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ON TESTING HYPOTHESES IN THE GENERALIZED SKILLINGS–MACK RANDOM BLOCKS SETTING

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The testing of the null hypothesis of no treatment effect against the alternative of increasing treatment effect by means of rank statistics is extended from the classical Friedman random blocks model into an unbalanced design allowing treatments not to be applied simultaneously in each random block. The asymptotic normality of the constructed rank test statistic is proved both in the setting not allowing ties and also for models with presence of ties. As a by-product of the proofs a multiple comparisons rule based on rank statistics is obtained for the case when the null hypothesis of no treatment effect is tested against the general alternative of its negation.

Keywords: rank test, random blocks, hypotheses testing, increasing treatment effect, asymptotic distribution

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1. INTRODUCTION AND MAIN RESULTS

In the scheme of random blocks the quality of k treatments is evaluated in such a way that the experimental units are partitioned into N groups of a homogeneous type (random blocks) and in each obtained block the examined treatments are assigned to its units so as the resulting effects would be stochastically independent. In the classical starting paper [3] on this topic, in each block the number of experimental units equals the number of treatments and each treatment is applied on one unit. Since sometimes one can face difficulties in ensuring that all the treatments will be applied in each block, the classical assumption has been in [8] modified in a way allowing the design where some of the treatments can be missing, i. e., the number of the units in the ith block equals the number k_i of the treatments applied in the ith block (no treatment is applied to more than one unit of the block) and $k_i \leq k$. The null hypothesis of no difference in the treatment effect is tested against the hypothesis that there exist treatments i, j such that their effects are different. A test of the hypothesis of no treatment effect against the alternative of the monotone increasing ordering in the response was presented in [7] in the classical Friedman's random block setting.

The aim of this paper is to extend these results into a scheme of (possibly) unbalanced designs with missing data, where the evaluated treatments are allowed to be applied to more than one experimental unit in the block, i. e., the block consists of cells of experimental units and in each cell the same treatment is applied. Rank test statistics for inference in this framework aimed at dealing with alternative of increasing treatment effect and also multiple comparisons method based on ranks are constructed in this section. The proofs of theorems of this section can be found in Section 2. The setting in which the assertions of the paper are derived is based on the following assumptions.

Let us assume in the theorems of this section that $\mathbf{X}_i = (\mathbf{X}_{i1}, \ldots, \mathbf{X}_{ik_i})$, $i = 1, \ldots, N$, are independent random vectors (here index i identifies the random block), and for $j = 1, \ldots, k_i$

$$\mathbf{X}_{ij} = \left(X_{ij}^{(1)}, \ldots, X_{ij}^{(d_{ij})}\right) \tag{1.1}$$

where $X_{ij}^{(1)}, \ldots, X_{ij}^{(d_{ij})}$ are one-dimensional random variables. We shall use also the notation

$$\mathbf{X}_i = (Z_{i1}, \ldots, Z_{id_{i.}}), \tag{1.2}$$

where $\{Z_{ir}\}$ are one-dimensional random variables and

$$d_{i.} = d_{i1} + \ldots + d_{ik_i} \tag{1.3}$$

denotes the number of observations in the ith block. The examined k treatments are in the N random blocks applied as follows.

(C1) *Let $i \in \{1, \ldots, N\}$ be an arbitrary fixed index. Each of the vectors $\mathbf{X}_{ij}$ is a result of the applications of some treatment $t_{ij} \in \{1, \ldots, k\}$, $k > 1$.*

(C2) *Each of the treatments $1, \ldots, k$ is in the same random block $\mathbf{X}_i$ applied at most in one of the vectors $\mathbf{X}_{i1}, \ldots, \mathbf{X}_{ik_i}$. Thus $k_i \leq k$ and with the notation from (C1) the set $\{t_{i1}, \ldots, t_{ik_i}\}$ consists of k_i elements. Throughout the paper we shall assume that for all i*

$$t_{i1} < \ldots < t_{ik_i}.$$

(C3) *The inequality $\min\{k_1, \ldots, k_N\} > 1$ holds, i. e., there are applied more than one treatment in every random block.*

(C4) *Use the notation (1.2). For every $i = 1, \ldots, N$ the random variables $Z_{i1}, \ldots, Z_{id_{i.}}$ are mutually different with probability 1.*

(C5) *For $r = 1, \ldots, k$, let n_{ir} denotes the number of applications of the treatment r in the ith block. Thus $n_{ir} = d_{ij}$ if $r = t_{ij}$ and if such a number t_{ij} does not exist, then $n_{ir} = 0$. Put*

$$m_{js} = \sum_{i=1}^{N} n_{ij} n_{is}. \tag{1.4}$$

For all $j \neq s$ belonging to $\{1, \ldots, k\}$ there exists a limit

$$p_{js} = \lim_{N \to \infty} \frac{m_{js}}{N} \tag{1.5}$$

and is a positive number. Moreover, for

$$D_N = \max\{d_{i.};\, i = 1, \ldots, N\} \tag{1.6}$$

the convergence

$$\lim_{N\to\infty} \frac{D_N^3}{N} = 0 \tag{1.7}$$

holds.

This setting is constructed for the sake of testing the null hypothesis

$$H_0: \quad \begin{array}{l} \textit{For all } i = 1,\ldots,N \textit{ the distribution of the random vector} \\ (Z_{ir_1},\ldots,Z_{ir_{d_{i.}}}) \textit{ is the same for every permutation } (r_1,\ldots,r_{d_{i.}}) \\ \textit{of the set } \{1,\ldots,d_{i.}\}. \end{array} \tag{1.8}$$

If the random vectors $\mathbf{X}_{ij}$, $j = 1,\ldots,k_i$ are independent and also their components $X_{ij}^{(1)},\ldots,X_{ij}^{(d_{ij})}$ are independent, then this null hypothesis coincides with the hypothesis (here $F_{ij}^{(v)}$ denotes the distribution function of $X_{ij}^{(v)}$)

$$H_0: \quad \begin{array}{l} F_{i1}^{(1)} = \ldots = F_{i1}^{(d_{i1})} = F_{i2}^{(1)} = \ldots = F_{i2}^{(d_{i2})} = \ldots = F_{ik_i}^{(1)} = \ldots = F_{ik_i}^{(d_{ik_i})} \\ \text{for } i = 1,\ldots,N \end{array}$$

of no treatments effect.

The condition (C5) will be used in establishing the asymptotic distribution of the presented test statistics. The setting (C1) – (C4) is a generalization of the scheme used by Skillings and Mack [8], who assume the vectors $\mathbf{X}_{ij}$ to be one-dimensional. The presented setting comprises also the framework used by Conover [2], pp. 383–384, who assumes that for all $i = 1,\ldots,N$ the equalities $k_i = k$ and $d_{ij} = m$ hold.

Suppose that (C4) holds, which means that for any fixed i the random variables $\{X_{ij}^{(v)}\}$ are mutually different almost surely. Therefore the ranks of these observations (1.2)

$$\mathbf{R}_i = (R_{i1}^{(1)},\ldots,R_{i1}^{(d_{i1})},R_{i2}^{(1)},\ldots,R_{i2}^{(d_{i2})},\ldots,R_{ik_i}^{(1)},\ldots,R_{ik_i}^{(d_{ik_i})}) \tag{1.9}$$

are uniquely determined with probability 1. As has been explained, n_{ir} denotes the number of applications of the treatment r in the ith block. In accordance with this let S_{ir} denote the sum of ranks of the rth treatment in the ith block, i. e.,

$$S_{ir} = \sum_{v=1}^{d_{ij}} R_{ij}^{(v)}, \qquad n_{ir} = d_{ij} \tag{1.10}$$

if $r = t_{ij}$ and if such an index t_{ij} does not exist, then

$$S_{ir} = 0, \qquad n_{ir} = 0. \tag{1.11}$$

Finally, let

$$\mathbf{A}^{(N)} = \Big(A_1^{(N)},\ldots,A_k^{(N)}\Big)', \quad A_r^{(N)} = \sum_{i=1}^{N} \sqrt{\frac{12}{d_{i.}+1}}\Big(S_{ir} - n_{ir}\frac{d_{i.}+1}{2}\Big). \tag{1.12}$$

Theorem 1.1. Suppose that the null hypothesis (1.8) holds, the conditions (C1)–(C4) are fulfilled and

$$\mathbf{\Sigma}_N = Var(\mathbf{A}^{(N)}) \tag{1.13}$$

denotes the covariance matrix of this random vector.

