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ROBUSTNESS OPTIMAL SPRING BALANCE WEIGHING DESIGNS FOR ESTIMATION TOTAL WEIGHT

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In this paper we develop the theory of spring balance weighing designs with non-positive correlated errors for that the lower bound of the variance of estimated total weight is attained.

Keywords: robustness, spring balance weighing design, total weight

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1. INTRODUCTION

We consider the standard Gauss–Markoff model $\mathbf{y} = \mathbf{X}\mathbf{w} + \mathbf{e}$, where $\mathbf{y}$ is the $n \times 1$ observation vector, $\mathbf{X} \in \mathbf{\Psi}_{n\times p,\,\xi}(0,1)$, $\mathbf{w}$ is a $p \times 1$ vector representing unknown parameters, $\mathbf{e}$ is an $n \times 1$ random vector of errors having the following properties: $\mathrm{E}(\mathbf{e}) = \mathbf{0}_n$, $\mathrm{E}(\mathbf{e}\mathbf{e}') = \sigma^2\mathbf{G}$, where σ^2 is the constant variance of errors, $\mathbf{0}_n$ is the $n \times 1$ vector of zeros, $\mathbf{G}$ is the $n \times n$ symmetric positive definite matrix of known elements. $\mathbf{\Psi}_{n\times p,\,\xi}(0,1)$ for fixed ξ, denotes the class of available $n \times p$ matrices such that

(i) $\mathbf{X} = (x_{ij}),\ x_{ij} = 0$ or $1,\ i = 1,2,\ldots,n,\ j = 1,2,\ldots,p$,

(ii) for given i, $\xi_i = \sum_{j=1}^{p} x_{ij},\ \max_i\{\xi_i\} = \xi,\ \xi < p$,

(iii) for every member of $\mathbf{\Psi}_{n\times p,\,\xi}(0,1)$, the total weight is estimable.

Following the usual terminology, the design matrix $\mathbf{X} \in \mathbf{\Psi}_{n\times p,\,\xi}(0,1)$ is called the weighing matrix and the experiment is called the spring balance weighing design. (See, for example, [12].) Some optimality criterions and the existence conditions determining such designs are given in [1, 8, 9].

Any spring balance weighing design is nonsingular if the matrix $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$ is nonsingular. It is obvious that if $\mathbf{G}$ is the positive definite matrix then any spring balance weighing design is nonsingular if and only if the matrix $\mathbf{X}'\mathbf{X}$ is nonsingular and then all parameters are estimable. Even $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$ is nonsingular, there exists a design which estimates the total weight with a smaller variance than the design which is most efficient for the estimation of individual weights. The examples of such designs are available in literature, see for instance [1]. When $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$ is singular, although unknown measurements of all objects are not estimable, but some linear functions of $\mathbf{w}$ may be estimable. One of the estimable function is the total weight (the sum of measurements) of all objects, i. e. $\mathbf{1}_p'\mathbf{w}$, where $\mathbf{1}_p$ denotes the $p \times 1$ vector of ones. Some examples of optimal singular spring balance weighing designs for estimated total weight are given in [4].

Suppose that we omit the assumption $\xi < p$ and we take $\xi = p$. Thus if $\mathbf{X} \in \mathbf{\Psi}_{n\times p,\, p}(0,1)$ then all the objects are weighted simultaneously in all the weighings and the design matrix $\mathbf{X}$ will contain only unities as its elements. Obviously such design will enable the estimation of total weight with minimum variance. However, due to practical limitation, it may be not possible to measure all objects together in each measurement operation.

It is, therefore, assumed that at most $\xi(< p)$ objects can be weighted simultaneously in each weighing. Under this restriction, a lower bound for variance of the estimated total weight is obtained using a singular spring balance weighing design permitting the estimation of total weight. Design for which the lower bound is attainable have been called "optimum".

In this paper we present the estimation of total weight of objects in the spring balance weighing design assuming that the errors are equal non-positive correlated and they have the same variance, i. e. for the random vector of errors $\mathbf{e}$, $\mathrm{E}(\mathbf{e}\mathbf{e}') = \sigma^2\mathbf{G}$, where

$$\mathbf{G} = g\left[(1-\rho)\mathbf{I}_n + \rho\mathbf{1}_n\mathbf{1}_n'\right], \quad g > 0, \; \frac{-1}{n-1} < \rho \le 0, \tag{1}$$

$\mathbf{I}_n$ is $n \times n$ identity matrix. The matrix $\mathbf{G}$ has compound symmetry structure. The compound symmetry correlation structure assumes equal correlation among all the measurements. Some examples of applying such matrix are available in [6]. Let note, for $g > 0$, $\frac{-1}{n-1} < \rho \le 0$, the matrix $\mathbf{G}$ is positive definite and moreover $\mathbf{G}^{-1} = \frac{1}{g(1-\rho)}\left[\mathbf{I}_n - \frac{\rho}{1+\rho(n-1)}\mathbf{1}_n\mathbf{1}_n'\right]$.

