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OBSERVABLES ON σ-MV ALGEBRAS AND σ-LATTICE EFFECT ALGEBRAS

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Effect algebras were introduced as abstract models of the set of quantum effects which represent sharp and unsharp properties of physical systems and play a basic role in the foundations of quantum mechanics. In the present paper, observables on lattice ordered σ-effect algebras and their "smearings" with respect to (weak) Markov kernels are studied. It is shown that the range of any observable is contained in a block, which is a σ-MV algebra, and every observable is defined by a smearing of a sharp observable, which is obtained from generalized Loomis–Sikorski theorem for σ-MV algebras. Generalized observables with the range in the set of sharp real observables are studied and it is shown that they contain all smearings of observables.

Keywords: lattice effect algebra, MV algebra, observable, state, Markov kernel, weak Markov kernel, smearing, generalized observable

Classification: 81P10, 81P15, 03G12

1. INTRODUCTION

Effect algebras [9] (equivalently, D-posets, [18], or weak orthoalgebras, [11]) were introduced as abstract models of the set of quantum effects (self-adjoint operators between the zero and identity operator in the usual ordering). Quantum effects represent sharp and unsharp properties of physical systems and play a basic role in the foundations of quantum mechanics [1]. They contain the usual quantum logics (orthomodular posets and lattices) as special subclasses. Also MV-algebras, introduced by Chang [5] as algebraic bases for many-valued logic, are a special subclass of effect algebras. In this paper, we consider lattice ordered effect algebras, which are a common generalization of MV-algebras and orthomodular lattices.

An *effect algebra* is an algebraic structure $(E; \oplus, 0, 1)$ where $\oplus$ is a partial binary operation and 0 and 1 are constants, such that for every $a, b, c \in E$, the following axioms hold:

(E1) $a \oplus b = b \oplus a$;

(E2) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$;

(E3) for every $a \in E$ there is a unique $a^\perp$ such that $a \oplus a^\perp = 1$;

(E4) if $a \oplus 1$ is defined, then $a = 0$.

The equalities in (E1) and (E2) mean that if one side is defined, so is the other and the equality holds. We write $a \perp b$ and say that a and b are *orthogonal* if $a \oplus b$ is defined. If we write $a \oplus b$, we tacitly assume $a \perp b$.

A partial order on E is defined by the relation $a \leq b$ iff there is c such that $a \oplus c = b$. If such element c exists, it is unique, and we write $c =: b \ominus a$. Thus $b \ominus a$ is defined and equals c iff $a \leq b$ and $a \oplus c = b$. The element $a^\perp$ in (E3) is called the *orthosupplement* of a, and we have $a^\perp = 1 \ominus a$. It can be shown that $a \perp b$ iff $a \leq b^\perp$ and $(a \oplus b)^\perp = a^\perp \ominus b$. Moreover we have $0 \leq a \leq 1$ for all $a \in E$.

The orthogonality relation can be naturally extended as follows. Any finite set of elements $a_1, a_2, \dots, a_n$ (not necessarily all different) are said to be *orthogonal* if $a_1 \oplus a_2 \oplus \cdots \oplus a_n$ exists, where the latter $\oplus$-sum is defined recurrently. Owing to (E1) and (E2), we can omit parentheses, and the resulting sum does not depend on the order of its summands. More generally, an arbitrary system $(a_i)_{i\in I}$ is said to be *orthogonal* if every its finite subsystem is orthogonal. If the element $\bigoplus_{i\in I} a_i := \bigvee_{F\subseteq I} \oplus_{i\in F} a_i$ exists, where the supremum on the right is taken over all finite subsystems $F \subseteq I$, we call it the *orthosum* of the system $(a_i)_{i\in I}$. An effect algebra E is *orthocomplete* if every orthogonal family admits an orthosum, and E is *σ-orthocomplete* if every countable orthogonal family admits an orthosum. Equivalently, E is orthocomplete if every ascending family has a supremum in E and E is σ-orthocomplete if every countable ascending family has a supremum in E.

A *subalgebra* of an effect algebra E is a subset $F \subseteq E$ such that $1 \in F$, $a \in F$ implies $a^\perp \in F$, and $a, b \in F$, $a \perp b$ implies $a \oplus b \in F$.

An effect algebra E which is lattice ordered with respect to its ordering, is called a *lattice effect algebra*. It is easy to see that a lattice effect algebra is σ-orthocomplete (orthocomplete) iff it is a σ-lattice (complete lattice).

Two elements a, b in an effect algebra E are said to be *compatible* (in Mackey's sense, written $a \leftrightarrow b$) if there are orthogonal elements a_1, b_1, c such that $a = a_1 \oplus c, b = b_1 \oplus c$. In a lattice effect algebra, compatibility is equivalent to the condition $(a \vee b) \ominus b = a \ominus (a \wedge b)$ [6].

We recall that an *MV-algebra* [4, 5, 8] is an algebraic structure $(M; \dot{+}, ^*, 0)$ with a binary operation $\dot{+}$[1], a unary operation * and a constant 0 such that $\dot{+}$ is commutative and associative with neutral element 0, $a \dot{+} a^* = 0^* =: 1$, $a \dot{+} 1 = 1$, $(a^*)^* = a$, and $(a^* \dot{+} b)^* \dot{+} b = (b^* \dot{+} a)^* \dot{+} a$, the last axiom is called the *Lukasziewicz* axiom. With the operations $a \vee b = (a^* \dot{+} b)^* \dot{+} b$, $a \wedge b = (a^* \vee b^*)^*$, M is a distributive lattice with 0 as the smallest and $0^* = 1$ as the greatest element. MV-algebras were introduced in [5] as algebraic bases for many-valued logic. In [19], categorical equivalence between unit intervals in abelian ℓ-groups with strong unit and MV-algebras was shown, as well as their importance in the K-theory of AF C*-algebras.

If E is a lattice effect algebra such that $a \leftrightarrow b$ for all $a, b \in E$, then $a \dot{+} b := (a \wedge b^\perp) \oplus b$ is a total binary operation such that $(E; \dot{+}, ^\perp, 0)$ is an MV-algebra [6]. Conversely, if in an MV-algebra $(M; \dot{+}, ^*, 0)$ we restrict the $\dot{+}$ operation to $\{(a, b) : a \leq b^*\}$, we obtain a lattice effect algebra in which any pair of elements is compatible. An effect algebra with the latter properties is called an *MV-effect alge-*

[1]Our notation differs from the one in [4, 5], as we reserved the symbol $\oplus$ for the operation in effect algebras

bra. Equivalently, MV-effect algebras can be characterized as lattice effect algebras satisfying Riesz decomposition properties, [8, Corollary 1.10.11]. MV-effect algebras are in one-to-one correspondence with MV-algebras [6].

An MV-algebra M is a *σ-MV algebra* if M is a σ-lattice.

It is well known that an orthomodular lattice can be covered by its blocks (i. e., maximal sets of pairwise compatible elements) which are boolean algebras. An analogous statement has been proved in [28] for lattice effect algebras: every lattice effect algebra E can be covered by its blocks, which are subalgebras and sublattices of E, and are MV-effect algebras in their own right. If E is a σ-orthocomplete lattice effect algebra, then blocks are σ-orthocomplete MV-effect algebras (equivalently, σ-MV algebras) [14].

An element a in an effect algebra E is *sharp* if $a \wedge a^{\perp} = 0$. The set of all sharp elements in a lattice effect algebra forms an orthomodular lattice (OML), which is a subalgebra and sublattice $Sh(E)$ of E [14]. If E is σ-orthocomplete, then $Sh(E)$ is a σ-OML.

A *state* on an effect algebra E is a mapping $m : E \mapsto [0,1]$ such that $m(a \oplus b) = m(a)+m(b)$ whenever $a \perp b$ and $m(1) = 1$. A state m is *σ-additive* if $m(a_n) \to m(a)$ whenever $a_n \uparrow a$. A state is *pure* if it is an extreme point in the convex set of states. We say that a state m on E is *faithful* if $m(a) = 0$ iff $a = 0$, $a \in E$. Let $\mathcal{S}(E)$ denote the set of all states on E, and $\mathcal{S}_\sigma(E)$ denote the set of all σ-additive states on E.

