KYBERNETIKA — VOLUME 47 (2011), NUMBER 5, PAGES 715–721

ON THE PROBLEM $Ax = \lambda Bx$ IN MAX ALGEBRA: EVERY SYSTEM OF INTERVALS IS A SPECTRUM

Sergeĭ Sergeev

We consider the two-sided eigenproblem $A \otimes x = \lambda \otimes B \otimes x$ over max algebra. It is shown that any finite system of real intervals and points can be represented as spectrum of this eigenproblem.

Keywords: extremal algebra, tropical algebra, generalized eigenproblem

Classification: 15A80, 15A22, 91A46, 93C65

1. INTRODUCTION

Max algebra is the analogue of linear algebra developed over the max-plus semiring, which is the set $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$ equipped with the operations of "addition" $a \oplus b := \max(a, b)$ and "multiplication" $a \otimes b := a + b$. This basic arithmetics is naturally extended to matrices and vectors. In particular, for matrices $A = (a_{ij}) \in \overline{\mathbb{R}}^{n\times m}$ and $B = (b_{ij}) \in \overline{\mathbb{R}}^{m\times k}$, their "product" $A \otimes B$ is defined by the rule $(A \otimes B)_{ij} = \bigoplus_{l=1}^{m} a_{il} \otimes b_{lj}$, for all $i = 1, \dots, n$ and $j = 1, \dots, k$.

One of the best studied problems in max algebra is the "eigenproblem": for given $A \in \overline{\mathbb{R}}^{n\times n}$ find $\lambda \in \overline{\mathbb{R}}$ and $x \in \overline{\mathbb{R}}^{n}$ with at least one finite entry, such that $A\otimes x = \lambda\otimes x$. This problem is very important for max-algebra and its applications [1, 2, 6, 7, 9, 15]. The theory of this problem has much in common with its counterpart in the nonnegative matrix algebra. In particular, there is exactly one eigenvalue ("max-algebraic Perron root") in the irreducible case, and in general, there may be several eigenvalues which correspond to diagonal blocks of the Frobenius normal form. There are efficient algorithms for computing both eigenvalues and eigenvectors [8, 12, 15].

We will consider the following generalization of the max algebraic eigenproblem:

$$A \otimes x = \lambda \otimes B \otimes x, \tag{1}$$

where $A, B \in \overline{\mathbb{R}}^{n\times m}$. The set of $\lambda \in \overline{\mathbb{R}}$ such that there exists x satisfying (1), with at least one finite entry, will be called the *spectrum* of (1) and denoted by $\sigma(A, B)$.

This problem is of interest as an analogue of matrix pencils in nonnegative matrix algebra, as studied in McDonald et al. [16], Mehrmann et al. [17]. Note that matrix pencils in linear algebra are very well-known, see Gantmacher [13] for basic reference, and their applications in control go back to Brunovsky [4].

Problem (1) can also be considered as a parametric extension of two-sided systems $A \otimes x = B \otimes x$. Importantly, such systems can be solved algorithmically, see Cuninghame Green and Butkovič [10].

Unlike the eigenproblem $A \otimes x = \lambda \otimes x$, the two-sided version does not seem to be well-known. Some results have been obtained by Binding and Volkmer [3], and Cuninghame-Green and Butkovič[11], mostly for special cases when both matrices are square, or when $A = B \otimes Q$. See also Butkovič [7]. In the latter case, it may be possible to reduce (1) to $Q \otimes x = \lambda \otimes x$. In general, however, it is nontrivial to decide whether the spectrum is nonempty, and some particular conditions have been studied by topological methods [3].

Further, the spectrum of (1) can be much richer, it may include intervals. Gaubert and Sergeev [14] came up with a general approach to the problem representing it in terms of parametric min-max functions and mean-payoff games, which allows to identify the whole spectrum in pseudo-polynomial time. The purpose of this note is more modest, it is to provide an example showing that any system of intervals and points can be realized as the spectrum of (1).

