma-1}}{x_t^\gamma (1 - \alpha\gamma)} x_t \right], \\ &= \frac{\alpha^t \gamma}{1 - \alpha\gamma}, \end{aligned}$$



then, letting  $t \rightarrow \infty$ , equality (27) yields.

Therefore, by Lemma 5.6 it is obtained that  $\tilde{f}$  is the optimal policy. Now since  $V$  satisfies (19), it follows that

$$\frac{\gamma}{1 - \alpha\gamma} \ln(x) + C = \sup_{a \in [0, x^\gamma]} \left\{ \ln(a) + \alpha \frac{\gamma}{1 - \alpha\gamma} E[\ln(\xi(x^\gamma - a))] + \alpha C \right\},$$

and it is obtained from (31) that  $C = \frac{1}{1-\alpha} \left[ \ln(1 - \alpha\gamma) + \frac{\alpha\gamma}{1-\alpha\gamma} (\mu_\gamma + \ln(\alpha\gamma)) \right]$ . □

### 6.2. Example: exponential utility

Now, consider the following:

$$U(a_t) = \frac{b}{\gamma} a_t^\gamma,$$

where  $b > 0$  and  $\gamma \in (0, 1)$ . The production function  $h(x) = x$ , and

$$x_{t+1} = \xi_t(x_t - a_t),$$

$a_t \in [0, x_t]$ ,  $t = 0, 1, 2, \dots$ ,  $x_0 = x \in X := [0, \infty)$ . It is assumed that  $\{\xi_t\}$  is a sequence of i.i.d. random variables independent of  $x_0$ , taking values in  $S = [0, \infty)$ . Let  $\xi$  be a generic element of the sequence  $\{\xi_t\}$ . Suppose that  $\mu_\gamma := E[\xi^\gamma] < \infty$ , with  $0 < \alpha\mu_\gamma < 1$ , where  $\delta := (\alpha\mu_\gamma)^{1/(\gamma-1)}$  and  $\Delta \in C^2((0, \infty))$  denote the density of  $\xi$ .

**Lemma 6.5.** The exponential utility example satisfies BA.

*Proof.* Notice that,  $A(x) = [0, x]$  is a compact set, for each  $x \in X$ , and the utility function is continuous. Now, let  $\theta$  be a bounded-continuous function on  $X$  and

$$\Theta(a) = \int \theta(sa)\Delta(s) ds,$$

$a \in [0, x]$  (see 18). Since  $\Delta \in C^2(0, \infty)$ , it is possible to obtain that  $\Theta$  is continuous in  $a \in [0, x]$  for each  $x \in X$ . This way the transition law is weakly continuous.

Furthermore, BA holds, due to (16) and (17) are satisfied (see Subsection 4.1, Remark 1b)), if

$$w(x) = \frac{b\mu_\gamma}{\gamma(1 - \alpha\mu_\gamma)} x^\gamma + 1, \quad c = \beta = 1.$$

□

**Lemma 6.6.** For each  $n = 1, 2, \dots$ ,

$$v_n(x) = \left( \frac{\delta^{n-1}(1 - \delta)}{1 - \delta^n} \right)^{\gamma-1} \frac{b}{\gamma} x^\gamma, \tag{32}$$

$x \in X$ .

Proof. Let  $x > 0$  be fixed. Then  $v_1(x) = \frac{b}{\gamma}x^\gamma$ . In this case,  $H^2$  (see (20)) is given by

$$H^2(x, a) = \frac{b}{\gamma}\mu_\gamma(x - a)^\gamma,$$

$a \in A(x)$ . Note that  $H^2 \in C^2(\widehat{\mathbb{K}}; \mathbb{R})$  and  $H^2_{aa}(x, a) < 0$ , then using Lemma 5.2,

$$v'_2(x) = b\delta^{\gamma-1} \left( x - \left( \frac{v'_2(x)}{b} \right)^{\frac{1}{\gamma-1}} \right)^{\gamma-1},$$

and it follows that

$$v'_2(x) = \left( \frac{\delta(1-\delta)}{1-\delta^2} \right)^{\gamma-1} bx^{\gamma-1}, \tag{33}$$

hence

$$v_2(x) = \left( \frac{\delta(1-\delta)}{1-\delta^2} \right)^{\gamma-1} \frac{b}{\gamma}x^\gamma.$$

Observe that  $v_2$  was obtained integrating (33) and taking the constant involved in the integration equal to zero, because for this example, for each  $n = 0, 1, 2, \dots$ ,  $v_n(0) = 0$  (see [8]).