2. LOOMIS–SIKORSKI THEOREM FOR σ-MV EFFECT ALGEBRAS

For the proof of the Loomis–Sikorski theorem for σ-MV-algebras, see [7, 20] (see also [3] for a different approach). In this paragraph, we briefly recall some basic facts that are used in the proof of this theorem.

The following notion is a direct generalization of a σ-algebra of sets. A *tribe of fuzzy sets* on a set $X \neq \emptyset$ is a nonempty system $\mathcal{T} \subseteq [0,1]^\Omega$ such that

(T1) $1_X \in \mathcal{T}$;

(T2) if $a \in \mathcal{T}$ then $1_X - a \in \mathcal{T}$;

(T3) $(a_n)_{n=1}^\infty \subseteq \mathcal{T}$ entails that the pointwise minimum

$$\min\Big(\sum_{n=1}^{\infty} a_n, 1\Big) \in \mathcal{T}.$$

Elements of $\mathcal{T}$ are called *fuzzy subsets* of X. Elements of $\mathcal{T}$ which are characteristic functions are called *crisp subsets* of X.

The basic properties of tribes are [8, Prop. 7.16]:

Proposition 2.1. Let $\mathcal{T}$ be a tribe of fuzzy subsets of X. Then

(i) $a \vee b = \max\{a, b\} \in \mathcal{T}$, $a \wedge b = \min\{a, b\} \in \mathcal{T}$;

(ii) $b - a \in \mathcal{T}$ if $a \leq b$, i. e. $a(x) \leq b(x)$ for all $x \in X$;

(iii) if $a_n \in \mathcal{T}$, $n \geq 1$, and $a_n \nearrow a$ (point-wise) then $a = \lim_n a_n \in \mathcal{T}$;

(iv) $\mathcal{T}$ is a σ-MV-algebra closed under point-wise suprema of sequences of its elements.

Denote by

$$Sh(\mathcal{T}) = \{A \subset X : \chi_A \in \mathcal{T}\},$$

i. e., $Sh(\mathcal{T})$ is the system of all crisp subsets in $\mathcal{T}$. According to [8, Th. 7.1.7], $Sh(\mathcal{T})$ is a σ-algebra of crisp subsets of X, and if $f \in \mathcal{T}$, then f is $Sh(\mathcal{T})$-measurable. That is, for every $f \in \mathcal{T}$ and every $B \in \mathcal{B}([0,1])$ (where $\mathcal{B}([0,1])$ denotes the Borel subsets of $[0,1]$), the pre-image $f^{-1}(B)$ belongs to $Sh(\mathcal{T})$. Moreover, the mapping

$$f^{-1} : \mathcal{B}([0,1]) \to Sh(\mathcal{T})$$

is a σ-homomorphism of Boolean σ-algebras.

Lemma 2.2. Let $\mathcal{T}$ be a tribe of fuzzy subsets of a set $X \neq \emptyset$. For every $f, g \in \mathcal{T}$, $f = g$ if and only if $f^{-1}(B) = g^{-1}(B)$ for all $B \in \mathcal{B}([0,1])$.

Proof. For the nontrivial direction, if $f \neq g$, there is $x \in X$ such that $f(x) \neq g(x)$. Assume $f(x) < g(x)$, then $g(x) > f(x) + \frac{1}{n}$ for some integer n. Putting $f(x) = \alpha$, we have $x \in f^{-1}[0,\alpha]$, while $x \notin g^{-1}[0,\alpha]$. □

A state on a tribe $\mathcal{T}$ is a mapping $m : \mathcal{T} \to [0,1]$ such that $m(1) = 1$, and $m(f+g) = m(f) + m(g)$ whenever $f, g \in \mathcal{T}$ with $f + g \leq 1$. A state m is σ-additive if for any nondecreasing sequence $(f_n)_n \subseteq \mathcal{T}$, if $f_n \nearrow f$ (point-wise), then $m(f_n) \to m(f)$.

By the Butnariu–Klement theorem [2], [27, Th. 8.1.12], [3] for every σ-additive state m on $\mathcal{T}$ we have

$$m(f) = \int_X f(x)\, \mathrm{d}\mu(x), \tag{1}$$

where μ is a probability measure on the σ-algebra $Sh(\mathcal{T})$ given by $\mu(A) = m(\chi_A)$, $A \in Sh(\mathcal{T})$.

Let $(M; \dot{+}, ^*, 0)$ be an MV-algebra. Recall that an element $a \in M$ is *idempotent* iff $a \oplus a = a$, and a is sharp iff $a \wedge a^* = 0$. It is well-known that idempotent and sharp elements in an MV-algebra coincide. By [5], for every MV-algebra, the set of idempotent elements $Sh(M)$ is a Boolean algebra. If M is a σ-MV-algebra, then $Sh(M)$ is a Boolean σ-algebra. Moreover, all countable suprema taken in $Sh(M)$ coincide with those taken in M [8, Th. 7.1.12].

On the set $\mathcal{M}(M)$ of all maximal ideals of M a topology τ_M is introduced as the collection of all subsets of the form

$$O(I) := \{A \in \mathcal{M}(M) : A \not\supseteq I\}, I \text{ is an ideal of } M.$$

For any $a \in M$, we put

$$M(a) := \{A \in \mathcal{M}(M) : a \notin A\}.$$

It is easy to see that for $a, b \in M$, (i) $M(0) = \emptyset$, (ii) $M(a) \subseteq M(b)$ whenever $a \leq b$, (iii) $M(a \wedge b) = M(a) \cap M(b)$, $M(a \vee b) = M(a) \cup M(b)$. Moreover, $\{M(a) : a \in M\}$ is a base of τ_M.

In addition, we have the following facts ([8, Prop. 7.1.13]): (i) $M(a)^c \subseteq M(a^*)$ ($M(a)^c$ is the set-theoretical complement of $M(a)$), (ii) if a is idempotent then $M(a)^c = M(a^*)$, (iii) if M is semisimple (in particular, if M is a σ-MV-algebra), then $M(a)^c = M(a^*)$ iff a is idempotent.

Denote by $Ext(\mathcal{S}(M))$ the set of all extremal states on M. Then by [21, Theorem 2.5], $Ext(\mathcal{S}(M)) \neq \emptyset$ and it is a compact Hausdorff space with respect to the weak topology of states (i. e., $m_\alpha \to m$ iff $m_\alpha(a) \to m(a)$ for all $a \in M$), and any state m on M is in the closure of the convex hull of $Ext(\mathcal{S}(M))$.

From [12, Theorem 15.32] and Mundici's ℓ-group representation of MV-algebras we have that there is a one-to-one correspondence between $Ext(\mathcal{S}(M))$ and $\mathcal{M}(M)$ given by the homeomorphism $m \mapsto Ker_m$ (see also [8, Th. 7.1.2]).

It follows that τ_M makes $\mathcal{M}(M)$ a compact Hausdorff topological space. Let M be a σ-MV-algebra (that is, M is a σ-lattice). With the topology τ_M, the space $X := \mathcal{M}(M)$ is basically disconnected (that is, the closure of every F_σ-subset of X is open) [8, Proposition 7.1.15].

For $a \in M$, define $a \mapsto \hat{a}$, where $\hat{a} \in [0, 1]^{Ext(\mathcal{S}(M))}$ by

$$\hat{a}(m) := m(a),\ m \in Ext(\mathcal{S}(M)).$$

Notice that by [8, Prop. 7.1.20], $a \in M$ is idempotent if and only if $\hat{a}$ is a characteristic function.

Let f be a real function on $X \neq \emptyset$. Define

$$N(f) := \{x \in X : |f(x)| > 0\}.$$

The following generalization of the Loomis–Sikorski theorem has been proved in [20] and, independently, in [7].