(I) The element $\mathbf{\Sigma}_N(j,s)$ of this matrix on the position (j,s)

$$\mathbf{\Sigma}_N(j,s) = \begin{cases} \sum_{i=1}^{N} n_{ij}(d_{i.} - n_{ij}) & j = s, \\ -m_{js} & j \neq s. \end{cases} \tag{1.14}$$

If all the numbers m_{js} defined in (1.4) are positive, then with the notation

$$\mathbf{f}_k = (1, 2, \ldots, k)' \tag{1.15}$$

the number

$$\sigma_N^2 = \mathbf{f}_k' \mathbf{\Sigma}_N \mathbf{f}_k \tag{1.16}$$

is positive.

(II) Suppose that also (C5) holds. Then the statistic (cf. (1.12), (1.16))

$$\tilde{T}_N = \frac{1}{\sigma_N} \sum_{j=1}^{k} j A_j^{(N)} \tag{1.17}$$

converges to $N(0,1)$ in distribution as $N \to \infty$.

Let H_1 denote the alternative hypothesis that for each $1 \leq j < j^* \leq k$ the effect of the treatment j is stochastically not larger than the effect of the treatment j^* and this ordering is strict for some $1 \leq j_0 < j_1 \leq k$ (i.e., every coordinates $X_{ij_0}^{(t)} \overset{st}{\prec} X_{ij_1}^{(t^*)}$ for all $i = 1, \ldots, N$ such that both treatments j_0, j_1 are applied in the ith block, the definition of the stochastic ordering can be found on p. 66 of [6]). In accordance with the previous theorem the null hypothesis H_0 is rejected in favor of H_1 whenever the statistic $\tilde{T}_N$ exceeds the $(1-\alpha)$th quantile of the normal $N(0,1)$ distribution.

Since (1.14) holds, the covariance matrix (1.13) can be written as

$$\mathbf{\Sigma}_N = \sum_{i=1}^{N} \mathbf{\Psi}_i, \tag{1.18}$$

where the $k \times k$ matrix

$$\mathbf{\Psi}_i = \begin{pmatrix} n_{i1}(d_{i.} - n_{i1}) & -n_{i1}n_{i2} & -n_{i1}n_{i3} & \ldots & -n_{i1}n_{ik} \\ -n_{i2}n_{i1} & n_{i2}(d_{i.} - n_{i2}) & -n_{i2}n_{i3} & \ldots & -n_{i2}n_{ik} \\ \vdots & & & & \vdots \\ -n_{ik}n_{i1} & -n_{ik}n_{i2} & -n_{ik}n_{i3} & \ldots & n_{ik}(d_{i.} - n_{ik}) \end{pmatrix}. \tag{1.19}$$

According to Theorem 5B on p. 20 of [4] the condition (C4) is fulfilled if for every $i = 1, \ldots, N$ the random variables $\{X_{ij}^{(v)}; j = 1, \ldots, k_i, v = 1, \ldots, d_{ij}\}$ are independent and their distribution functions are continuous. In dealing with the case of ties inside blocks the following assumption will be useful.

(C4*) *There exists a number $\delta > 0$ such that for all $i = 1, 2, \ldots$ with the notation (1.2)*

$$1 - P(Z_{i1} = Z_{i2} = \ldots = Z_{id_{i.}}) \geq \delta\,. \tag{1.20}$$

Now assume that (1.9) denotes the vector of midranks of the numbers (1.2), the quantities S_{ir} are again defined by means of (1.10) and (1.11),

$$\tilde{\sigma}_i^2 = \frac{1}{d_{i.} - 1} \sum_{j=1}^{k_i} \sum_{v=1}^{d_{ij}} (R_{ij}^{(v)} - \frac{d_{i.} + 1}{2})^2, \quad i = 1, \ldots, N, \tag{1.21}$$

and

$$\tilde{\mathbf{A}}^{(N)} = (\tilde{A}_1^{(N)}, \ldots, \tilde{A}_k^{(N)})' = \sum_{i=1}^{N} \tilde{\mathbf{B}}_i\,, \tag{1.22}$$

where

$$\tilde{\mathbf{B}}_i = (\tilde{B}_1^{(i)}, \ldots, \tilde{B}_k^{(i)})', \quad \tilde{B}_r^{(i)} = \begin{cases} \sqrt{\dfrac{d_{i.}}{\tilde{\sigma}_i^2}} \Big(S_{ir} - n_{ir} \dfrac{d_{i.} + 1}{2}\Big) & \tilde{\sigma}_i^2 > 0, \\ 0 & \tilde{\sigma}_i^2 = 0. \end{cases} \tag{1.23}$$

If $x^{(1)} \leq \ldots \leq x^{(m)}$ denotes the ordering of the coordinates of the vector $\mathbf{x} = (x_1, \ldots, x_m)$ according to the magnitude and

$$x^{(1)} = \ldots = x^{(\tau_1)} < x^{(\tau_1+1)} = \ldots = x^{(\tau_1+\tau_2)},$$
$$x^{(\tau_1+\ldots+\tau_{j-1})} < x^{(\tau_1+\ldots+\tau_{j-1}+1)} = \ldots = x^{(\tau_1+\ldots+\tau_{j-1}+\tau_j)}, \quad j = 3, \ldots, L,$$
$$\tau_1 + \ldots + \tau_L = m,$$

then in accordance with [4] the vector $\tau(\mathbf{x}) = (\tau_1, \ldots, \tau_L)$ is called the vector of number of ties in $\mathbf{x}$.

Now let $\tau(\mathbf{X}_1, \ldots, \mathbf{X}_N) = (\tau(\mathbf{X}_1), \ldots, \tau(\mathbf{X}_N))$ and $\tau(\mathbf{X}_i)$ denotes the vector of number of ties in $\mathbf{X}_i$.

Theorem 1.2. Suppose that the conditions (C1) – (C3), (C4*) are fulfilled and the null hypothesis (1.8) holds.

(I) Let $\tau_N = (\tau_1^{(N)}, \ldots, \tau_N^{(N)})$ be such a vector that $P(\tau(\mathbf{X}_i) = \tau_i^{(N)}) > 0$, $i = 1, \ldots, N$. For the conditional covariance matrix

$$\tilde{\mathbf{\Sigma}}_N = Var(\tilde{\mathbf{A}}^{(N)} \,|\, \tau(\mathbf{X}_1, \ldots, \mathbf{X}_N) = \tau_N) \tag{1.24}$$

the equality

$$\tilde{\mathbf{\Sigma}}_N = \sum_{i=1}^{N} (1 - \delta_{0,\tilde{\sigma}_i}) \mathbf{\Psi}_i \tag{1.25}$$

holds. Here $\delta_{0,\tilde{\sigma}_i}$ denotes the Kronecker delta and the $k \times k$ matrix $\mathbf{\Psi}_i$ is defined in (1.19).

If for each $j \neq s$ belonging to $\{1, \dots, k\}$ there exists i such that $n_{ij} n_{is} (1 - \delta_{0,\tilde{\sigma}_i}) > 0$, then the number (cf. (1.15))

$$\sigma_N^2 = \mathbf{f}_k' \tilde{\mathbf{\Sigma}}_N \mathbf{f}_k \tag{1.26}$$

is positive.

(II) Suppose that also (C5) holds and (cf. (1.6))

$$\lim_{N\to\infty} \frac{D_N^4}{N} = 0. \tag{1.27}$$

Since the number $\delta > 0$ in (1.20) is the same for all N, the statistic (cf. (1.26))

$$\tilde{T}_N = \begin{cases} \dfrac{1}{\sigma_N} \displaystyle\sum_{j=1}^{k} j \tilde{A}_j^{(N)} & \sigma_N > 0, \\ 0 & \text{otherwise} \end{cases} \tag{1.28}$$

converges to $N(0,1)$ in distribution as $N \to \infty$.

In accordance with the previous theorem the null hypothesis H_0 is rejected in favor of the alternative H_1 of the increasing treatment effect whenever the statistic $\tilde{T}_N$ from (1.28) exceeds the $(1-\alpha)$th quantile of the normal $N(0,1)$ distribution.

The results obtained in the proofs of the previous theorems can be used in constructing tests of the null hypothesis (1.8) against the general alternative of its negation. In the wording of the next theorem by the upper left submatrix we understand the matrix described by (2.1).