Let us note a possible application of (1) in scheduling in the spirit of Cuninghame-Green [9]. See also Burns [5]. Suppose that the products $P_1, \dots, P_n$ are prepared using m machines (or, say, processors), where every machine contributes to the completion of each product by producing a partial product. Let a_{ij} be the duration of the work of the jth machine needed to complete the partial product for P_i. Let us denote by x_j the starting time of the jth machine, then all partial products for P_i will be ready by the time $\max(x_1 + a_{i1}, \dots, x_m + a_{im})$. Now suppose that m other machines prepare partial products for products $Q_1, \dots, Q_n$, and the duration and starting times are b_{ij} and y_j respectively. If the machines are linked then it may be required that $y_j - x_j$ is a constant time λ. Now consider a synchronization problem: to find λ and starting times of all $2m$ machines so that each pair P_i, Q_i is completed at the same time. Algebraically, we have to solve

$$\begin{aligned} &\max(x_1 + a_{i1}, \dots, x_m + a_{im}) = \max(\lambda + x_1 + b_{i1}, \dots, \lambda + x_m + b_{im}), \\ &\forall\, i = 1, \dots, n, \end{aligned} \tag{2}$$

which is clearly the same as (1).

2. PRELIMINARIES

We begin with some definitions and notation. The *max algebraic column span* of $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times m}$ is defined by

$$\operatorname{span}_{\oplus}(A) = \left\{ \bigoplus_{i=1}^{m} \alpha_i A_{\cdot i} \mid \alpha_i \in \overline{\mathbb{R}} \right\}.$$

For $y \in \overline{\mathbb{R}}^n$ denote $\operatorname{supp}(y) = \{i : \ y_i \neq -\infty\}$, and for $y, z \in \overline{\mathbb{R}}^n$ denote

$$T(y, z) := \arg\min\{y_i - z_i \mid i \in \operatorname{supp}(y) \cap \operatorname{supp}(z)\}.$$

In max algebra, *one-sided* systems $A \otimes x = b$ can be easily solved, and the solvability criterion is as follows. By $A_{i\cdot}$ (resp. $A_{\cdot i}$) we denote the ith row (resp. the ith column) of $A \in \overline{\mathbb{R}}^{n\times m}$.

Theorem 2.1. (Butkovič [6], Theorem 2.2) Let $A \in \overline{\mathbb{R}}^{n\times m}$ and $b \in \mathbb{R}^n$. The following statements are equivalent.

1. $b \in \text{span}_\oplus(A)$.
2. $A \otimes x = b$ is solvable.
3. $\bigcup_{i=1}^m T(b, A_{\cdot i}) = \{1, \ldots, n\}$.

The author is not aware of any such criterion for two-sided systems $A\otimes x = B\otimes x$. However, the following *cancellation law* can be useful in their analysis ($a, b, c, d \in \overline{\mathbb{R}}$):

$$\begin{aligned} &\text{if } a < c \text{ then} \\ &a \otimes x \oplus b = c \otimes x \oplus d \quad \Leftrightarrow \quad b = c \otimes x \oplus d. \end{aligned} \tag{3}$$

Consider a particular application of this law. In what follows we write $x < y$ also for two *vectors* x and y, if $x_i < y_i$ holds for all their components.

Lemma 2.2. Let $A, B \in \overline{\mathbb{R}}^{n\times m}$ and let $A_{i\cdot} < B_{i\cdot}$ for some i. Then $A \otimes x = B \otimes x$ does not have nontrivial solution.

Proof. Applying cancellation (3), we obtain that the ith equation of $A\otimes x = B\otimes x$ is equivalent to $B_{i\cdot} \otimes x = -\infty$. Note that all entries of $B_{i\cdot}$ are finite, hence $x_j = -\infty$ for all j. □

When A, B have finite entries only, Lemma 2.2 can be used [11] to obtain bounds for the spectrum of (1):

$$\sigma(A, B) \subseteq \left[\max_i \min_j (a_{ij} - b_{ij}), \min_i \max_j (a_{ij} - b_{ij})\right]. \tag{4}$$

The cancellation law also allows to replace the finiteness restriction by requiring that a_{ij} *or* b_{ij} is finite for all i and j.