Theorem 2.3. For every σ-MV-algebra M there exist a tribe $\mathcal{T}$ of fuzzy sets and an MV-σ-homomorphism h from $\mathcal{T}$ onto M.

This can be briefly described as follows: Let M be a σ-MV-algebra. Let $\mathcal{T}$ be the tribe of fuzzy sets defined on $X := Ext(\mathcal{S}(M))$ generated by the set $\{\hat{a} : a \in M\}$. Denote by $\mathcal{T}'$ the class of all functions $f \in \mathcal{T}$ with the property that for some $b \in M$, $N(f - \hat{b})$ is a meager set. It can be shown that if for some b_1 and b_2 and $f \in \mathcal{T}'$ we have $N(f - \hat{b}_i)$ is a meager set for $i = 1, 2$, then $b_1 = b_2$. Moreover, $\mathcal{T}' = \mathcal{T}$. Due to the definition of $\mathcal{T}'$, for any $f \in \mathcal{T}$ there is a unique element $h(f) := b \in M$ such that $N(f - \hat{b})$ is meager.

Moreover, h maps $Sh(\mathcal{T})$ onto $Sh(M)$ and $h(f) = 0$ iff $h(\chi_{N(f)}) = 0$, [23, § 5]. The triple $(X, \mathcal{T}, h)$ ($X = Ext(\mathcal{S}(M))$), described in the preceding paragraph, is called the *standard Loomis–Sikorski representation* of M.

3. SHARP AND UNSHARP OBSERVABLES

Let $(\Omega, \mathcal{A})$ be a measurable space, where Ω is a nonempty set and $\mathcal{A}$ is a σ-algebra of subsets of Ω. In accordance with [27], a $(\Omega, \mathcal{A})$-*observable* on a σ-orthocomplete effect algebra E is a mapping $\xi : \mathcal{A} \to E$ such that (i) $\xi(\Omega) = 1$; (ii) $\xi(\bigcup_{i=1}^{\infty} A_i) = \bigoplus_{i=1}^{\infty} \xi(A_i)$ whenever $(A_i)_{i=1}^{\infty}$ is a sequence of mutually disjoint elements of $\mathcal{A}$. $(\Omega, \mathcal{A})$ is called the *value space* of the observable ξ. If m is a σ-additive state, then $m \circ \xi$ is a probability measure on the measurable space $(\Omega, \mathcal{A})$, which is called the *distribution* of the observable ξ in the state m. We will assume that $\mathcal{S}_\sigma(E) \neq \emptyset$.

A *real* observable is an observable with value space $(\mathbb{R}, B(\mathbb{R}))$, where $\mathbb{R}$ is the set of real numbers and $B(\mathbb{R})$ is the σ-algebra of Borel sets.

An observable is *sharp* if its range consists of sharp elements. Observables which are not sharp are called *unsharp*.

In what follows, E will denote a σ-orthocomplete lattice effect algebra.

Theorem 3.1. The range of every observable on E is contained in a block of E.

Proof. Let ξ be an $(\Omega, \mathcal{A})$-observable on E. Let $A, B \in \mathcal{A}$. Then $A \cap B, A \setminus (A \cap B), B \setminus (A \cap B)$ are pairwise orthogonal sets in $\mathcal{A}$, and from $A = (A \setminus (A \cap B)) \cup (A \cap B)$ we get $\xi(A) = \xi(A \setminus (A \cap B)) \oplus \xi(A \cap B)$, and similarly $\xi(B) = \xi(B \setminus (A \cap B)) \oplus \xi(A \cap B)$. Since $\xi(A \setminus (A \cap B)) \oplus \xi(B \setminus (A \cap B) \oplus \xi(A \cap B) = \xi(A \cup B)$, we see that $\xi(A), \xi(B)$ are compatible. Since A and B are arbitrary, the range of ξ consists of pairwise compatible elements of E. Consequently, the range of ξ is contained in block of E. □

Remark 3.2. Since by Theorem 3.1 the range of every observable ξ on E is contained in a block M of E, we may consider ξ as an observable on the σ-MV algebra M.

Since the restriction of a σ-additive state m on E to M is a σ-additive state on M, we have $\{m \circ \xi : m \in \mathcal{S}_\sigma(E)\} \subseteq \{m \circ \xi : m \in \mathcal{S}_\sigma(M)\}$. Similarly, the restriction of a faithful state on E to M is a faithful state on M.

Let $(Z, \mathcal{F})$ and $(Y, \mathcal{G})$ be measurable spaces. We recall that a mapping $\nu : Z \times \mathcal{G} \to [0, 1]$ is a *Markov kernel* (MK) (see e. g. [30]) if

(i) for any fixed $G \in \mathcal{G}$, the mapping $\nu_G(\cdot) := \nu(\cdot, G) : Z \to [0, 1]$ is $\mathcal{F}$-measurable;

(ii) for any fixed $z \in Z$, $\nu_z(\cdot) := \nu(z, .) : \mathcal{G} \to [0, 1]$ is a probability measure.

The notion of a Markov kernel can be weakened as follows (see [15]). Let $(Z, \mathcal{F})$ and $(Y, \mathcal{G})$ be measurable spaces. Let $\mathcal{P}$ be a family of probability measures on $(Z, \mathcal{F})$. We will say that $\nu : Z \times \mathcal{G} \to [0, 1]$ is a *weak Markov kernel (WMK) with respect to* $\mathcal{P}$ if

(i) $z \mapsto \nu(z, G)$ is $\mathcal{F}$-measurable;

(ii) for every $G \in \mathcal{G}$, $0 \leq \nu(z, G) \leq 1$ $\mathcal{P}$-a.e.;

(iii) $\nu(z, Y) = 1$ $\mathcal{P}$-a.e. and $\nu(z, \emptyset) = 0$ $\mathcal{P}$-a.e.;

(iv) if $\{G_n\}$ is a sequence in $\mathcal{G}$ such that $G_n \cap G_m = \emptyset$ whenever $n \neq m$, then

$$\nu(z, \bigcup_n G_n) = \sum_n \nu(z, G_n), \ \mathcal{P} - \text{a.e.}$$

Let $(Z, \mathcal{F})$ and $(Y, \mathcal{G})$ be measurable spaces, and let $\nu : Z \times \mathcal{G} \to [0,1]$ be a WMK w.r. $\mathcal{P}$. It is easy to see that the mapping

$$G \mapsto \int_Z \nu(z, G) P(\mathrm{d}z)$$

is a probability measure on $(Y, \mathcal{G})$ for every probability measure P in $\mathcal{P}$.

For a $(Z, \mathcal{F})$-observable ξ, with range in a block M, denote $\mathcal{P}(\xi) := \{m \circ \xi : m \in \mathcal{S}_\sigma(M)\}$. Then $\mathcal{P}(\xi)$ is a subset of the set of probability measures on $(Z, \mathcal{F})$.

Definition 3.3. Let $(Z, \mathcal{F})$ and $(Y, \mathcal{G})$ be measurable spaces, let ξ be $(Z, \mathcal{F})$-observable on E and let $\nu : Z \times \mathcal{G} \to [0,1]$ be a WMK with respect to $\mathcal{P}(\xi)$. The mapping ψ from $\mathcal{P}(\xi)$ to the set of probability measures on $(Y, \mathcal{G})$ defined by

$$\psi(m \circ \xi)(G) := \int_Z \nu(z, G) m \circ \xi(\mathrm{d}z)$$

will be called the *smearing* of the observable ξ with respect to ν.

If, in addition, there is an $(Y, \mathcal{G})$-observable η on E such that

$$m(\eta(G)) = \psi(m \circ \xi)(G), G \in \mathcal{G},$$

we say that η is *defined by the smearing of* ξ *with respect to* ν.

We note that sometimes the observable η is also called a smearing of ξ with respect to ν.