Theorem 1.3. (I) Suppose that for each $j \neq s$ belonging to $\{1, \dots, k\}$ there exists i such that $n_{ij} n_{is} \tilde{\sigma}_i^2 > 0$. Then the rank of upper left $(k-1)\times(k-1)$ submatrix $\tilde{\mathbf{\Sigma}}_{N[11]}$ of the matrix $\tilde{\mathbf{\Sigma}}_N$

$$rank(\tilde{\mathbf{\Sigma}}_{N[11]}) = rank(\tilde{\mathbf{\Sigma}}_N) = k-1, \tag{1.29}$$

the matrix

$$\tilde{\mathbf{\Sigma}}_N^* = \begin{pmatrix} \left(\tilde{\mathbf{\Sigma}}_{N[1,1]}\right)^{-1} & \mathbf{0}_{k-1\times 1} \\ \mathbf{0}_{1\times k-1} & 0 \end{pmatrix} \tag{1.30}$$

is the generalized inverse of the matrix $\tilde{\mathbf{\Sigma}}_N$ and the equality

$$\tilde{\mathbf{A}}^{(N)\prime} \tilde{\mathbf{\Sigma}}_N^- \tilde{\mathbf{A}}^{(N)} = \tilde{\mathbf{A}}^{(N)\prime} \tilde{\mathbf{\Sigma}}_N^* \tilde{\mathbf{A}}^{(N)} \tag{1.31}$$

holds for any generalized inverse $\tilde{\mathbf{\Sigma}}_N^-$ of the matrix $\tilde{\mathbf{\Sigma}}_N$.

(II) Define the test statistic by the formula

$$\tilde{Q}_N = \frac{1}{\sqrt{N}} (\tilde{A}_1^{(N)}, \dots, \tilde{A}_{k-1}^{(N)}) \Big(\frac{1}{N} \tilde{\mathbf{\Sigma}}_{N[1,1]}\Big)^{-1} \frac{1}{\sqrt{N}} (\tilde{A}_1^{(N)}, \dots, \tilde{A}_{k-1}^{(N)})' \tag{1.32}$$

provided that (1.29) is true and put $\tilde{Q}_N = 0$ otherwise. Let the null hypothesis (1.8) and the conditions (C1) – (C3), (C5) hold. Suppose further that either (C4*) and (1.27) hold or (C4) is fulfilled. Then $\tilde{Q}_N$ converges to χ^2_{k-1} distribution with $k-1$ degrees of freedom in distribution as $N \to \infty$. Moreover, if (C4*) holds with $\delta = 1$ and for every $j \neq s$ belonging to $\{1, \ldots, k\}$ the limit (cf. (1.5))

$$p_{js} = p \tag{1.33}$$

does not depend on $j \neq s$, then

$$\lim_{N\to\infty} P\Big(\max_{j\neq s} \frac{\sqrt{2}\,|\tilde{A}_j^{(N)} - \tilde{A}_s^{(N)}|}{\sqrt{m_{jj} + m_{ss} + 2m_{js}}} > t(k, 1-\alpha)\Big) = \alpha. \tag{1.34}$$

In this formula $t(k, 1-\alpha)$ denotes $(1-\alpha)$th quantile of the modulus of the normal $N_k(\mathbf{0}, \mathbf{I}_k)$ distribution, i. e.,

$$P\Big(\max_{i,j} |x_i - x_j| < t(k, 1-\alpha)\,\Big|\,\mathbf{x} \sim N_k(\mathbf{0}, \mathbf{I}_k)\Big) = 1 - \alpha, \tag{1.35}$$

the quantity m_{js} is for $j \neq s$ defined in (1.4), and for $j = 1, \ldots, k$

$$m_{jj} = \sum_{i=1}^{N} n_{ij}(d_{i.} - n_{ij}). \tag{1.36}$$

In accordance with the previous theorem the null hypothesis (1.8) is rejected whenever the statistic $\tilde{Q}_N$ exceeds the $(1-\alpha)$th quantile of the chi-square distribution with $k-1$ degrees of freedom. If this test rejects the null hypothesis (1.8) and the numbers $\{\frac{m_{js}}{N};\ 1 \leq j < s \leq k\}$ are not strikingly different, then the setting of the experiment can be perceived as a member of the limiting set-up in which (1.33) holds, and therefore in accordance with (1.34) the treatments j and s are declared to be different if

$$\frac{\sqrt{2}\,|\tilde{A}_j^{(N)} - \tilde{A}_s^{(N)}|}{\sqrt{m_{jj} + m_{ss} + 2m_{js}}} > t,\ \ t = t(k, 1-\alpha). \tag{1.37}$$

If the numbers $\{\frac{m_{js}}{N};\ 1 \leq j < s \leq k\}$ are perceived as strikingly different, then the critical constant t in (1.37) should be obtained by means of simulation.

Rank test statistics for mentioned hypotheses can be constructed also in a different way. Put

$$\hat{\mathbf{A}}_N = (\hat{A}_1^{(N)}, \ldots, \hat{A}_k^{(N)})',\ \ \hat{A}_r^{(N)} = \sum_{i=1}^{N} \hat{B}_r^{(i)},\ \ \hat{B}_r^{(i)} = \Big(S_{ir} - n_{ir}\frac{d_{i.}+1}{2}\Big), \tag{1.38}$$

$$\hat{\mathbf{\Sigma}}_N = \sum_{i=1}^{N} \frac{\tilde{\sigma}_i^2}{d_{i.}} \mathbf{\Psi}_i, \tag{1.39}$$

where Ψ_i is the matrix (1.19).

Theorem 1.4. (I) Suppose that for each $j \neq s$ belonging to $\{1, \ldots, k\}$ there exists i such that $n_{ij} n_{is} \tilde{\sigma}_i^2 > 0$. Then the rank of upper left $(k-1) \times (k-1)$ submatrix $\hat{\boldsymbol{\Sigma}}_{N[11]}$ of the matrix $\hat{\boldsymbol{\Sigma}}_N$

$$rank(\hat{\boldsymbol{\Sigma}}_{N[11]}) = rank(\hat{\boldsymbol{\Sigma}}_N) = k - 1, \tag{1.40}$$

the matrix

$$\hat{\boldsymbol{\Sigma}}_N^* = \begin{pmatrix} \left(\hat{\boldsymbol{\Sigma}}_{N[1,1]}\right)^{-1} & \mathbf{0}_{k-1\times 1} \\ \mathbf{0}_{1\times k-1} & 0 \end{pmatrix} \tag{1.41}$$

is the generalized inverse of the matrix $\hat{\boldsymbol{\Sigma}}_N$ and the equality

$$\hat{\mathbf{A}}^{(N)\prime} \hat{\boldsymbol{\Sigma}}_N^- \hat{\mathbf{A}}^{(N)} = \hat{\mathbf{A}}^{(N)\prime} \hat{\boldsymbol{\Sigma}}_N^* \hat{\mathbf{A}}^{(N)} \tag{1.42}$$

holds for any generalized inverse $\hat{\boldsymbol{\Sigma}}_N^-$ of the matrix $\hat{\boldsymbol{\Sigma}}_N$. Moreover, with the notation (1.15) the number $\mathbf{f}_k' \hat{\boldsymbol{\Sigma}}_N \mathbf{f}_k$ is positive.

(II) Define the test statistics by the formulas

$$\hat{T}_N = \frac{1}{\sqrt{\mathbf{f}_k' \hat{\boldsymbol{\Sigma}}_N \mathbf{f}_k}} \sum_{j=1}^k j \hat{A}_j^{(N)}, \tag{1.43}$$

$$\hat{Q}_N = \frac{1}{\sqrt{N}} (\hat{A}_1^{(N)}, \ldots, \hat{A}_{k-1}^{(N)}) \left(\frac{1}{N} \hat{\boldsymbol{\Sigma}}_{N[1,1]}\right)^{-1} \frac{1}{\sqrt{N}} (\hat{A}_1^{(N)}, \ldots, \hat{A}_{k-1}^{(N)})', \tag{1.44}$$

provided that (1.40) is true and put $\hat{T}_N = \hat{Q}_N = 0$ otherwise. Let the null hypothesis (1.8) and the conditions (C1)–(C3), (C4*), (C5) hold. If $\sup\{d_{i.}; i = 1, 2, \ldots\} < +\infty$, then the test statistic $\hat{T}_N$ converges to $N(0, 1)$ and $\hat{Q}_N$ converges to χ^2_{k-1} distribution with $k - 1$ degrees of freedom in distribution as $N \to \infty$.