Suppose that for some  $n > 2$ ,  $v_{n-1}$  satisfies (32). Then  $H^n$  (see (20)), is as follows:

$$H^n(x, a) = \left( \frac{\delta^{n-1}(1-\delta)}{1-\delta^n} \right)^{\gamma-1} \frac{b\mu_\gamma}{\gamma}(x - a)^\gamma.$$

Note that  $H^n \in C^2(\widehat{\mathbb{K}}; \mathbb{R})$  and  $H^n_{aa}(x, a) < 0$ , then using Lemma 5.2,

$$v'_n(x) = \alpha E \left[ v'_{n-1} \left( \xi \left( x - \left( \frac{v'_n(x)}{b} \right)^{\frac{1}{\gamma-1}} \right) \right) \xi \right]$$

and a straightforward computation allows to obtain that

$$v'_n(x) = \left( \frac{\delta^{n-1}(1-\delta)}{1-\delta^n} \right)^{\gamma-1} bx^{\gamma-1}. \tag{34}$$

Finally, integrating (34) allows to prove that (32) holds. □

**Corollary 6.7.** For each  $n = 1, 2, \dots$ ,

$$f_n(x) = \left( \frac{\delta^{n-1}(1-\delta)}{1-\delta^n} \right) x,$$

$x \in X$ .

Proof. Fix  $x > 0$ . For Lemma 6.5 it follows that

$$v_n(x) = \max_{a \in [0, x^\gamma]} \left\{ \frac{b}{\gamma}a^\gamma + \alpha \left( \frac{\delta^{n-1}(1-\delta)}{1-\delta^n} \right)^{\gamma-1} \frac{b\mu_\gamma}{\gamma}(x - a)^\gamma \right\}.$$

Let  $\bar{g}$  be defined as

$$\bar{g}(a) := \frac{b}{\gamma} a^\gamma + \alpha \left( \frac{\delta^{n-1}(1-\delta)}{1-\delta^n} \right)^{\gamma-1} \frac{b\mu_\gamma}{\gamma} (x-a)^\gamma,$$

$a \in (0, x)$ . Using the first order condition for the optimality of  $\bar{g}$  and the fact that  $f_n(x) \in (0, h(x))$ , it is possible to obtain that

$$f_n(x) = \left( \frac{\delta^{n-1}(1-\delta)}{1-\delta^n} \right) x.$$

Observe that for each  $x \geq 0$ ,  $f_n(x) \rightarrow \tilde{f}(x)$ , where

$$\tilde{f}(x) := \left( \frac{\delta-1}{\delta} \right) x, \tag{35}$$

and  $\tilde{f}(x) \in [0, x]$ , i. e.  $\tilde{f}$  is an admissible deterministic stationary policy. □

**Corollary 6.8.** For the exponential example,

$$V(x) = \left( \frac{\delta-1}{\delta} \right)^{\gamma-1} \frac{b}{\gamma} x^\gamma, \tag{36}$$

and  $\tilde{f}$  is the optimal policy defined in (35).

*Proof.* Letting  $n \rightarrow \infty$  in (32), (36) follows. On the other hand, since  $U'(a) = ba^{\gamma-1}$  and  $h'(x) = 1$ , then

$$\begin{aligned} & \alpha E \left[ h'(\xi(h(x) - \tilde{f}(x))) U' \left( \tilde{f} \left( \xi \left( h(x) - \tilde{f}(x) \right) \right) \right) \xi \right] \\ &= \alpha b E \left[ \left( \xi \left( x - x \frac{\delta-1}{\delta} \right) \frac{\delta-1}{\delta} \right)^{\gamma-1} \xi \right] \end{aligned}$$

and a straightforward computation allows to obtain that (26) holds. Now, observe that

$$\alpha^t E \left[ h'(x_t) U'(\tilde{f}(x_t)) x_t \right] = \alpha^t b \left( \frac{\delta-1}{\delta} \right)^{\gamma-1} E[x_t^\gamma], \tag{37}$$

with  $x_0 = x \in X$ . Since  $\{\xi_n\}$  is a sequence of i.i.d. random variables independent of  $x_0$ , then

$$E[x_t^\gamma] = \left( \frac{x}{\delta} \right)^\gamma \mu_\gamma^t. \tag{38}$$

Substituting (38) in (37) and since  $0 < \alpha\mu_\gamma < 1$ , it follows that

$$\lim_{t \rightarrow \infty} \alpha^t E \left[ h'(x_t) U'(\tilde{f}(x_t)) x_t \right] = 0.$$

Therefore, by Lemma 5.6 it is obtained that  $f$  is the optimal policy. □

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