Theorem 3.4. Every observable on a σ-lattice effect algebra is defined by a smearing of a sharp observable.

P r o o f. Let ξ be an $(\Omega, \mathcal{A})$-observable on E. By Theorem 3.1, the range $\{\xi(A) : A \in \mathcal{A}\}$ is contained in a block M of E, and M is a σ-MV algebra. Let $(X, \mathcal{T}, h)$ be the standard LS-representation of M, where X is a nonempty set, $\mathcal{T}$ is a tribe of functions $f : X \to [0,1]$ and $h : \mathcal{T} \to M$ is a σ-MV algebra homomorphism which maps $\mathcal{T}$ onto M and maps the sharp elements $Sh(\mathcal{T})$ of $\mathcal{T}$ onto sharp elements $Sh(M)$ of M. For every $A \in \mathcal{A}$ there is an $f_A \in \mathcal{T}$ with $h(f_A) = \xi(A)$, where f_A is $Sh(\mathcal{T})$-measurable, and is unique up to h-null sets. Define $\nu : X \times \mathcal{A} \to [0,1]$ by $\nu(x, A) = f_A(x)$. It was proved in [15, Examples 3.3, 4.2], that $\nu(x, A)$ is a weak Markov kernel with respect to $\{m \circ h : m \in \mathcal{S}_\sigma(M)\}$. Owing to [2] we have

$$m(\xi(A)) = m(h(f_A)) = \int_X f_A(x) P(\mathrm{d}x),$$

where $P := m \circ h/Sh(\mathcal{T}) = m((h/Sh(\mathcal{T}))$. The restriction $h/Sh(\mathcal{T}) : Sh(\mathcal{T}) \to Sh(M)$ can be considered as a sharp $(X, Sh(\mathcal{T}))$-observable on M, hence also on E. By Definition 3.3, the observable ξ is defined by a smearing of the sharp observable $h/Sh(\mathcal{T})$. □

Let for every block M, $(X_M, \mathcal{T}_M, h_M)$ denote its standard LS representation. By Theorems 3.1 and 3.4, every observable ξ is defined by a smearing of $h_M/Sh(\mathcal{T}_M)$ for some block M. Observe that the functions $f_A \in \mathcal{T}_M$ such that $\xi(A) = h_M(f_A)$, $A \in \mathcal{A}$, are defined up to h_M-null sets. Thus we may say that $\nu(x, A) = f_A(x)$ is a WMK with respect to h_M, independently of states.

In what follows, we denote $h_M^R := h_M/Sh(\mathcal{T}_M)$, and we call h_M^R a *basic observable*. Next we show some conditions under which a smearing of a basic observable defines an observable.

Let M be a block of E, and let $(\Omega, \mathcal{A})$ be a measurable space. Assume that $\nu : X_M \times \mathcal{A} \to [0, 1]$ is a WMK with respect to $\mathcal{P}(h_M^R) = \{m \circ h_M^R : m \in \mathcal{S}_\sigma(M)\}$. Suppose further that $\nu_A \in \mathcal{T}_M$. Then for $m \in \mathcal{S}_\sigma(M)$,

$$\psi(m \circ h_M^R)(A) = \int_{X_M} \nu_A(x) m \circ h_M^R(\mathrm{d}x) = m(h_M(\nu_A))$$

Clearly, if the mapping $\eta : A \mapsto h_M(\nu_A)$ is an observable, then η is defined by a smearing of h_M^R and we write $\eta = \nu \circ h_M^R$.

Theorem 3.5. Let M be a block of E and $(X_M, \mathcal{T}_M, h_M)$ be its standard LS-representation. Let $(\Omega, \mathcal{A})$ be a measurable space and let $\nu : X_M \times \mathcal{A} \to [0, 1]$ be a mapping with the property that for all $A \in \mathcal{A}$, the function $\nu_A : X \to [0, 1]$ belongs to $\mathcal{T}_M$. Then:

(1) If ν is a Markov kernel, then $\eta : A \mapsto h_M(\nu_A)$ is an $(\Omega, \mathcal{A})$-observable with range in M.

(2) If ν is a weak MK with respect to $\mathcal{P}(h_M^R)$, and there is a faithful σ-additive state on M, then $\eta : A \mapsto h_M(\nu_A)$ is an $(\Omega, \mathcal{A})$-observable with range in M.

Proof. (1) Let ν be a MK. Since $\nu_A \in \mathcal{T}$, we have, by the BK-theorem, that for every $m \in \mathcal{S}_\sigma(M)$,

$$m \circ h_M(\nu_A) = \int_X \nu_A(x) m \circ h_M(\mathrm{d}x).$$

Define $\eta(A) := h_M(\nu_A) \in M$, $A \in \mathcal{A}$. Since ν is MK, we have $\eta(\Omega) = h_M(\nu_\Omega) = h(1) = 1$. If (A_i) is any sequence of pairwise disjoint sets in $\mathcal{A}$ with $A = \bigcup_{i=1}^\infty A_i$, then $\nu_A = \sum_{i=1}^\infty \nu_{A_i}$ (for all $x \in X$), whence $\sum_{i=1}^n \nu_{A_i} \le \nu_A \le 1$, which entails that $\dot{+}_{i=1}^n \nu_{A_i} = \min(\sum_{i=1}^n \nu_{A_i}, 1) = \sum_{i=1}^n \nu_{A_i}$, so that ν_{A_i} are orthogonal and, since h_M is a σ-MV algebra morphism, we obtain $h_M(\nu_A) = h_M(\sum_{i=1}^\infty \nu_{A_i}) = h_M(\lim_{n\to\infty} \sum_{i=1}^n \nu_{A_i}) = \bigvee_{n=1}^\infty h_M(\sum_{i=1}^n \nu_{A_i}) = \bigvee_{n=1}^\infty \oplus_{i=1}^n h_M(\nu_{A_i}) = \oplus_{i=1}^\infty h_M(\nu_{A_i})$. Hence $\eta(A) = \oplus_{i=1}^\infty \eta(A_i)$, which proves that η is an observable. By the definition, $\eta = \nu \circ h_M^R$.

(2) Let ν be a weak MK, and let m_0 be a faithful state in $\mathcal{S}_\sigma(M)$. Let $a \in M$, then $a = h_M(f_a)$ for some $f_a \in \mathcal{T}_M$, and $m_0(a) = m_0(h_M(f_a))$, hence $m_0(a) = 0$

iff $a = h_M(f_a) = 0$. Moreover, for any $a \in M$, $m_0(a) = 0$ implies $m(a) = 0$ for all $m \in \mathcal{S}_\sigma(M)$. It follows that ν is a weak MK with respect to $\{m \circ h_M^R : m \in \mathcal{S}_\sigma(M)\}$ iff ν is a weak MK with respect to $m_0 \circ h_M^R$. Since ν is a WMK, ν_A is $Sh(\mathcal{T})$-measurable function for every $A \in \mathcal{A}$, and by supposition, $\nu_A \in \mathcal{T}$. Therefore $h_M(\nu_A) \in M$. We have to prove that $A \mapsto h_M(\nu_A)$ is an observable. Let $(A_n)_{n=1}^\infty \subseteq \mathcal{A}$, $A_n \cap A_m = \emptyset$, $m \neq n$, $A = \bigcup_n A_n$. Since ν is a WMK, $\nu_A = \sum_n \nu_{A_n}$ a.e. $\{m \circ h_M : m \in \mathcal{S}_\sigma(M)\}$. It follows that for all $n \in \mathbb{N}$, $\sum_{i=1}^n \nu_{A_i} \leq \nu_A$ a.e. $m_0 \circ h_M$, and therefore $h_M(\{x : \sum_{i=1}^n \nu_{A_i}(x) \not\leq 1\}) = 0$. Put $g_1 = \nu_{A_1}$, $g_i = \nu_{A_i} \wedge (g_1 + \cdots + g_{i-1})^\perp$, $i = 2, \ldots, n$. Then $g_i \in \mathcal{T}$, g_i, $i = 1, \ldots, n$ are pairwise orthogonal, and $\nu_{A_i} = g_i$ a.e. $m_0 \circ h_M$. It follows that $h_M(\dot{+}_{i=1}^n \nu_{A_i}) = h_M(\min(\sum_{i=1}^n \nu_{A_i}, 1) = h_M(\sum_{i=1}^n g_i) = \oplus_{i=1}^n h_M(g_i) = \oplus_{i=1}^n h_M(\nu_{A_i})$. Now $\nu_A = \lim_n \sum_{i=1}^n \nu_{A_i}$ a.e. $m_0 \circ h$, so that $h_M(\nu_A) = h_M(\lim_n \sum_{i=1}^n g_i) = \oplus_{i=1}^\infty h_M(g_i) = \oplus_{i=1}^\infty h_M(\nu_{A_i})$. Hence $\eta(A) = \oplus_{i=1}^\infty \eta(A_i)$, and η is an observable. □