The hat test statistics $\hat{T}_N$, $\hat{Q}_N$ from the previous theorem are constructed for using in tests in the same way as their tilde counterparts $\tilde{T}_N$, $\tilde{Q}_N$.

Now the power of the mentioned tests is illustrated by means of simulation estimates. Let us consider the random block scheme of $k = 5$ treatments with the following number of observations per cell.

Block	Number of observations				
i	n_{i1}	n_{i2}	n_{i3}	n_{i4}	n_{i5}
1	1	2	5	0	0
2	1	2	0	0	9
3	0	1	4	7	0
4	0	1	3	0	8
5	1	0	0	7	8
6	0	2	2	8	6
7	2	0	4	5	6
8	2	3	0	6	8
9	2	1	6	0	6
10	3	2	4	7	0

Suppose that this scheme of treatments holds, the effect of the jth treatment is $N(\mu_j, 1)$ distributed for the block index $i = 1, \dots, 7$ and $E(\mu_j, 1)$ distributed for $i = 8, 9, 10$, where $E(\mu_j, 1)$ denotes the exponential distribution with the density $exp(-(x-\mu_j))$, $x \geq \mu_j$. The simulations are carried out for 6 possible configurations of the these parameters μ_j, $j = 1, \dots, k$, which are described in the following table (obviously the configuration (I) means that the null hypothesis (1.8) holds).

	μ_1	μ_2	μ_3	μ_4	μ_5
(I)	2	2	2	2	2
(II)	2	2.1	2.2	2.2	2.2
(III)	2	2.1	2.2	2.2	2.3
(IV)	2	2.5	2.6	2.6	2.6
(V)	2	2.5	2.6	2.7	2.7
(VI)	2	2.5	2.6	2.8	2.9

Simulation estimates of the power given in the following table are in each case based on 10000 trials. For typographical reasons the notation $P_\alpha(\tilde{T}) = P(\tilde{T}_N > u_{1-\alpha})$, $P_\alpha(\hat{T}) = P(\hat{T}_N > u_{1-\alpha})$, $P_\alpha(\tilde{Q}) = P(\tilde{Q}_N > \chi^2_{k-1}(1-\alpha))$, $P_\alpha(\hat{Q}) = P(\hat{Q}_N > \chi^2_{k-1}(1-\alpha))$ is used, where $u_{1-\alpha}$ and $\chi^2_{k-1}(1-\alpha)$ is $(1-\alpha)$th quantile of the $N(0,1)$ distribution and of the chi-square distribution with $k-1$ degrees of freedom, respectively.

		$P_\alpha(\tilde{T})$	$P_\alpha(\hat{T})$	$P_\alpha(\tilde{Q})$	$P_\alpha(\hat{Q})$
(I)	$\alpha = 0.05$	0.049	0.049	0.044	0.043
	$\alpha = 0.10$	0.095	0.095	0.096	0.097
(II)	$\alpha = 0.05$	0.185	0.185	0.092	0.092
	$\alpha = 0.10$	0.293	0.291	0.170	0.169
(III)	$\alpha = 0.05$	0.311	0.313	0.134	0.133
	$\alpha = 0.10$	0.449	0.445	0.229	0.230
(IV)	$\alpha = 0.05$	0.493	0.494	0.374	0.372
	$\alpha = 0.10$	0.633	0.632	0.500	0.499
(V)	$\alpha = 0.05$	0.678	0.679	0.472	0.470
	$\alpha = 0.10$	0.795	0.791	0.599	0.600
(VI)	$\alpha = 0.05$	0.902	0.902	0.682	0.678
	$\alpha = 0.10$	0.950	0.947	0.794	0.790

Similar results hold when the treatment effects have the Cauchy or exponential or binomial distribution, in the sense that the values of the probabilities of rejection will be different but mutual power ordering remains the same. Thus the simulation results suggest that the tests based on the tilde and the hat variants of the statistics are practically equivalent.

When the quadratic statistics are considered, the use of the $\tilde{Q}_N$ test has the advantage that in the case of the rejection of H_0 the multiple comparisons procedure, based on (1.37) can be used. While the statistics $\tilde{Q}_N$, $\hat{Q}_N$ are constructed for testing the hypothesis H_0 of no treatment effect against the general alternative of its negation, the test statistics $\tilde{T}_N$, $\hat{T}_N$ are constructed for testing H_0 against the alternative H_1 of increasing treatment effect. The results of simulations support this construction and suggest that for testing H_0 against this special alternative H_1 the test based on $\tilde{T}_N$ should be used, because under validity of H_1 it has clearly better power than the test based on $\tilde{Q}_N$.

Now we are going to illustrate in this setting the performance of the multiple comparisons rule based on (1.37). In the following table $P_\alpha(+)$, $P_\alpha(-)$ are simulation estimates of the correct and false detection of different treatments, respectively. Thus $P_\alpha(+)$ denotes the estimate of the probability that (1.37) holds for at least one pair j, s of the indices such that $\mu_j \neq \mu_s$, $P_\alpha(-)$ is the estimate of the probability that (1.37) holds for at least one pair j, s of the indices such that $\mu_j = \mu_s$. The estimates are based on 10000 trials in each case.

		$P_\alpha(+)$	$P_\alpha(-)$
(I)	$\alpha = 0.05$	0	0.043
	$\alpha = 0.10$	0	0.092
(II)	$\alpha = 0.05$	0.072	0.015
	$\alpha = 0.10$	0.134	0.035
(III)	$\alpha = 0.05$	0.113	0.005
	$\alpha = 0.10$	0.197	0.013
(IV)	$\alpha = 0.05$	0.306	0.014
	$\alpha = 0.10$	0.436	0.033
(V)	$\alpha = 0.05$	0.385	0.006
	$\alpha = 0.10$	0.521	0.013
(VI)	$\alpha = 0.05$	0.584	0
	$\alpha = 0.10$	0.714	0

The results of simulations suggest that the detection of different treatments by means of (1.37) is trustworthy, because under the validity of alternative the probability of the false detection is fractional when compared to the probability of the the correct detection.

The following example is based on artificial data.

Example. A fertilizer producing company decided to verify the effectiveness of its products in agricultural practice and asked wheat producing farmers in 10 areas to apply some of its 5 products. However, not each of the addressed farmers was willing to take part in this experiment and the company received the following data on the yield of wheat.

	The yield of wheat for particular fertilizer				
Area	Type 1	Type 2	Type 3	Type 4	Type 5
1	28.5	35.1 31.6	29.8 30.8 30.1 32.7 30.7		
2	31.6	32.1 30.2			30.7 31.3 29.4 30.5 30.8 29.0 32.9 28.7 29.6
3		28.4	32.8 29.2 27.2 32.3	33.9 32.2 31.3 29.3 29.5 33.5 33.3	
4		29.5	32.3 31.1 30.0		29.3 31.6 27.4 27.9 35.0 30.9 36.3 30.5

	The yield of wheat for particular fertilizer				
Area	Type 1	Type 2	Type 3	Type 4	Type 5
5	30.3			32.8 30.6 30.0 33.9 29.2 32.6 29.9	32.5 32.7 29.3 32.1 33.1 28.9 31.7 30.9
6		28.2 30.1	30.4 28.5	27.1 29.2 31.0 27.6 33.5 29.8 26.9 31.2	34.0 31.1 31.3 30.9 31.4 32.8
7	30.4 31.5		31.6 31.0 31.3 28.7	29.1 30.9 31.7 29.2 32.6	31.2 36.5 31.8 28.4 31.9 32.0
8	30.2 31.0	32.9 32.2 31.6		34.5 31.3 31.1 31.9 38.1 31.2	32.0 31.7 35.8 32.7 33.0 34.6 31.5 31.4
9	31.6 30.1	34.5	31.0 31.2 32.3 31.7 30.8 31.5		33.8 33.1 34.3 32.4 32.8 32.5
10	31.0 35.3 32.3	31.6 34.2	31.7 33.5 31.8 32.1	32.5 32.2 31.1 33.1 32.7 33.4 32.8	

The fertilizers were labelled from 1 to 5 in such a way that the producer has reasons to assume that their effectiveness is increasing with the increasing label of the type. The average yield in various areas may differ because of the soil and weather conditions, but if the fertilizers have equal effect on the yield of wheat, then the distribution of the yield in the particular area does not depend on the order in which the fields are numbered, and the condition (1.8) is fulfilled. Thus the decision whether the fertilizers do not have the same effect can be based on the testing by means of the statistic (1.17).

The ranks (1.9) of the observations are as follows.