For the case $\mathbf{G} = \mathbf{I}_n$, some problems concerned on the spring balance weighing designs have been considered in literature: [1, 4]. For some patterns of $\mathbf{G}$ the conditions determining optimal design for estimating total weight were given in [5].

2. OPTIMAL DESIGN

Assume that $\mathbf{X} \in \mathbf{\Psi}_{n\times p,\, \xi}\,(0,1)$ and $\mathbf{G}$ is of (1). According to the literature (see for example [8]) the following definition can be establish.

Definition 2.1. Any spring balance weighing design $\mathbf{X} \in \mathbf{\Psi}_{n\times p,\, \xi}(0,1)$ with the covariance matrix $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is of (1), is said to be optimal for the estimated total weight if the variance of its estimator attains the lower bound.

We will denote by $(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})^{-}$ a generalized inverse (g-inverse) of $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$, i.e. $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})^{-}\mathbf{X}'\mathbf{G}^{-1}\mathbf{X} = \mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$. As mentioned in Section 1, a parametric function of interest is the total weight and this function will be estimable if and only if there exists an $n \times 1$ vector $\mathbf{a}$ such that $\mathbf{a}'\mathbf{X} = \mathbf{1}_p'$. This condition is equivalent to the $\mathbf{1}_p'(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})^{-}\mathbf{X}'\mathbf{G}^{-1}\mathbf{X} = \mathbf{1}_p'$.

The following lemma given in [5] will be required to prove the main result of next theorem.

Lemma 2.2. For any symmetric positive definite $n \times n$ matrix $\mathbf{G}$, any $n \times p$ matrix $\mathbf{X}$ and any vector $\mathbf{c} \neq \mathbf{0}_p$ satisfying $\mathbf{c}'(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})^{-}\mathbf{X}'\mathbf{G}^{-1}\mathbf{X} = \mathbf{c}'$,

$$\mathbf{c}'(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})^{-}\mathbf{c} \geq \frac{(\mathbf{c}'\mathbf{c})^2}{\mathbf{c}'\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\mathbf{c}}. \tag{2}$$

Equality holds in (2) if and only if $\mathbf{c}$ is eigenvector of $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$.

Theorem 2.3. In any spring balance weighing design $\mathbf{X} \in \mathbf{\Psi}_{n\times p,\,\xi}(0,1)$ with the covariance matrix $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is of (1), the variance of the estimator of total weight is given as

$$\operatorname{Var}\left(\widehat{\mathbf{1}_p'\mathbf{w}}\right) \geq \frac{\sigma^2 p^2 g(1+\rho(n-1))}{n\xi^2}. \tag{3}$$

The design is optimal for the estimated total weight if and only if

(i) $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\mathbf{1}_p = \mu\mathbf{1}_p$ and

(ii) $\xi_i = \xi$ for all $i = 1, 2, \ldots, n$,

where μ is non-negative scalar.

Proof. The proof falls naturally into two parts. Under the above assumptions and considering Lemma 2.2 it would be noticed that $\operatorname{Var}\left(\widehat{\mathbf{1}_p'\mathbf{w}}\right) = \operatorname{Var}\left(\mathbf{1}_p'(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})^{-}\mathbf{X}'\mathbf{G}^{-1}\mathbf{y}\right) = \sigma^2\mathbf{1}_p'(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})^{-}\mathbf{1}_p \geq \frac{\sigma^2p^2}{\mathbf{1}_p'\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\mathbf{1}_p}$. The equality holds if and only if $\mathbf{1}_p$ is eigenvector of $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$. Furthermore,

$$\begin{aligned}\mathbf{1}_p'\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\mathbf{1}_p &= \frac{1}{g(1-\rho)}\left(\mathbf{1}_p'\mathbf{X}'\mathbf{X}\mathbf{1}_p - \frac{\rho}{1+\rho(n-1)}\mathbf{1}_p'\mathbf{X}'\mathbf{1}_n\mathbf{1}_n'\mathbf{X}\mathbf{1}_p\right)\\ &= \frac{1}{g(1-\rho)}\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{p}x_{ij}\right)^2 - \frac{\rho}{1+\rho(n-1)}\left(\sum_{i=1}^{n}\sum_{j=1}^{p}x_{ij}\right)^2\right)\\ &\leq \frac{1}{g(1-\rho)}\left(n\cdot\xi^2 - \frac{\rho}{1+\rho(n-1)}n^2\xi^2\right) = \frac{n\xi^2}{g(1+\rho(n-1))}.\end{aligned}$$

The equality holds if and only if $\xi_i = \xi$ for all $i = 1, 2, \ldots, n$. Thus $\operatorname{Var}\left(\widehat{\mathbf{1}_p'\mathbf{w}}\right) \geq \sigma^2\frac{p^2g(1+\rho(n-1))}{n\xi^2}$, which completes the first part of the proof.