It will be also useful that (1) is equivalent to the following system with separated variables:

$$\begin{aligned} &C(\lambda) \otimes x = D \otimes y, \text{ where} \\ &C(\lambda) = \begin{pmatrix} A \\ \lambda \otimes B \end{pmatrix}, \quad D = \begin{pmatrix} I \\ I \end{pmatrix}, \end{aligned} \tag{5}$$

and $I = (\delta_{ij}) \in \overline{\mathbb{R}}^{n\times n}$ denotes the max-plus identity matrix with entries

$$\delta_{ij} = \begin{cases} 0, & \text{if } i = j, \\ -\infty, & \text{if } i \neq j. \end{cases} \tag{6}$$

The finite vectors belonging to $\text{span}_\oplus(D)$ can be easily described.

Lemma 2.3. $z \in \mathbb{R}^{2n}$ belongs to $\operatorname{span}_{\oplus}(D)$ if and only if $z_i = z_{n+i}$ for all $i = 1, \ldots, n$.

3. MAIN RESULTS

Let $\{[a_i, c_i],\ i = 1, \ldots, m\}$ be a finite system of intervals on the real line, where $a_i \le c_i < a_{i+1}$ for $i = 1, \ldots, m-1$, with possibility that $a_i = c_i$. Define matrices $A \in \mathbb{R}^{2\times 3m}$, $B \in \mathbb{R}^{2\times 3m}$:

$$\begin{aligned} A &= \begin{pmatrix} \ldots & a_i & b_i & c_i & \ldots \\ \ldots & 2a_i & 2b_i & 2c_i & \ldots \end{pmatrix}, \\ B &= \begin{pmatrix} \ldots & 0 & 0 & 0 & \ldots \\ \ldots & a_i & c_i & b_i & \ldots \end{pmatrix}, \end{aligned} \tag{7}$$

where $b_i := \frac{a_i + c_i}{2}$.

Theorem 3.1. With A, B defined by (7),

$$\sigma(A, B) = \bigcup_{i=1}^{m} [a_i, c_i]. \tag{8}$$

P r o o f. First we show that any λ outside the system of intervals is not an eigenvalue.

Case 1. $\lambda < a_1$, resp. $\lambda > c_m$. In these cases $\lambda \otimes B_{1\cdot} < A_{1\cdot}$, resp. $\lambda \otimes B_{1\cdot} > A_{1\cdot}$, hence by Lemma 2.2 $A \otimes x = \lambda \otimes B \otimes x$ cannot hold with nontrivial x.

Case 2. $c_k < \lambda < a_{k+1}$. Using cancellation law (3), we obtain that the first equation of $A \otimes x = \lambda \otimes B \otimes x$ is equivalent to

$$\bigoplus_{i=k}^{m-1} (a_{i+1} \otimes x_{3i+1} \oplus b_{i+1} \otimes x_{3i+2} \oplus c_{i+1} \otimes x_{3i+3}) = \lambda \otimes \bigoplus_{i=1}^{3k} x_i. \tag{9}$$

For the second equation of $A \otimes x = \lambda \otimes B \otimes x$, observe that $2a_i > \lambda + a_i$, $2b_i > \lambda + c_i$ and $2c_i > \lambda + b_i$ for all $i \ge k+1$. After cancellation (3), the l.h.s. and the r.h.s. of this equation turn into max-linear forms $u(x)$ and $v(x)$ respectively, such that

$$\begin{aligned} & u(x) = v(x), \\ & u(x) \ge \bigoplus_{i=k}^{m-1} (2a_{i+1} \otimes x_{3i+1} \oplus 2b_{i+1} \otimes x_{3i+2} \oplus 2c_{i+1} \otimes x_{3i+3}), \\ & v(x) \le \lambda \otimes \bigoplus_{i=0}^{k-1} (a_{i+1} \otimes x_{3i+1} \oplus c_{i+1} \otimes x_{3i+2} \oplus b_{i+1} \otimes x_{3i+3}). \end{aligned} \tag{10}$$