Area	Type 1	Type 2	Type 3	Type 4	Type 5
1	1	8 6	2 5 3 7 4		
2	10	11 5			7 9 3 6 8 2 12 1 4
3		2	9 3 1 8	12 7 6 4 5 11 10	
4		4	10 8 5		3 9 1 2 11 7 12 6
5	6			14 7 5 16 2 12 4	11 13 3 10 15 1 9 8
6		4 8	9 5	2 6 11 3 17 7 1 13	18 12 14 10 15 16
7	5 10		11 7 9 2	3 6 12 4 16	8 17 13 1 14 15
8	1 2	14 12 8		16 5 3 10 19 4	11 9 18 13 15 17 7 6
9	6 1	15	3 4 8 7 2 5		13 12 14 9 11 10
10	1 16 8	3 15	4 14 5 6	9 7 2 12 10 13 11	

There are no ties within the blocks and we shall use the test based on Theorem 1.1. The number of observations per cell is the same as in the simulation study. Thus the covariance matrix from (1.14) and the norming variance (1.16) (here $\mathbf{f}_k =$

$(1, 2, 3, 4, 5)')$

$$\mathbf{\Sigma}_N = \begin{pmatrix} 162 & -18 & -37 & -50 & -57 \\ -18 & 176 & -35 & -55 & -68 \\ -37 & -35 & 260 & -92 & -96 \\ -50 & -55 & -92 & 379 & -182 \\ -57 & -68 & -96 & -182 & 403 \end{pmatrix}, \qquad \sigma_N^2 = \mathbf{f}_k' \mathbf{\Sigma}_N \mathbf{f}_k = 3053.$$

The matrix of differences $(S_{ir} - n_{ir} * (d_{i.} + 1)/2)$ is

$$\begin{pmatrix} -3.5 & 5.0 & -1.5 & 0 & 0 \\ 3.5 & 3.0 & 0 & 0 & -6.5 \\ 0 & -4.5 & -5.0 & 9.5 & 0 \\ 0 & -2.5 & 3.5 & 0 & -1 \\ -2.5 & 0 & 0 & 0.5 & 2.0 \\ 0 & -7 & -5 & -16 & 28 \\ -3.0 & 0 & -7.0 & -4.0 & 14.0 \\ -17.0 & 4.0 & 0 & -3.0 & 16.0 \\ -9.0 & 7.0 & -19.0 & 0 & 21.0 \\ -0.5 & 1.0 & -5.0 & 4.5 & 0 \end{pmatrix}.$$

Therefore the observed value of the statistic (1.12) is

$$\mathbf{A}^{(N)} = \left(A_1^{(N)}, \ldots, A_k^{(N)}\right)' = \Big(-26.6111,\ 6.3681,\ -33.5176,\ -4.9771,\ 58.7377\Big)'$$

and the test statistic (1.17)

$$\tilde{T}_N = \frac{1}{\sqrt{3053}} \sum_{j=1}^{5} j A_j^{(N)} = 2.8840\,.$$

But the 95% quantile of $N(0,1)$ distribution $u_{0.95} = 1.644854$, hence $\tilde{T}_N > u_{0.95}$ and we reject at the significance level $\alpha = 0.05$ the hypothesis of the equality of the effectiveness of these types of fertilizers in favour of the alternative, that for $j = 1, 2, 3, 4$ the average yield μ_j obtained by application of the jth type $\mu_j \leq \mu_{j+1}$ and the inequality $\mu_j < \mu_{j+1}$ holds for some j.

For $\alpha = 0.05$ the equality (1.35) holds with $t(5, 0.95) = 3.8577$ and since for the quantity $\kappa_{js} = \sqrt{2}|A_j^{(N)} - A_s^{(N)}|/\sqrt{m_{jj} + m_{ss} + 2m_{js}}$ the inequality $\kappa_{js} > 3.8577$ holds only for the pairs $j = 1$, $s = 5$ and $j = 3$, $s = 5$, the method of multiple comparisons based on (1.37) detects as different the types 1 and 5 and the types 3 and 5.

2. PROOFS

Suppose that $\mathbf{V} = \left(v_{js}\right)_{j,s=1}^{k}$ is a $k \times k$ matrix and

$$\mathbf{V}_{[11]} = \left(v_{js}\right)_{j,s=1}^{k-1} \tag{2.1}$$

denotes its left upper $(k-1)\times(k-1)$ submatrix. In the proofs of this paper the following Lemma will be useful. We remark that its assertions (I) and (II) have already been used in [8].

Lemma 2.1. Let us assume that $\mathbf{V}$ is a symmetric positive semidefinite $k\times k$ matrix and the sum of its columns is zero vector.

(I) Suppose that all off-diagonal elements of $\mathbf{V}$ are negative numbers. Then

$$rank(\mathbf{V}) = rank(\mathbf{V}_{[11]}) = k-1. \tag{2.2}$$

(II) If (2.2) holds, then the matrix

$$\mathbf{V}^* = \begin{pmatrix} \left(\mathbf{V}_{[11]}\right)^{-1} & \mathbf{0}_{k-1\times 1} \\ \mathbf{0}_{1\times k-1} & 0 \end{pmatrix} \tag{2.3}$$

is the generalized inverse of the matrix $\mathbf{V}$. Moreover, if $\mathbf{x}\in\mathbb{R}^k$ and the sum of its coordinates is zero, then the equality

$$\mathbf{x}'\mathbf{V}^-\mathbf{x} = \mathbf{x}'\mathbf{V}^*\mathbf{x} = (x_1,\ldots,x_{k-1})\left(\mathbf{V}_{[11]}\right)^{-1}(x_1,\ldots,x_{k-1})' \tag{2.4}$$

holds for any generalized inverse $\mathbf{V}^-$ of the matrix $\mathbf{V}$.

(III) If (2.2) is fulfilled then for the vector (1.15) the inequality

$$\mathbf{f}_k'\mathbf{V}\mathbf{f}_k > 0 \tag{2.5}$$

holds.

P r o o f. (III) Since the matrix $\mathbf{V}$ is symmetric and positive semidefinite, there exists a random vector $\boldsymbol{\xi} = (\xi_1,\ldots,\xi_k)'$ having the normal $N_k(\mathbf{0},\mathbf{V})$ distribution. Suppose that

$$\mathbf{f}_k'\mathbf{V}\mathbf{f}_k = 0. \tag{2.6}$$

Then obviously almost surely

$$\sum_{j=1}^k j\xi_j = 0. \tag{2.7}$$

On the other hand, since the sum of the columns of the matrix $\mathbf{V}$ is the zero vector,

$$\sum_{j=1}^k \xi_j = 0 \tag{2.8}$$

almost surely. Combining (2.7) and (2.8) one obtains that almost surely

$$\sum_{j=1}^k \xi_j = \frac{1}{k}\sum_{j=1}^k j\xi_j, \quad \sum_{j=1}^{k-1}(k-j)\xi_j = 0. \tag{2.9}$$

Put $\mathbf{g} = (k-1, k-2, \ldots, 1)'$. Then (2.9) means that the equality $\mathbf{g}'\mathbf{V}_{[11]}\mathbf{g} = 0$ holds. But this is a contradiction, because $\mathbf{g}$ is a non-zero vector and by (2.2) the matrix $\mathbf{V}_{[11]}$ is regular. Thus (2.6) cannot happen and since $\mathbf{V}$ is positive semidefinite, the inequality (2.5) holds. □

Proof of Theorem 1.1. (I) Obviously

$$\mathbf{A}^{(N)} = \sum_{i=1}^{N} \mathbf{B}_i, \qquad \mathbf{B}_i = \sqrt{\frac{12}{d_{i.}+1}} \left(S_{i1} - n_{i1}\frac{d_{i.}+1}{2}, \ldots, S_{ik} - n_{ik}\frac{d_{i.}+1}{2} \right)'. \quad (2.10)$$

Since the null hypothesis (1.8) holds, by (C4) the random vector $\mathbf{R}_i$ is uniformly distributed over the set $\mathcal{R}^{(d_{i.})}$ of all permutations of $\{1, \ldots, d_{i.}\}$. Therefore making use of Theorem 3 and Theorem 4 from Section 3.3 of Hájek, Šidák and Sen (1999) one obtains the formula

$$Var(\mathbf{B}_i) = \mathbf{\Psi}_i, \quad (2.11)$$

where $\mathbf{\Psi}_i$ is the matrix (1.19). The independence of the blocks implies the formula for $\mathbf{\Sigma}_N$.