Let note, $\mathbf{u} = \mathbf{G}^{-\frac{1}{2}}\mathbf{X}\mathbf{1}_p$ and $\mathbf{v} = \mathbf{G}^{-\frac{1}{2}}\mathbf{X}\left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)^{-}\mathbf{1}_p$. Applying the Cauchy–Schwarz inequality on $\mathbf{u}$ and $\mathbf{v}$ we have $(\mathbf{u}'\mathbf{v})^2 \leq (\mathbf{u}'\mathbf{u})(\mathbf{v}'\mathbf{v})$. Equality holds if and only if $\mathbf{u} = \mu\mathbf{v}$ for some real scalar μ. Substituting for $\mathbf{u}$ and $\mathbf{v}$ the condition $\mathbf{u} = \mu\mathbf{v}$ reduced to $\mathbf{G}^{-\frac{1}{2}}\mathbf{X}\mathbf{1}_p = \mu\mathbf{G}^{-\frac{1}{2}}\mathbf{X}\left(\mathbf{X}'\mathbf{G}^{-\frac{1}{2}}\mathbf{X}\right)^{-}\mathbf{1}_p$ which is equivalent to $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\mathbf{1}_p = \mu\mathbf{1}_p$. The proof is completed by showing that the equality in (3) is attained if and only if $\xi_i = \xi$ for all $i = 1, 2, \ldots, n$, i.e. the (ii) is true. □

Also note the following remark.

Remark 2.4. For $\mathbf{G} = \mathbf{I}_n$ Theorem 2.3 was given in [4], whereas in [2] was proved under assumption $\mathbf{G} = diag(g_1, g_2, \ldots, g_n),\ g_i > 0$ for all i's.

Theorem 2.5. In any spring balance weighing design $\mathbf{X} \in \mathbf{\Psi}_{n\times p,\,\xi}(0,1)$ with the covariance matrix $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is of (1), the conditions (i) and (ii) of Theorem 2.3 are equivalent to

(i) $\mathbf{X}'\mathbf{G}^{-1}\mathbf{1}_n = \vartheta\mathbf{1}_p$ and

(ii) $\mathbf{X}\mathbf{1}_p = \xi\mathbf{1}_n$,

where $\vartheta = \frac{\mu}{\xi}$.

P r o o f. To prove the theorem we first observe that from (ii) of Theorem 2.3 we obtain $\mathbf{X}\mathbf{1}_p = \xi\mathbf{1}_n$. Considering (i) of the theorem 2.3 we conclude $\xi\mathbf{X}'\mathbf{G}^{-1}\mathbf{1}_n = \mu\mathbf{1}_p$. Next, it implies $\mathbf{X}'\mathbf{G}^{-1}\mathbf{1}_n = \mu\xi^{-1}\mathbf{1}_p$. On the other hand, we assume the conditions given in Theorem 2.5 are true. From $\mathbf{X}'\mathbf{G}^{-1}\mathbf{1}_n = \vartheta\mathbf{1}_p$ we have $\mathbf{X}'\mathbf{G}^{-1}\xi\mathbf{1}_n = \vartheta\xi\mathbf{1}_p$. Taking $\mathbf{X}\mathbf{1}_p$ for $\xi\mathbf{1}_n$ we obtain $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\mathbf{1}_p = \vartheta\xi\mathbf{1}_p = \mu\mathbf{1}_p$. Moreover $\mathbf{X}\mathbf{1}_p = \xi\mathbf{1}_n$ is equivalent to (ii) of Theorem 2.3 and we get the required result. □

Remark 2.6. In particular case $\mathbf{G} = \mathbf{I}_n$, the theorem 2.5 was given as Lemma 2.1 in [11].

Let note, for $\mathbf{G}$ in (1), $\mathbf{G}\mathbf{1}_n = \alpha\mathbf{1}_n$ and $\mathbf{G}^{-1}\mathbf{1}_n = \frac{1}{\alpha}\mathbf{1}_n$, where $\alpha = g(1 + \rho(n-1))$.