Notice that if $(\Omega, \mathcal{A})$ is a standard Borel space (in particular, if $(\Omega, \mathcal{A}) \equiv (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, then under conditions (2), there is a Markov kernel ν^* such that, for every $A \in \mathcal{A}$, $\nu(x, A) = \nu^*(x, A)$ a.e. $m_0 \circ h_M^R$ (see, e. g., [29, pp. 338–340], [26, Appendix] or [30, Theorem 6.11]).

In the next theorem, a joint distribution of real observables on a σ-MV algebra is constructed by means of Markov kernels. The method applies also to observables on lattice σ-effect algebras, since the range of an observable is contained in a block, which is a σ-MV algebra.

Notice that a standard LS representation need not contain all $Sh(\mathcal{T})$-measurable functions. Indeed, by [8, Theorem 7.1.7 (3)], a tribe $\mathcal{T}$ contains all $Sh(\mathcal{T})$-measurable functions iff $\mathcal{T}$ contains all constant $[0, 1]$-valued functions.

Theorem 3.6. Let M be a σ-MV algebra with a faithful state $m_0 \in \mathcal{S}_\sigma(M)$ which has a standard LS-representation $(X, \mathcal{T}, h)$ such that $\mathcal{T}$ contains all $Sh(\mathcal{T})$ -measurable functions. Let ξ, η be two real observables on M. Then there is an observable ρ with value space $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$ such that $\rho(B \times \mathbb{R}) = \xi(B)$ and $\rho(\mathbb{R} \times B) = \eta(B)$ for all $B \in \mathcal{B}(\mathbb{R})$.

Proof. Since the observables are real, and there is a faithful state, we may deal with MKs instead of WMKs. Let $\mu : X \times \mathcal{B}(\mathbb{R}) \to [0, 1]$, $\nu : X \times \mathcal{B}(\mathbb{R}) \to [0, 1]$ be MKs such that $m(\xi(A)) = \int_X \mu(x, A) m(h^R(\mathrm{d}x))$ and $m(\eta(B)) = \int_X \nu(x, B) m(h^R(\mathrm{d}x))$, $A, B \in \mathcal{B}(\mathbb{R})$, $m \in \mathcal{S}_\sigma(E)$. For any fixed $x \in X$, $\mu(x, .)$ and $\nu(x, .)$ are probability measures on $\mathcal{B}(\mathbb{R})$. Define $\lambda(x, A \times B) = \mu(x, A)\nu(x, B)$, then $\lambda(x, .)$ extends to a probability measure on $\mathcal{B}(\mathbb{R}^2)$ for every $x \in X$.

Moreover, for every fixed $D \in \mathcal{B}(\mathbb{R}^2)$, $\lambda_D := \lambda(\cdot, D)$ is $Sh(\mathcal{T})$-measurable. Indeed, the family $\{D \in \mathcal{B}(\mathbb{R}^2) : \lambda_D \text{ is measurable}\}$ is a monotone system and contains the algebra generated by rectangle sets. Hence λ is a Markov kernel.

By our assumptions, $\lambda_D \in \mathcal{T}$ for every D. Define $\rho(D) = h(\lambda_D)$, by Theorem 3.5, ρ is an observable on M with the desired properties. □

Let $\xi : (\Omega, \mathcal{A}) \to E$ be a sharp observable with the range $\mathcal{R}(\xi)$. Then there is a block M such that $\mathcal{R}(\xi)$ is contained in the Boolean σ-algebra $Sh(M)$ of sharp elements of M. Since $Sh(M)$ is the range of the basic observable h^R_M, we have $\mathcal{R}(\xi) \subseteq \mathcal{R}(h^R_M)$, whence by [31, Theorem 1.4], there is a measurable function $f : X \to \Omega$ such that $\xi(A) = h^R_M \circ f^{-1}(A)$, $A \in \mathcal{A}$ (where $f^{-1}(A) \in Sh(\mathcal{T}_M)$, owing to measurability). In other words, ξ is the function f of h^R_M. For every $m \in \mathcal{S}_\sigma(M)$ we then have $m(\xi(A)) = \int_X \chi_{f^{-1}(A)}(x) m \circ h^R_M(\mathrm{d}x)$. If we put $\lambda(x, A) := \chi_{f^{-1}(A)}(x)$, then λ is a MK such that $\xi = \lambda \circ h^R_M$, and for every $A \in \mathcal{A}$, $\lambda(x, A) \in \{0, 1\}$. This proves the following statement:

Theorem 3.7. An observable ξ from $(\Omega, \mathcal{A})$ to E is sharp if and only if there is a block M of E and a measurable function $f : X_M \to \Omega$ such that $\xi(A) = h^R_M \circ f^{-1}(A)$ for every $A \in \mathcal{A}$. Moreover, ξ is defined by a smearing of h^R_M with respect to the MK $\lambda(x, A) := \chi_{f^{-1}}(A)(x)$.

The method of the proof of the following theorem is a modification of the last part of the proof of [16, Theorem 3.3].

Theorem 3.8. Let m_0 be a faithful σ-additive state on E. Let $\xi : (\Omega, \mathcal{A}) \to E$, $\eta : (\Sigma, \mathcal{B}) \to E$ be observables, and let ξ be defined by a smearing of η with respect to a WMK ν with values in $\{0, 1\}$. If $(\Omega, \mathcal{A})$ is a standard Borel space, then there is a measurable function $f : \Sigma \to \Omega$ such that for every state m and every $A \in \mathcal{A}$, $m(\xi(A)) = m(\eta \circ f^{-1}(A))$.

Proof. Since $(\Omega, \mathcal{A})$ is standard Borel space, by [26, Appendix], there is a MK $\lambda : \Sigma \times \mathcal{A} \to [0, 1]$ such that, for every $A \in \mathcal{A}$, $\lambda(y, A) \in \{0, 1\}$ modulo $m_0 \circ \eta$. Put $\pi(A) := \{y \in \Sigma : \lambda(y, A) = 1\}$, $A \in \mathcal{A}$. Then $\pi(A) \cap \pi(A^c) = \emptyset$, since $\lambda(y, .)$ is a probability measure. This yields a partition $\Sigma = \pi(A) \cup \pi(A^c) \cup C_A$, where $\eta(C_A) = 0$ by the assumptions. Then, for every state m,

$$m(\xi(A)) = \int_{\pi(A)} \lambda(y, A) m(\eta(\mathrm{d}y)) + \int_{\pi(A^c)} \lambda(y, A) m(\eta(\mathrm{d}y)) + \int_{C_A} \lambda(y, A) m(\eta(\mathrm{d}y))$$

The first integral is $m(\eta(\pi(A)))$, the other two are 0. Next we show that π is a σ-homomorphism of sets modulo $m_0 \circ \eta$.

(1) $\pi(A \cap B) = \{y : \lambda(y, A \cap B) = 1\}$. Since $\lambda(y, .)$ is a probability measure, $\lambda(y, A \cap B) = 1$ iff $\lambda(y, A) = 1 = \lambda(y, B)$. This entails $\pi(A \cap B) = \pi(A) \cap \pi(B)$.