We claim that (9) and (10) cannot hold at the same time with a nontrivial x. Using that $\lambda < a_{i+1} \le b_{i+1} \le c_{i+1}$ for all $i \ge k$, and that the l.h.s. of (9) attains maximum at a particular term, we deduce from (9) that

$$\bigoplus_{i=k}^{m-1} (2a_{i+1} \otimes x_{3i+1} \oplus 2b_{i+1} \otimes x_{3i+2} \oplus 2c_{i+1} \otimes x_{3i+3}) > 2\lambda \otimes \bigoplus_{i=1}^{3k} x_i. \tag{11}$$

(Note that both sides of (9) are finite, since all coefficients of A and B are finite and x is nontrivial.) The l.h.s. of (11) is the same as the r.h.s. of the second statement of (10). Therefore, combining (11) and the first two statements of (10), we obtain

$$v(x) = u(x) > 2\lambda \otimes \bigoplus_{i=1}^{3k} x_i. \tag{12}$$

Now, since the coefficients a_{i+1}, b_{i+1} and c_{i+1} on the r.h.s. of the last statement of (10) do not exceed λ, the r.h.s. of that statement does not exceed the r.h.s. of (12). But combining (12) with that last statement of (10) we obtain just the opposite. This contradiction shows that $A \otimes x = \lambda \otimes B \otimes x$ cannot have nontrivial solutions in case 2.

Now we prove that any λ in the intervals is an eigenvalue, by guessing a vector that belongs to $\mathrm{span}_\oplus(C(\lambda)) \cap \mathrm{span}_\oplus(D)$. The columns of $C(\lambda)$ will be denoted by

$$\begin{aligned} u^i(\lambda) &= (a_i \quad 2a_i \quad \lambda \quad a_i + \lambda)^T, \\ v^i(\lambda) &= (b_i \quad 2b_i \quad \lambda \quad c_i + \lambda)^T, \\ w^i(\lambda) &= (c_i \quad 2c_i \quad \lambda \quad b_i + \lambda)^T. \end{aligned} \tag{13}$$

Case 3. $a_i \le \lambda \le b_i$. We take

$$z^\lambda = (0 \quad \lambda + b_i - a_i \quad 0 \quad \lambda + b_i - a_i)^T. \tag{14}$$

By Lemma 2.3 $z^\lambda \in \mathrm{span}_\oplus(D)$. It suffices to check that $z^\lambda \in \mathrm{span}_\oplus(C(\lambda))$. We write

$$T(z^\lambda, u^i(\lambda)) = \arg\min(-a_i,\ \lambda + b_i - 3a_i,\ -\lambda,\ b_i - 2a_i), \tag{15}$$

$$T(z^\lambda, v^i(\lambda)) = \arg\min(-b_i,\ \lambda - b_i - a_i,\ -\lambda,\ -b_i), \tag{16}$$

$$T(z^\lambda, w^i(\lambda)) = \arg\min(-c_i,\ \lambda - b_i - c_i,\ -\lambda,\ -a_i). \tag{17}$$

In (16) and (17) we used that $2b_i = a_i + c_i$. The inequalities $a_i \le \lambda \le b_i$ imply that

$$-\lambda \le -a_i \le b_i - 2a_i \le \lambda + b_i - 3a_i, \tag{18}$$

hence the minimum in (15) is attained by the 3rd component. Analogously, the minimum in (16) is attained by the 4th and 1st components, and the minimum in (17) is attained by the 2nd component. By Theorem 2.1 $z^\lambda \in \mathrm{span}_\oplus(C(\lambda))$.