Corollary 2.7. In any spring balance weighing design $\mathbf{X} \in \mathbf{\Psi}_{n\times p,\,\xi}(0,1)$ with the covariance matrix $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is of (1), the conditions (i) of Theorem 2.5 is equivalent to $\mathbf{X}'\mathbf{1}_n = \omega\mathbf{1}_p$, where $\omega = \frac{\mu\alpha}{\xi}$.

Above consideration imply that $\mathbf{X}'\mathbf{1}_n = \omega\mathbf{1}_p$, i.e. the sum of elements in each column of the design matrix $\mathbf{X}$ is the same. On the other hand $\mathbf{X}'\mathbf{1}_n = \frac{\mu\alpha}{\xi}\mathbf{1}_p$, i.e. the sum of elements in each column of the design matrix $\mathbf{X}$ depends on the matrix $\mathbf{G}$. Comparing equalities $\omega\mathbf{1}_p$ and $\frac{\mu\alpha}{\xi}\mathbf{1}_p$ and place forms of $\alpha,\ \mu,\ \xi$ and $\mathbf{G}^{-1}$ we obtain identity. Here is way we conclude that the spring balance weighing design $\mathbf{X} \in \mathbf{\Psi}_{n\times p,\,\xi}(0,1)$ is optimal for the estimated total weight for any $\rho,\ \frac{-1}{n-1} < \rho \leq 0$, i.e. this design is robust for different ρ. The results given in the above theorems imply the next corollary.

Corollary 2.8. Any spring balance weighing design $\mathbf{X} \in \mathbf{\Psi}_{n\times p,\,\xi}(0,1)$ with the covariance matrix $\sigma^2\mathbf{I}_n$ is optimal for the estimated total weight if and only if such design is optimal for the estimated total weight with the covariance matrix $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is of (1).

Let us consider any ρ_u, $\frac{-1}{n-1} < \rho_u \leq 0$, $u = 1,2$, $\rho_1 \neq \rho_2$. It is worth pointing out that the design $\mathbf{X} \in \mathbf{\Psi}_{n\times p,\,\xi}(0,1)$ satisfying Theorem 2.5 is optimal for the estimated total weight in the sense of attaining minimal variance of the estimator of total weight for the covariance matrix $\sigma^2\mathbf{G}$ for ρ_1 and for ρ_2. Simutaneously the lower boud of variance given in (3) is not the same for different numbers of ρ. For a deeper discussion of robustness optimal designs we refer the reader to [7].

3. CONSTRUCTION OF THE DESIGN MATRIX

Let $\mathbf{N}$ denote the usual $v \times b$ binary incidence matrix of block design where v and b mean the number of treatments and number of blocks, respectively. Let $\mathbf{N1}_b = r\mathbf{1}_v$ and $\mathbf{N}'\mathbf{1}_v = k\mathbf{1}_b$, where r is the number of replications of the ith treatment and k is the size of jth block, $i = 1,2,\ldots,v$, $j = 1,2,\ldots,b$.

Theorem 3.1. Any spring balance weighing design $\mathbf{X} \in \mathbf{\Psi}_{n\times p,\,\xi}(0,1)$, $\mathbf{X} = \mathbf{N}$ (or $\mathbf{X} = \mathbf{N}'$) with the covariance matrix $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is of the form (1), is optimal for estimated total weight of $p = b$ (or $p = v$) objects in $n = v$ (or $n = b$) weighings.

P r o o f. Let note, if $\mathbf{X} = \mathbf{N}$ then $\xi = r$, if $\mathbf{X} = \mathbf{N}'$ then $\xi = k$. Taking $\mathbf{a} = \frac{1}{k}\mathbf{1}_v$ (or $\mathbf{a} = \frac{1}{r}\mathbf{1}_b$) it is clear that condition $\mathbf{a}'\mathbf{N} = \mathbf{1}'_p$ is satisfied for $\mathbf{X} = \mathbf{N}$ (or $\mathbf{X} = \mathbf{N}'$). The condition given in Corollary 2.7 and the condition (ii) of Theorem 2.5 follow from the equalities $\mathbf{N1}_b = r\mathbf{1}_v$ and $\mathbf{N}'\mathbf{1}_v = k\mathbf{1}_b$. □

Remark 3.2. Following the standard notation given, for example in [3, 9, 10], it is clear that $\mathbf{N}$ could be the incidence matrix of the balanced incomplete block design or one of partially balanced incomplete block design with two associated classes:

(i) group divisible design consisting of three subtypes: singular, semi-regular, regular,

(ii) triangular,

(iii) Latin square types,

(iv) cyclic,

(v) partial geometry,

(vi) miscellaneous.