(2) If $A \subseteq B$, then $\pi(A) \subseteq \pi(B)$ follows from $\lambda(y, A) \leq \lambda(y, B)$. It is also clear that $\pi(B \setminus A) = \pi(B) \setminus \pi(A)$.

(3) Let $A_n \in \mathcal{A}$,$n \in \mathbb{N}$, $A_n \cap A_m = \emptyset$ $m \neq n$. Put $A := \bigcup_n A_n$. Then $A_n \subseteq A \implies \pi(A_n) \subseteq \pi(A)$, hence $\bigcup_n \pi(A_n) \subseteq \pi(A)$. Let $C := \pi(A) \setminus \bigcup_n \pi(A_n)$. If $y \in C$, then $\lambda(y, A) = 1$ while $\lambda(y, A_n) \neq 1$ for all n, so that either $\lambda(y, A_n) = 0$ or $y \in C_{A_n}$. Since $\lambda(y, A) = \sum_n \lambda(y, A_n) = 1$, there is n with $\lambda(y, A_n) \neq 0$, hence $y \in C_{A_n}$. Therefore $C \subseteq \bigcup_n C_{A_n}$, so that $\eta(C) = 0$, hence $\pi(A) = \bigcup_n \pi(A_n)$ modulo η-null sets.

This concludes the proof that π is a set σ-homomorphism modulo η.

Recall that $\mu := m_0 \circ \eta$ is a probability measure on $\mathcal{B}$. Put $I := \{B \in \mathcal{B} : \mu(B) = 0\}$, then I is a σ-ideal of sets, and $\mathcal{B}/I$ is a Boolean algebra. Let $p : B \mapsto [B]$ be the canonical homomorphism. Put $\pi_1 : \mathcal{A} \xrightarrow{\pi} \mathcal{B} \xrightarrow{p} \mathcal{B}/I$. Then π_1 is a σ-homomorphism. The triple $(\Sigma, \mathcal{B}, p)$, where $p : \mathcal{B} \to \mathcal{B}/I$ is surjective, satisfies conditions of [22, Lemma 4.1.8], or [31, Theorem 1.4], and hence there is $f : \Sigma \to \Omega$, measurable, and such that $\pi_1(A) = p \circ f^{-1}(A)$, $A \in \mathcal{A}$. By definition of I, $B_1 \in [B]$ iff $\mu(B_1 \Delta B) = 0$, and since m_0 is faithful, $\eta(B_1) = \eta(B)$. Hence $\pi_1(A) = [f^{-1}(A)] = [\pi(A)]$ implies $m(\xi(A)) = m(\eta(\pi(A)) = m(\eta(f^{-1}(A)), m \in \mathcal{S}_\sigma(E)$. □

A set $\mathcal{S}$ of states on E is said to be *order determining* iff for $a, b \in E$, $m(a) \le m(b)$ for all $m \in \mathcal{S}$ implies $a \le b$ [8, p. 254].

Corollary 3.9. If there is an order determining set of states on E, then under the conditions of Theorem 3.8, $\xi(A) = \eta \circ f^{-1}(A)$ for every $A \in \mathcal{A}$. Consequently, $\mathcal{R}(\xi) \subseteq \mathcal{R}(\eta)$. In particular, if η is sharp, then ξ is sharp as well.

In what follows, we will consider ordering with respect to smearings of observables with ranges in the same block. For a block M, let $\mathcal{R}(M)$ denote the set of observables with ranges in M.

If ξ and η are observables in $\mathcal{R}(M)$, we will write $\xi \preceq \eta$ if η is defined by a smearing of ξ.

If $\xi \preceq \eta$ and $\eta \preceq \xi$, we will say that ξ and η are *equivalent* and write $\xi \sim \eta$. An observable ξ is called *minimal* if $\eta \preceq \xi$ implies $\xi \sim \eta$. By Theorem 3.4, we have for every block M, $h_M^R \in \mathcal{R}(M)$ and $h_M^R \preceq \xi$ for every $\xi \in \mathcal{R}(M)$. Moreover, h_M^R is minimal in $\mathcal{R}(M)$.

Let us recall the following facts (cf. [17]). Let $(Z, \mathcal{F})$, $(Y, \mathcal{G})$ and $(V, \mathcal{H})$ be measurable spaces, $\nu : Z \times \mathcal{G} \to [0,1]$ be a WMK with respect to a probability measure τ on $(Z, \mathcal{F})$, $\mu : Y \times \mathcal{H} \to [0,1]$ be a WMK with respect to a probability measure ρ on $(Y, \mathcal{G})$ such that $T_\nu(\tau)$ is absolutely continuous with respect to ρ, where $T_\nu(\tau)$ is the smearing of τ by ν, i. e., $T_\nu(\tau)(G) = \int_Z \nu(z, G)\tau(\mathrm{d}z)$, $G \in \mathcal{G}$. By [17, § 2.1], then there is a WMK $\lambda : Z \times \mathcal{H} \to [0,1]$, such that for every probability measure $p \ll \tau$, $T_\mu(T_\nu(p)) = T_\lambda(p)$. This WMK λ is called the *composition* of μ and ν, and will be denoted by $\lambda = \mu \circ \nu$.

Assume that there is a faithful state m_0 on M. Let $\xi \preceq \eta$ w.r. ν, and $\eta \preceq \zeta$ w.r. μ. Putting $\tau = m_0 \circ \xi$, $\rho = m_0 \circ \eta$, we have $\rho(G) = m_0(\eta(G)) = \int_Z \nu(z, G) m_0(\xi(\mathrm{d}z)) = T_\nu(\tau)(G)$, and for every $m \in \mathcal{S}_\sigma(M)$, $m \circ \xi \ll m_0 \circ \xi = \tau$. This entails that $\zeta \preceq \xi$ w.r. λ, where $\lambda = \mu \circ \nu$. It follows that $\preceq$ is reflexive and transitive, hence a preorder.

4. OBSERVABLES WITH THE RANGE IN THE SET OF SHARP REAL OBSERVABLES

For every $a \in E$, there is a block M with $a \in M$, hence by the LS theorem, $a = h_M(f_a)$, where $f_a \in \mathcal{T}_M$. Put $\Lambda_a : \mathcal{B}(\mathbb{R}) \to E$, $\Lambda_a(B) = h_M^R(f_a^{-1}(B))$. Since f_a is $Sh(\mathcal{T}_M)$-measurable, it follows that Λ_a is a sharp real observable. It was proved in [23, Theorem 4.1] that the map $a \mapsto \Lambda_a$ is one-to-one, i. e. it does not depend on the choice of the representative f_a of the element a. Owing to [2], for every σ-additive state m, $m(a) = m(h_M(f_a)) = \int_{X_M} f_a(x) m \circ h_M^R(\mathrm{d}x)$. By the integral transformation theorem, $\int_{X_M} f_a(x) m \circ h_M^R(\mathrm{d}x) = \int_0^1 t m \circ h_M^R(f^{-1}(\mathrm{d}t)) = \int_0^1 t m(\Lambda_a(\mathrm{d}t))$.

It was proved in [10], that every element of a Dedekind σ-complete unital ℓ-group admits a unique rational spectral resolution. Since σ-MV algebras in the Mundici representation are categorically equivalent with Dedekind σ-complete ℓ-groups with strong unit [8, Prop. 6.22], we may derive from [10, Corollary 4.8] that for every element a in a σ-MV algebra M, the observable Λ_a is uniquely determined by its rational spectral resolution, and hence does not depend on the choice of the LS representation, and conversely, every element $a \in M$ is uniquely defined by the observable Λ_a [24]. Moreover, it was proved in [25], that for any $a \in E$, the observable Λ_a does not depend on the choice of the block M to which a belongs. Consequently, elements of E are in one-to-one correspondence with special sharp real observables of E.