Since the sum of coordinates of $\mathbf{A}^{(N)}$ is zero vector, the sum of columns of its covariance matrix $\mathbf{\Sigma}_N$ is also zero vector. If all the numbers m_{js} defined in (1.4) are positive, then $\mathbf{\Sigma}_N$ is a symmetric positive semidefinite matrix with negative off-diagonal elements and according to the Lemma 2.1 the number (1.16) is positive.

(II) Since (C5) and (1.14) hold and the sum of columns of $\mathbf{\Sigma}_N$ is zero vector, there exists a finite limit

$$\mathbf{W} = \lim_{N\to\infty} \frac{\mathbf{\Sigma}_N}{N}. \quad (2.12)$$

Now we are going to prove that

$$\frac{\mathbf{A}^{(N)}}{\sqrt{N}} \to N_k(\mathbf{0}, \mathbf{W}) \quad (2.13)$$

in distribution as $N \to \infty$. Taking into account (2.10), the independence of random blocks and (2.12), one obtains the convergence

$$\sum_{i=1}^{N} Var\left(\frac{\mathbf{B}_i}{\sqrt{N}}\right) = \frac{\mathbf{\Sigma}_N}{N} \to \mathbf{W}.$$

Since $E(\mathbf{B}_i) = \mathbf{0}$ and (2.10) holds, according to the multivariate Lindeberg theorem from p. 390 of [6] the convergence (2.13) will be proved by showing that for every fixed $\varepsilon > 0$

$$\lim_{N\to\infty} \sum_{i=1}^{N} \int_{\{\|\mathbf{x}\|>\varepsilon\}} \|\mathbf{x}\|^2 \mathrm{d}F_{i,N}(\mathbf{x}) = 0,$$

where $F_{i,N}$ denotes the distribution function of the random vector $\mathbf{B}_i/\sqrt{N}$. This condition is equivalent to

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \int\limits_{\{\|\mathbf{B}_i\|>\sqrt{N}\varepsilon\}} \|\mathbf{B}_i\|^2 \mathrm{d}P = 0\,. \tag{2.14}$$

It is obvious that there exists a positive constant K such that $\|\mathbf{B}_i\| \le K d_{i.}^{3/2}$. Hence if $\varepsilon > 0$, then by (1.7) the set $\{\|\mathbf{B}_i\| > \sqrt{N}\varepsilon\}$ is empty for $i = 1, \dots, N$ and all N sufficiently large. Thus (2.14) holds and (2.13) is proved.

As we have already mentioned, the matrix $\boldsymbol{\Sigma}_N$ is symmetric, p.s.d. and the sum of its columns is zero vector. This together with (2.12), (1.14), the condition (C5) and Lemma 2.1 means that

$$\mathbf{f}_k' \mathbf{W} \mathbf{f}_k > 0. \tag{2.15}$$

But $\mathbf{f}_k' \mathbf{A}^{(N)}/\sqrt{N} \to N(0, \mathbf{f}_k' \mathbf{W} \mathbf{f}_k)$ in distribution by (2.13) and $\mathbf{f_k}'(\boldsymbol{\Sigma}_\mathbf{N}/N)\mathbf{f_k} \to \mathbf{f_k}'\mathbf{W}\mathbf{f_k}$ by (2.12), which together with (2.15) and

$$T_N = \frac{1}{\sqrt{\mathbf{f}_k'(\boldsymbol{\Sigma}_N/N)\mathbf{f}_k}} \frac{\mathbf{f}_k' \mathbf{A}^{(N)}}{\sqrt{N}}$$

means that the assertion (II) of Theorem 1.1 holds. □

The proof of Theorem 1.2 will be based on the following lemma.

Lemma 2.2. Suppose that the conditions (C1) – (C3), (C4*) are fulfilled and the null hypothesis (1.8) holds. Let $\tau_N = (\tau_1^{(N)}, \dots, \tau_N^{(N)})$ be such a vector that $P(\tau(\mathbf{X}_i) = \tau_i^{(N)}) > 0$, $i = 1, \dots, N$.

(I) The conditional expectations and the conditional covariance matrices (cf. (1.23))

$$E\Big(\tilde{\mathbf{B}}_i \,|\, \tau(\mathbf{X}_i) = \tau_i^{(N)}\Big) = \mathbf{0}, \quad Var\Big(\tilde{\mathbf{B}}_i \,|\, \tau(\mathbf{X}_i) = \tau_i^{(N)}\Big) = (1 - \delta_{0,\tilde{\sigma}_i})\boldsymbol{\Psi}_i\,, \tag{2.16}$$

where Ψ_i is the matrix (1.19). Therefore (1.25) holds.

(II) Suppose that also the condition (C5) is fulfilled. Let $\mathcal{M}^+(k-1)$ denote the set of all symmetric positive definite $(k-1) \times (k-1)$ matrices and (cf. (1.24))

$$\tilde{\mathbf{W}}_N = \frac{\tilde{\boldsymbol{\Sigma}}_N}{N}, \quad \overline{\mathbf{W}}_N = E(\tilde{\mathbf{W}}_N). \tag{2.17}$$

Then

$$\overline{\mathbf{W}}_N = \frac{1}{N} \sum_{i=1}^{N} \boldsymbol{\Psi}_i \gamma_i, \quad \gamma_i = P(\tilde{\sigma}_i^2 > 0) \tag{2.18}$$

and there exists a compact subset $\mathcal{K} \subset \mathcal{M}^+(k-1)$ and an index N_0 such that for all integers $N \ge N_0$ and the upper left $(k-1) \times (k-1)$ submatrix of the matrix $\overline{\mathbf{W}}_N$ the inclusion

$$\overline{\mathbf{W}}_{N[11]} \in \mathcal{K} \tag{2.19}$$

holds.

P r o o f. (I) Let the number $\tilde{\sigma}_i^2$ be positive. Since it is constant on the set $\{\tau(\mathbf{X}_i) = \tau_i^{(N)}\}$, taking into account (1.10) one obtains by the results from Section 4 of [1]

$$E\Big(\tilde{B}_r^{(i)} \,|\, \tau(\mathbf{X}_i) = \tau_i^{(N)}\Big) = \sqrt{\frac{d_{i.}}{\tilde{\sigma}_i^2}} \sum_{v=1}^{n_{ir}} E\Big(R_{ij}^{(v)} - \frac{d_{i.}+1}{2}\Big) = 0\,,$$

which proves the first formula in (2.16), the second one can be proved similarly.

If the sets $D_i \in \mathcal{B}^k$, $i = 1, \ldots, N$, then

$$P\Big(\tilde{\mathbf{B}}_i \in D_i\,,\ i = 1, \ldots, N \,\Big|\, \tau(\mathbf{X}_1, \ldots, \mathbf{X}_N) = \tau_N\Big)$$

$$= \prod_{i=1}^{N} P\bigg(\Big(S_{i1} - n_{i1}\frac{d_{i.}+1}{2}, \ldots, S_{ik} - n_{ik}\frac{d_{i.}+1}{2}\Big)' \in \sqrt{\frac{\tilde{\sigma}_i^2}{d_{i.}}} D_i \,\Big|\, \tau(\mathbf{X}_i) = \tau_i^{(N)}\bigg)$$

$$= \prod_{i=1}^{N} P\Big(\tilde{\mathbf{B}}_i \in D_i \Big|\, \tau(\mathbf{X}_1, \ldots, \mathbf{X}_N) = \tau_N\Big) \tag{2.20}$$

which together with (2.16) and (1.22) yields (1.25).