Case 4. $b_i \le \lambda \le c_i$. We take

$$z^\lambda = (0 \quad c_i \quad 0 \quad c_i)^T, \tag{19}$$

By Lemma 2.3 $z^\lambda \in \text{span}_\oplus(D)$, and we claim again that $z^\lambda \in \text{span}_\oplus(C(\lambda))$. We compute

$$T(z^\lambda, v^i(\lambda)) = \arg\min(-b_i,\ c_i - 2b_i,\ -\lambda,\ -\lambda), \tag{20}$$

$$T(z^\lambda, w^i(\lambda)) = \arg\min(-c_i,\ -c_i,\ -\lambda,\ c_i - b_i - \lambda). \tag{21}$$

We observe that the minimum in (20) is attained by the 3rd and 4th components, while the minimum in (21) is attained by the 1st and 2nd components. The claim follows by Theorem 2.1. □

ACKNOWLEDGEMENT

This work was supported by EPSRC grant RRAH12809, RFBR grant 08-01-00601 and RFBR-CRNF grant 11-01-93106.
The author is grateful to Peter Butkovič for valuable discussions concerning this paper and the two-sided eigenproblem, to Stéphane Gaubert for showing him an example that the spectrum of this eigenproblem can be disconnected and contain both an interval and a point, and to Jean-Jacques Loiseau for suggesting references [4] and [13]. The author wishes to thank the anonymous referees for their careful reading and useful remarks.

(Received January 12, 2011)

REFERENCES

[1] M. Akian, R. Bapat, and S. Gaubert: Max-plus algebras. In: Handbook of Linear Algebra (L. Hogben, ed.), Discrete Math. Appl. *39*, Chapter 25, Chapman and Hall 2006.

[2] F. L. Baccelli, G. Cohen, G.-J. Olsder, and J.-P. Quadrat: Synchronization and Linearity: An Algebra for Discrete Event Systems. Wiley 1992.

[3] P. A. Binding and H. Volkmer: A generalized eigenvalue problem in the max algebra. Linear Algebra Appl. *422* (2007), 360–371.

[4] P. Brunovsky: A classification of linear controllable systems. Kybernetika *6* (1970), 173–188.

[5] S. M. Burns: Performance Analysis and Optimization of Asynchronous Circuits. PhD Thesis, California Institute of Technology 1991.

[6] P. Butkovič: Max-algebra: the linear algebra of combinatorics? Linear Algebra Appl. *367* (2003), 313–335.

[7] P. Butkovič: Max-linear Systems: Theory and Algorithms. Springer 2010.

[8] J. Cochet-Terrasson, G. Cohen, S. Gaubert, M. M. Gettrick, and J. P. Quadrat: Numerical computation of spectral elements in max-plus algebra. In: Proc. IFAC Conference on Systems Structure and Control, IRCT, Nantes 1998, pp. 699–706.

[9] R. A. Cuninghame-Green: Minimax Algebra. Lecture Notes in Econom. and Math. Systems *166*, Springer, Berlin 1979.

[10] R. A. Cuninghame-Green and P. Butkovič: The equation $A \otimes x = B \otimes y$ over (max,+). Theoret. Comput. Sci. *293* (2003), 3–12.

[11] R. A. Cuninghame-Green and P. Butkovič: Generalised eigenproblem in max algebra. In: Proc. 9th International Workshop WODES 2008, pp. 236–241.

[12] L. Elsner and P. van den Driessche: Modifying the power method in max algebra. Linear Algebra Appl. *332–334* (2001), 3–13.

[13] F. R. Gantmacher: The Theory of Matrices. Chelsea, 1959.

[14] S. Gaubert and S. Sergeev: The level set method for the two-sided eigenproblem. E-print `http://arxiv.org/pdf/1006.5702.`

[15] B. Heidergott, G.-J. Olsder, and J. van der Woude: Max-plus at Work. Princeton Univ. Press, 2005.

[16] J. J. McDonald, D. D. Olesky, H. Schneider, M. J. Tsatsomeros, and P. van den Driessche: Z-pencils. Electron. J. Linear Algebra *4* (1998), 32–38.

[17] V. Mehrmann, R. Nabben, and E. Virnik: Generalization of Perron-Frobenius theory to matrix pencils. Linear Algebra Appl. *428* (2008), 20–38.

Sergeĭ Sergeev, INRIA and CMAP École Polytechnique, 91128 Palaiseau Cédex. France.
e-mail: sergeev@cmap.polytechnique.fr