As a special case of Theorem 3.7 and taking into account the proof of [26, Theorem 4.1], we obtain the following result.

Theorem 4.1. A real observable ξ on E is sharp if and only if there is a block M and a measurable function $f : X_M \to [0,1]$ such that $\xi(B) = h_M^R(f^{-1}(B))$, $B \in \mathcal{B}(\mathbb{R})$. In addition, if $f \in \mathcal{T}_M$, then $\xi = \Lambda_a$, where $a = h_M(f)$.

Let M be a block of E, which is σ-MV algebra with standard LS-representation $(X_M, \mathcal{T}_M, h_M)$. Let $\xi : (\Omega, \mathcal{A}) \to E$ be an observable with range in M. For every $A \in \mathcal{A}$, $\xi(A) = h_M(f_A)$ with $f_A \in \mathcal{T}_M$, and $\Lambda_{\xi(A)} = h_M^R \circ f_A^{-1} : \mathcal{B}([0,1]) \to Sh(M) \subseteq Sh(E)$ is a sharp real observable on E. Let us write $\tilde{\xi}$ for the map $A \mapsto \Lambda_{\xi(A)}$. Then $\tilde{\xi}$ maps $\mathcal{A}$ to the set of sharp real observables on M with the value space $([0,1], \mathcal{B}([0,1])$. For every $A \in \mathcal{A}$, $\tilde{\xi}(A) = \Lambda_{\xi(A)}$ can be considered as a real observable on the σ-OML $Sh(E)$ of all sharp elements on E.

Let L be a σ-OML. Observables $(u_i)_{i \in I}$ associated with L are compatible iff their ranges are contained in the same block B of L. We recall some basic facts about compatible observables on σ-OMLs and their functional calculus, see [31, Th. 1.4, Th. 1.6].

Let $\xi_1, \xi_2, \ldots, \xi_n$ be compatible sharp real observables on E. Then their ranges are contained in $Sh(M)$ for a block M. Put $\mathcal{L} = Sh(M)$, $\mathcal{S} = Sh(\mathcal{T}_M)$. Then h_M^R maps $\mathcal{S}$ onto $\mathcal{L}$. By Theorem 4.1, there are measurable functions $f_i : X_M \to [0,1]$ such that $\xi_i = h_M^R \circ f_i^{-1}$. Starting with $(X_M, Sh(\mathcal{T}_M), h_M^R)$ as $(X, \mathcal{S}, h)$, we will follow the construction in the proof of [31, Th. 1.6 (ii)]. Let $\mathcal{L}_0$ denote the smallest sub-σ-algebra of $Sh(M)$ containing the ranges of $\xi_i, i = 1, 2, \ldots, n$. Define $\bar{f} : X_M \to \mathbb{R}^n$ by $\bar{f}(x) = (f_1(x), f_2(x), \ldots, f_n(x))$. Then $\bar{f}$ is $Sh(\mathcal{T})$-measurable. Let $u := h_M^R \circ \bar{f}^{-1}$. Then $u : \mathcal{B}(\mathbb{R}^n) \to Sh(M)$ is a σ-homomorphism (of boolean σ-algebras) and $\xi_i(B) = u(p_i^{-1}(B))$, $i = 1, 2, \ldots, n$ for all $B \in \mathcal{B}(\mathbb{R}^n)$, where p_i is the projection from $\mathbb{R}^n$ to $\mathbb{R}$. Since $\mathcal{B}(\mathbb{R}^n)$ is the smallest σ-algebra of subsets of $\mathbb{R}^n$ containing all the sets $p_i^{-1}(B), i = 1, 2, \ldots, n$, the range of u is $\mathcal{L}_0$. For any Borel function $\phi : \mathbb{R}^n \to \mathbb{R}$, the mapping $u \circ \phi^{-1}$ is an observable on $Sh(M)$ whose range is contained in $\mathcal{L}_0$. Conversely, if η is any sharp real observable with range in $\mathcal{L}_0$, there exists a Borel function $\phi : \mathbb{R}^n \to \mathbb{R}$ such that $\eta(B) = u(\phi^{-1}(B))$, for all B.

By [31, Remark p. 17], $u \circ \phi^{-1}$ is interpreted as the function ϕ of the observables $\xi_i, i = 1, 2, \ldots, n$, in symbols $\phi(\xi_i, \xi_2, \ldots, \xi_n)$. Notice that the function of observables does not depend on the choice of $(X, \mathcal{S}, h)$, see [31, Remark on p. 17].

Thus for every $B \in \mathcal{B}(\mathbb{R})$,

$$\phi(\xi_1, \xi_2, \ldots, \xi_n)(B) = h_M^R \circ \bar{f}^{-1}(\phi^{-1}(B)) = h_M^R(\phi \circ (f_1, \ldots, f_n))^{-1}(B). \tag{2}$$

We will also need the following definition. Let $\eta_n, \eta (n \in \mathbb{N})$ be real observables on L with ranges in a block B. We say that the sequence η_n converges to η *everywhere* if for every $\epsilon > 0$ we have $\liminf((\eta_n - \eta)(-\epsilon, +\epsilon)) = 1$, [22, Def. 6.1.2]. Assume that for some observable z and measurable functions f_n, f we have $\eta_n = z \circ f_n^{-1}, \eta = z \circ f^{-1}$. Then the sequence η_n converges to η everywhere iff there is a subset $Z \in \mathcal{B}(\mathbb{R})$ such that $z(Z) = 1$ and $f_n \to f$ everywhere on Z, [22, Th. 6.1.3].

Lemma 4.2. (i) Let $a_1, a_2, \ldots, a_n$ be orthogonal elements contained in a block M of E. Then $\Lambda_{a_1 \oplus a_2 \cdots \oplus a_n} = \phi(\Lambda_{a_1}, \ldots, \Lambda_{a_n})$, where $\phi(t_1, \ldots, t_n) = t_1 + t_2 + \cdots + t_n$.
(ii) Let $a, b \in M$, $b \leq a$. Then $\Lambda_{a \ominus b} = \phi(\Lambda_a, \Lambda_b)$, where $\phi(t_1, t_2) = t_1 - t_2$.

Proof. (i) Let $\mathcal{L}_0$ be the smallest sub-σ-algebra of $Sh(M)$ containing the ranges of $\Lambda_{a_i}, i = 1, 2, \ldots, n$. Put $a := a_1 \oplus a_2 \oplus \cdots \oplus a_n$. For every $x \in X_M = Ext(\mathcal{S}(M))$, $x(a) = \sum_{i \leq n} x(a_i)$, hence $\hat{a}(x) = \sum_{i \leq n} \hat{a}_i(x)$, and $a_i = h_M(\hat{a_i})$, $i = 1, 2, \ldots, n$, $a = h_M(\hat{a})$. Then for all $B \in \mathcal{B}(\mathbb{R})$, $\Lambda_{a_i}(B) = h_M \circ \hat{a_i}^{-1}(B)$, $i = 1, 2, \ldots, n$, $\Lambda_a = h_M \circ \hat{a}^{-1}(B)$. Put $\phi(t_1, \ldots, t_n) = \sum_{i \leq n} t_i$, then by (2), $\Lambda_{a_1} + \cdots + \Lambda_{a_n} = h_M^R \circ (\sum_{i \leq n} \hat{a}_i)^{-1} = h_M^R \circ \hat{a}^{-1} = \Lambda_a$.

(ii) The proof follows the same pattern as the proof of (i) with $\phi(t_1, t_2) = t_1 - t_2$ and taking into account that $x(a \ominus b) = x(a) - x(b)$ for all $x \in X_M = Ext\mathcal{S}(M)$. □

Theorem 4.3. For every observable ξ, the mapping $\tilde{\xi} : A \mapsto \Lambda_{\xi(A)}$ has the following properties:

(GO1) $\tilde{\xi}(\Omega) = \Lambda_1$, where Λ_1 is the (unique) observable on $Sh(E)$ with $\Lambda_1(\{1\}) = 1$;

(GO2) if $A_1, \ldots, A_n$ are pairwise disjoint elements of $\mathcal{A}$, then $\tilde{\xi}(A_1 \cup \cdots \cup A_n) = \sum_{i=1}^n \tilde{\xi}(A_i)$, where the latter sum is given by the functional calculus for compatible observables;

(GO3) for any $A_n, A \in \mathcal{A}$, $n \in \mathbb{N}$ such that $A_n \nearrow A$, we have $\tilde{\xi}(A_n) \to \tilde{\xi}(A)$ everywhere.