Moreover, if $\mathbf{N}$ is the incidence matrix of $\alpha-$ resolvable block designs, $(\alpha_1, \alpha_2, \ldots, \alpha_t)-$ resolvable block designs, singular group divisible designs or semi-regular group divisible designs, appropriate constructions are given in [2].

As counter-example let as consider the experiment in which we determine total weight of $p = 3$ objects in $n = 6$ measurements operations. Let us consider design matrix $\mathbf{X} \in \mathbf{\Psi}_{6\times 3,2}(0,1)$ and the covariance matrix $\sigma^2\mathbf{G}$ for $g = 1$, where

$$\mathbf{X} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \qquad \mathbf{G} = \begin{bmatrix} 1 & -\frac{1}{7} & -\frac{1}{7} & -\frac{1}{7} & -\frac{1}{7} & -\frac{1}{7} \\ -\frac{1}{7} & 1 & -\frac{1}{7} & -\frac{1}{7} & -\frac{1}{7} & -\frac{1}{7} \\ -\frac{1}{7} & -\frac{1}{7} & 1 & -\frac{1}{7} & -\frac{1}{7} & -\frac{1}{7} \\ -\frac{1}{7} & -\frac{1}{7} & -\frac{1}{7} & 1 & -\frac{1}{7} & -\frac{1}{7} \\ -\frac{1}{7} & -\frac{1}{7} & -\frac{1}{7} & -\frac{1}{7} & 1 & -\frac{1}{7} \\ -\frac{1}{7} & -\frac{1}{7} & -\frac{1}{7} & -\frac{1}{7} & -\frac{1}{7} & 1 \end{bmatrix}.$$

We have $\alpha = \frac{2}{7}$, $\mathbf{X1}_3 = 2 \cdot \mathbf{1}_6$ hence $\xi = 2$, $\mathbf{X}'\mathbf{1}_6 = 4 \cdot \mathbf{1}_3$ thus $\omega = 4$, $\mathbf{X}'\mathbf{G}^{-1}\mathbf{1}_6 = 14 \cdot \mathbf{1}_3$ and $\vartheta = 14$. Since $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X1}_3 = 28 \cdot \mathbf{1}_3$, then $\mu = 28$ and $4 = \omega = \frac{\mu\alpha}{\xi}$. Moreover, for the design $\mathbf{X}$ with $\mathbf{G}$, $\mathrm{Var}\big(\widehat{\mathbf{1}_3'\mathbf{w}}\big) = 0,107\sigma^2$. It is easy to see that for covariance matrix of errors $\sigma^2\mathbf{G}$, the design $\mathbf{X}$ that satisfies Theorem 2.5 is optimal for estimation of the total weight.

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REFERENCES

[1] K. S. Banerjee: Weighing Designs for Chemistry, Medicine, Economics, Operations Research, Statistics. Marcel Dekker Inc., New York 1975.

[2] B. Ceranka and K. Katulska: Optimum singular spring balance weighing designs with non-homogeneity of the variances of errors for estimating the total weight. Austral. J. Statist. *28* (1986), 200–205.

[3] W. H. Clatworthy: Tables of two-associate-class partially balanced designs. NBS Appl. Math. *63* (1973).

[4] A. Dey and S. C. Gupta: Singular weighing designs and the estimation of total weight. Comm. Statist. Theory Methods *7* (1977), 289–295.

[5] K. Katulska: On the estimation of total weight in singular spring balance weighing designs under the covariance matrix of errors $\sigma^2\mathbf{G}$. Austral. J. Statist. *31* (1989), 277–286.

[6] M. Krzyśko and M. Skorzybut: Dysciminant analysis of multivariate repeated measures data with Kronecker product structured covariance matrices. Statist. Papers *50* (2009), 817–835.

[7] J. Masaro and C. S. Wong: Robustness of A-optimal designs. Linear Algebra Appl. *429* (2008), 1392–1408.

[8] F. Pukelsheim: Optimal Design of Experiment. John Wiley and Sons, New York 1993.

[9] D. Raghavarao: Constructions and Combinatorial Problems in designs of Experiments. John Wiley Inc., New York 1971.

[10] D. Raghavarao and L. V. Padgett: Block Designs, Analysis, Combinatorics and Applications. Series of Applied Mathematics 17, Word Scientific Publishing Co. Pte. Ltd., 2005

[11] B. K. Sinha: Optimum spring balance weighing designs. In: Proc. All India Convention on Quality and Reliability. Indian Inst. Tech., Kharagpur 1972.

[12] K. R. Shah and B. K. Sinha: Theory of Optimal Designs. Springer-Verlag, Berlin 1989.

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