(II) The equality (2.18) follows from (2.17) and (1.25). To prove (2.19) put

$$\mathbf{\Sigma}_N = \sum_{i=1}^{N} \mathbf{\Psi}_i.$$

Since the sum of the columns of this matrix is zero vector, the validity of the condition (C5) and Lemma 2.1 imply that there exists a finite limit

$$\mathbf{W} = \lim_{N\to\infty} \frac{\mathbf{\Sigma}_N}{N} \tag{2.21}$$

and the equality

$$\mathit{rank}(\mathbf{W}) = \mathit{rank}(\mathbf{W}_{[11]}) = k - 1 \tag{2.22}$$

holds. Let $\mathbf{x} \in \mathbb{R}^k$. Then

$$\mathbf{x}'\overline{\mathbf{W}}_N\mathbf{x} = \frac{1}{N}\sum_{i=1}^{N} \mathbf{x}'\mathbf{\Psi}_i\mathbf{x}\gamma_i.$$

Since (1.20) holds, there exist positive numbers δ_1, δ_2 such that $\delta_1 < \gamma_i < \delta_2$ for all $i = 1, 2, \ldots$. However by (2.11) the matrix $\mathbf{\Psi}_i$ is positive semidefinite and therefore

$$\delta_2\mathbf{x}'\Big(\frac{1}{N}\mathbf{\Sigma}_N\Big)\mathbf{x} \geq \mathbf{x}'\overline{\mathbf{W}}_N\mathbf{x} \geq \delta_1\mathbf{x}'\Big(\frac{1}{N}\mathbf{\Sigma}_N\Big)\mathbf{x}. \tag{2.23}$$

But if $\mathcal{G} \subset \mathcal{M}^+(k-1)$ is a compact set, then the set

$$\mathcal{K} = \{\mathbf{V} \in \mathcal{M}^+(k-1);\ \text{there exists a } \mathbf{W}^* \in \mathcal{G} \text{ such that } \delta_2\mathbf{W}^* \gg \mathbf{V} \gg \delta_1\mathbf{W}^*\} \tag{2.24}$$

is also compact. But by (2.21) and (2.22) the matrix $\mathbf{W}_{[11]}$ is positive definite and there exist a compact set $\mathcal{G} \subset \mathcal{M}^+(k-1)$ and a positive integer N_0 such that

$$\frac{\mathbf{\Sigma}_{N[11]}}{N} \in \mathcal{G}$$

for all integers $N \geq N_0$. This together with (2.23) and (2.24) means that the lemma is true. □

P r o o f o f T h e o r e m 1.2. (I) Equality (1.25) follows from Lemma 2.2(I). Hence if for each $j \neq s$ belonging to $\{1, \ldots, k\}$ there exists i such that $n_{ij}n_{is}(1 - \delta_{0,\tilde{\sigma}_i}) > 0$, then all off-diagonal elements of the matrix $\tilde{\mathbf{\Sigma}}_N$ are negative and as this matrix is p.s.d and the sum of its columns is zero vector, the number (1.26) is positive by Lemma 2.1.

(II) Let $\{N_v\}_{v=1}^{\infty}$ be an increasing sequence of positive integers. Taking into account the previous lemma and the fact that the sum of columns of the matrix $\overline{\mathbf{W}}_N$ from (2.17) is zero vector, we see, that there exists its subsequence $\{N_{v_u}\}_{u=1}^{\infty}$ such that

$$\overline{\mathbf{W}}_{N_{v_u}} \to \overline{\mathbf{W}}, \tag{2.25}$$

$\overline{\mathbf{W}}$ is a symmetric p.s.d. matrix, the sum of its columns is zero vector and its upper left submatrix $\overline{\mathbf{W}}_{[11]}$ is regular. This together with Lemma 2.1 means, that

$$\mathbf{f}_k'\overline{\mathbf{W}}\mathbf{f}_k > 0. \tag{2.26}$$

But for the matrix $\tilde{\mathbf{W}}_N$ from (2.17) owing to the independence of $\tilde{\sigma}_1^2, \ldots, \tilde{\sigma}_N^2$ we have (the notation (j,s) denotes the position in the matrix)

$$\begin{aligned} var(\tilde{\mathbf{W}}_N(j,s)) &= \frac{1}{N^2}\, var(\tilde{\mathbf{\Sigma}}_N(j,s)) = \frac{1}{N^2}\sum_{i=1}^{N} \mathbf{\Psi}_i^2(j,s) var(\delta_{0,\tilde{\sigma}_i^2}) \\ &\leq \frac{1}{N^2}\sum_{i=1}^{N} d_{i.}^4 \leq \frac{D_N^4}{N} \to 0 \end{aligned}$$

because (1.27) holds. This together with Tchebychev's inequality implies that the difference $\tilde{\mathbf{W}}_N - \overline{\mathbf{W}}_N \to 0$ in probability as $N \to \infty$. But any sequence converging to zero in probability contains a subsequence converging to zero almost surely. Thus neglecting a set of probability 0 we see that there exists a subsequence $\{N^*\}$ of the sequence $\{N_{v_u}\}_{u=1}^{\infty}$ such that

$$\tilde{\mathbf{W}}_{N^*} \to \overline{\mathbf{W}} \tag{2.27}$$

as $N^* \to \infty$. Hence if we prove the convergence in distribution

$$\tilde{\mathbf{A}}^{(N^*)}/\sqrt{N^*} \to N_k(\mathbf{0}, \overline{\mathbf{W}}), \tag{2.28}$$

then by means of (2.26), (2.27) one can easily prove the convergence in distribution

$$\frac{1}{\sqrt{\mathbf{f}_k'\tilde{\mathbf{W}}_{N^*}\mathbf{f}_k}}\sum_{j=1}^{k} j\tilde{A}_j^{(N^*)} \to N(0,1),$$

which means that the convergence in distribution $\tilde{T}_N \to N(0,1)$ holds.

To prove (2.28) put

$$C_i = \{(z_1, z_2, \dots, z_{d_{i.}})' \in \mathbb{R}^{d_{i.}};\ \text{there exist } j,\ \tilde{j} \text{ such that } z_j \neq z_{\tilde{j}}\}. \tag{2.29}$$

According to the assumptions $P((\mathbf{X}_{i1}, \dots, \mathbf{X}_{ik_i}) \in C_i) \geq \delta > 0$. Hence for the set

$$H_N = \{(\mathbf{x}_1, \dots, \mathbf{x}_N) \in \mathbb{R}^{d_{1.}} \times \dots \times \mathbb{R}^{d_{N.}};\ \frac{1}{N}\sum_{i=1}^{N} \chi_{C_i}(\mathbf{x}_i) > \frac{\delta}{2}\},$$

by the strong law of large numbers

$$\lim_{N\to\infty} \chi_{H_N}((\mathbf{X}_1, \dots, \mathbf{X}_N)) = 1 \tag{2.30}$$

almost surely. Let $F_N(\mathbf{t}|\tau(\mathbf{X}_1, \dots, \mathbf{X}_N) = \tau_N)$ denotes the conditional distribution function of the random vector $\mathbf{A}^{(N)}/\sqrt{N}$ given $\tau(\mathbf{X}_1, \dots, \mathbf{X}_N) = \tau_N$. Put

$$G_N = \Big\{\tau_N;\ \tau_N = \tau(\mathbf{X}_1, \dots, \mathbf{X}_N),\ (\mathbf{X}_1, \dots, \mathbf{X}_N) \in H_N,\ P(\tau(\mathbf{X}_1, \dots, \mathbf{X}_N) = \tau_N) > 0\Big\}.$$

Let $\mathbf{t} \in \mathbb{R}^k$ be an arbitrary fixed vector. If we prove that for each $\tau_{N^*} \in G_{N^*}$

$$\lim_{N^*\to\infty} F_{N^*}(\mathbf{t}|\tau(\mathbf{X}_1, \dots, \mathbf{X}_{N^*}) = \tau_{N^*}) = F(\mathbf{t}), \tag{2.31}$$

where F denotes the distribution function of the normal $N_k(\mathbf{0}, \overline{\mathbf{W}})$ distribution, then for the quantity

$$\Delta_{N^*}(\tau_{N^*}) = \Big|F_{N^*}(\mathbf{t}\,|\,\tau(\mathbf{X}_1, \dots, \mathbf{X}_{N^*}) = \tau_{N^*}) - F(\mathbf{t})\Big|$$

almost surely

$$\lim_{N^*\to\infty} \Delta_{N^*}\Big(\tau(\mathbf{X}_1, \dots, \mathbf{X}_{N^*})\Big)\chi_{H_{N^*}}(\mathbf{X}_1, \dots, \mathbf{X}_{N^*}) = 0 \tag{2.32}$$

(the events $\{(\mathbf{X}_1, \dots \mathbf{X}_N \in H_N,\ \tau(\mathbf{X}_1, \dots, \mathbf{X}_N) \in G_N\}$ and $\{(\mathbf{X}_1, \dots, \mathbf{X}_N \in H_N\}$ have the same probability). Thus by (2.30) for distribution function $F_N(\mathbf{t})$ of the random vector $\tilde{\mathbf{A}}^{(N)}/\sqrt{N}$ one obtains that

$$|\, F_{N^*}(\mathbf{t}) - F(\mathbf{t})\,| \leq \int_{H_{N^*}} \Delta_{N^*}\Big(\tau(\mathbf{x}_1, \dots, \mathbf{x}_{N^*})\Big)\, \mathrm{d}P^{\infty}(x^{\infty}) + o(1) \to 0 \tag{2.33}$$

because (2.32) and the Lebesgue theorem hold. Convergence (2.33) means that (2.28) is proved.

Hence it is sufficient to prove (2.31). Suppose that $\tau_{N^*} = (\tau_1^{(N^*)}, \dots, \tau_{N^*}^{(N^*)}) \in G_{N^*}$. According to (2.20) the random vectors $\tilde{\mathbf{B}}_1, \dots, \tilde{\mathbf{B}}_{N^*}$ are independent under the probability $P\Big(.\Big|\, \tau(\mathbf{X}_1,\ldots,\mathbf{X}_{N^*}) = \tau_{N^*}\Big)$ and by (1.22)

$$\sum_{i=1}^{N^*} Var\left(\frac{\tilde{\mathbf{B}}_i}{\sqrt{N^*}} \,|\, \tau(\mathbf{X}_1,\ldots,\mathbf{X}_{N^*}) = \tau_{N^*}\right)$$
$$= \quad Var\left(\frac{\tilde{\mathbf{A}}^{(N^*)}}{\sqrt{N^*}} \,|\, \tau(\mathbf{X}_1,\ldots,\mathbf{X}_{N^*}) = \tau_{N^*}\right) \to \overline{\mathbf{W}}$$

because the convergence (2.27) holds. Hence the multivariate Lindeberg theorem implies that (2.31) will be proved by verifying for every $\varepsilon > 0$ the equality

$$\lim_{N^*\to\infty} \sum_{i=1}^{N^*} \int_{\{\|\mathbf{x}\|>\varepsilon\}} \|\mathbf{x}\|^2 \mathrm{d}F_{i,N^*}(\mathbf{x}|\, \tau(\mathbf{X}_1,\ldots,\mathbf{X}_{N^*}) = \tau_{N^*}) = 0\,,$$

where $F_{i,N^*}(\mathbf{x}|\, \tau(\mathbf{X}_1,\ldots,\mathbf{X}_{N^*}) = \tau_{N^*})$ stands for the conditional distribution function of the random vector $\tilde{\mathbf{B}}_i/\sqrt{N^*}$. Obviously it is sufficient to prove the equality

$$\lim_{N^*\to\infty} \frac{1}{N^*} \sum_{i=1}^{N^*} \int_{\{\|\tilde{\mathbf{B}}_\mathbf{i}\|>\sqrt{N^*}\varepsilon\}} \|\tilde{\mathbf{B}}_i\|^2 \,\mathrm{d}P(\omega|\, \tau(\mathbf{X}_1,\ldots,\mathbf{X}_{N^*}) = \tau_{N^*}) = 0\,, \tag{2.34}$$

where $P(\cdot|\, \tau(\mathbf{X}_1,\ldots,\mathbf{X}_{N^*}) = \tau_{N^*})$ denotes this conditional probability. Since the function $g(x) = x^2$ is convex, for $n_{ir} = d_{ij} > 0$

$$\Big(S_{ir} - n_{ir}\frac{d_{i.}+1}{2}\Big)^2 \le n_{ir} \sum_{v=1}^{n_{ir}} \Big(R_{ij}^{(v)} - \frac{d_{i.}+1}{2}\Big)^2 \le d_{i.}^2 \tilde{\sigma}_i^2 .$$

Thus there exists a positive number M such that $\|\tilde{\mathbf{B}}_i\| \le M d_{i.}^{3/2}$ for all i. This together with (1.27) means that the set $\{\|\tilde{\mathbf{B}}_i\| > \sqrt{N^*}\varepsilon\}$ is empty if the integer N^* is sufficiently large, and the convergence (2.34) is proved. □

P r o o f o f T h e o r e m 1.3. (I) This assertion is an immediate consequence of Theorem 1.2(I) and Lemma 2.1.

(II) Since the model based on (C4) can be handled similarly, we shall prove the assertion for the case with tied observations. The convergence in distribution $\tilde{Q}_N \to \chi^2_{k-1}$ can be proved by means of (2.27) and (2.28), because the matrix $\overline{\mathbf{W}}$ is symmetric, p.s.d., the sum of its columns is zero vector, its upper left submatrix $\overline{\mathbf{W}}_{[11]}$ is regular and Lemma 2.1 holds.

Now we are going to prove (1.34). Making use of the assumptions of the theorem and of the fact that the sum of the columns of $\tilde{\mathbf{\Sigma}}_N$ is zero vector, one can prove the convergence

$$\frac{1}{N}\tilde{\mathbf{\Sigma}}_N \to \overline{\mathbf{W}} = kp\mathbf{V}, \quad \mathbf{V} = \mathbf{I}_k - \sqrt{\mathbf{c}}(\sqrt{\mathbf{c}})', \;\; (\sqrt{\mathbf{c}})' = (\frac{1}{\sqrt{k}},\ldots,\frac{1}{\sqrt{k}}), \tag{2.35}$$

where p is the number (1.33). Thus by (2.28)

$$\tilde{\mathbf{A}}^{(N)}/\sqrt{N} \to N_k(\mathbf{0}, \overline{\mathbf{W}}), \tag{2.36}$$

and therefore

$$\tilde{\mathbf{A}}^{(N)}/\sqrt{N} = \mathcal{O}_P(1). \tag{2.37}$$

But (1.33) holds and the sum of columns of $\tilde{\mathbf{\Sigma}}_N$ is zero vector. Therefore

$$\frac{m_{jj}}{N} = (k-1)p + o(1)$$

which together with (2.37), (2.36) and (2.35) means, that

$$\max_{j \neq s} \frac{\sqrt{2}\,|\tilde{A}_j^{(N)} - \tilde{A}_s^{(N)}|}{\sqrt{m_{jj} + m_{ss} + 2m_{js}}} = \max_{j \neq s} \frac{|\tilde{A}_j^{(N)} - \tilde{A}_s^{(N)}|}{\sqrt{Nkp}} + o_P(1) \longrightarrow \max_{1 \leq j < s \leq k} |x_j - x_s| \tag{2.38}$$

in distribution, where the random vector $(x_1, \ldots, x_k)'$ is $N_k(\mathbf{0}, \mathbf{V})$ distributed. Taking into account (2.35) it is easy to see that the normally distributed random vector $(x_j - x_s; 1 \leq j < s \leq k)'$ has the same covariance matrix as the random vector $(y_j - y_s; 1 \leq j < s \leq k)'$, where $(y_1, \ldots, y_k)'$ is $N_k(\mathbf{0}, I_k)$ distributed. This together with (2.38) implies (1.34). □

Proof of Theorem 1.4. (I) This assertion is an immediate consequence of Lemma 2.1.

(II) Since the model based on (C4) can be handled similarly, we shall prove the assertion for the case with tied observations. By (2.20) and (2.16)

$$Var\Big(\hat{\mathbf{A}}_N \,|\, \tau(\mathbf{X}_i) = \tau_i^{(N)} \Big) = \hat{\mathbf{\Sigma}}_N$$

is the matrix (1.39). Thus for

$$\hat{\mathbf{W}}_N = \frac{\hat{\mathbf{\Sigma}}_N}{N}, \qquad \overline{\mathbf{W}}_N = E(\hat{\mathbf{W}}_N)$$

the equality

$$\overline{\mathbf{W}}_N = \frac{1}{N}\sum_{i=1}^{N} \mathbf{\Psi}_i \gamma_i, \qquad \gamma_i = \frac{1}{d_{i.}} E(\tilde{\sigma}_i^2) \tag{2.39}$$

holds. Hence similarly as in the proof of Lemma 2.2 one can prove that there exist a compact subset $\mathcal{K} \subset \mathcal{M}^+(k-1)$ and an index N_0 such that for all integers $N \geq N_0$ the upper left $(k-1) \times (k-1)$ submatrix $\overline{\mathbf{W}}_{N[11]} \in \mathcal{K}$. Since

$$Var(\hat{\mathbf{W}}_N(j,s)) = \frac{1}{N^2}\, var(\hat{\mathbf{\Sigma}}_N(j,s)) = \frac{1}{N^2} \sum_{i=1}^{N} \frac{1}{d_i^2} \mathbf{\Psi}_i^2(j,s) Var(\tilde{\sigma}_i^2) \leq \frac{K}{N} \to 0,$$

the convergence of hat statistics in distribution can be proved similarly as the convergence in distribution of their tilde counterparts. □

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