Proof. Write $h = h_M$. (GO1): we have $\xi(\Omega) = 1 = h(\hat{1})$, where $\hat{1}(x) = x(1) = 1$ for all $x \in X_M = Ext\mathcal{S}(M)$. Then $\hat{1}^{-1}(\{1\}) = X_M$, hence $\Lambda_{\xi(\Omega)}(\{1\}) = h(\hat{1}^{-1}(\{1\}) = h(X_M) = h(\chi_{X_M}) = 1$.

(GO2): As $A_1, \ldots, A_n$ are disjoint, $\xi(A_1), \xi(A_2), \ldots, \xi(A_n)$ are orthogonal, and $\xi(\cup_{i=1}^n A_i) = \oplus_{i=1}^n \xi(A_i)$. Put $a_i := \xi(A_i), i = 1, 2, \ldots, n$, $a := \xi(\cup_{i=1}^n A_i) = \oplus_{i=1}^n a_i$, then $a_i = h(\hat{a_i}), a = h(\hat{a}), i = 1, 2, \ldots, n$, and $\Lambda_{a_i} = h \circ \hat{a_i}^{-1}$, $\Lambda_a = h \circ \hat{a}^{-1}$.

The proof follows by Lemma 4.2 (i).

(GO3): If $A_n \nearrow A$, then $\xi(A_n) \nearrow \xi(A)$. Let $\xi(A_i) = a_i = h(\hat{a_i})$, $\xi(A) = a = h(\hat{a})$. Put $f_n := \sup\{\hat{a_i} : i \leq n\}$, then $(f_n)_n$ is a nondecreasing sequence of functions in $\mathcal{T}_M$ with $h(f_n) = \bigvee_{i\le n} h(\hat{a_i}) = a_n$ for every n. Put $V_n := \{x \in X_M : f_n(x) \neq \hat{a_n}(x)\}$, then $h(V_n) = 0$.

Let $f = \lim_n f_n = \sup_n f_n$ be their pointwise limit. Then $f \in \mathcal{T}_M$, $h(f) = h(\sup_n f_n) = \bigvee_n h(f_n) = \bigvee_n a_n = a$. Put $V = \{x \in X_M : f(x) \neq \hat{a}(x)\}$, then $h(V) = 0$.

So we have $\hat{a_n} \to \hat{a}$ pointwise on $Z := X_M \setminus (V \cup \bigcup_n V_n)$, $h(Z) = 1$. This entails that for every $\epsilon > 0$, $\bigcup_n \bigcap_{k\ge n} (\hat{a} - \hat{a_n})^{-1}(-\epsilon, +\epsilon) = Z$.

By Lemma 4.2 (ii), $\Lambda_a - \Lambda_{a_n} = h \circ (\hat{a} - \hat{a_n})^{-1}$. Thus

$$\begin{aligned}\liminf(\Lambda_a - \Lambda_{a_n})(-\epsilon, +\epsilon) &= \liminf h \circ (\hat{a} - \hat{a_n})^{-1}(-\epsilon, +\epsilon) \\ &= h(\bigcup_n \bigcap_{k\le n} (\hat{a} - \hat{a_n})^{-1}(-\epsilon, +\epsilon)) \\ &= h(Z) = 1.\end{aligned}$$

This concludes the proof that $\tilde{\xi}(A_n) \to \tilde{\xi}(A)$ everywhere. □

Definition 4.4. Let $(\Omega, \mathcal{A})$ be a measurable space. A mapping Ξ from $\mathcal{A}$ to a compatible set of real observables on $Sh(E)$ with properties (GO1), (GO2), (GO3) will be called an $(\Omega, \mathcal{A})$-*generalized observable* on E.

If $\Xi = \tilde{\xi}$ for an observable ξ, we say that Ξ is *associated* with ξ.

Let Ξ be an $(\Omega, \mathcal{A})$-generalized observable on E and let M be a block that contains the range of $\Xi(A)$, $\forall A \in \mathcal{A}$. Define

$$P^m_\Xi(A) = \int_{\mathbb{R}} tm \circ \Xi(A)(\mathrm{d}t), A \in \mathcal{A}, m \in \mathcal{S}_\sigma(M). \tag{3}$$

Notice that if Ξ is a generalized observable associated with an $(\Omega, \mathcal{A})$-observable ξ, then $\Xi(A) = \tilde{\xi}(A) = h^R_M \circ f_A^{-1}$, $A \in \mathcal{A}$, where $f_A \in \mathcal{T}_M$ is such that $\xi(A) = h_M(f_A)$. Then we have

$$\begin{aligned}P^m_{\tilde{\xi}}(A) &= \int_{\mathbb{R}} tm \circ \tilde{\xi}(A)(\mathrm{d}t) = \int_{\mathbb{R}} tm \circ h^R_M(f_A^{-1}(\mathrm{d}t)) \\ &= \int_{X_M} f_A(x) m \circ h^R_M(\mathrm{d}x),\end{aligned}$$

By (1), the identity (3) can be equivalently written as

$$P^m_{\tilde{\xi}}(A) = m(\xi(A)). \tag{4}$$

Lemma 4.5. For every $m \in \mathcal{S}_\sigma(M)$, the mapping $A \mapsto P^m_\Xi(A)$ is a probability measure on $(\Omega, \mathcal{A})$.

Proof. Recall that for all $A \in \mathcal{A}$, $\Xi(A)$ is a sharp real observable with range in M. By Theorem 4.1, there is a measurable function $f_A : X_M \to [0, 1]$ such that

$$\Xi(A)(B) = h^R_M(f_A^{-1}(B)), B \in \mathcal{B}(\mathbb{R}).$$

By (GO1), $\Xi(\Omega)(B) = 1$ if $1 \in B$, and $\Xi(\Omega)(B) = 0$ otherwise. From this we obtain that $h_M^R(\{x \in X_M : f_\Omega(x) \neq 1\}) = 0$, and

$$\begin{aligned} P_\Xi^m(\Omega) &= \int_{\mathbb{R}} tm \circ \Xi(\Omega)(\mathrm{d}t) \\ &= \int_{\mathbb{R}} tm \circ h_M^R(f_\Omega^{-1}(\mathrm{d}t)) \\ &= \int_{X_M} f_\Omega(x) m \circ h_M^R(\mathrm{d}x) = 1. \end{aligned}$$

Let $A_i, i \in \mathbb{N}$, be a sequence of pairwise disjoint elements of $\mathcal{A}$, and $A = \bigcup_{i\in\mathbb{N}} A_i$. From (GO2) and (GO3) we get

$$\Xi(A) = \sum_{i\in\mathbb{N}} \Xi(A_i), \tag{5}$$

where the series on the right converges everywhere. Now let $\Xi(A_i) = h_M^R \circ f_{A_i}^{-1}$, $\Xi(A) = h_M^R \circ f_A^{-1}$ (cf. Theorem 4.1). By the functional calculus, equation (5) entails that

$$h_M^R(\{x \in X_M : \sum_{i\in\mathbb{N}} f_{A_i}(x) \neq f_A(x)\}) = 0.$$

Therefore for all $m \in \mathcal{S}_\sigma(M)$,

$$\begin{aligned} P_\Xi^m(A) &= \int_{\mathbb{R}} tm \circ \Xi(A)(\mathrm{d}t) = \int_{X_M} f_A(x) m \circ h_M^R(\mathrm{d}x) \\ &= \int_{X_M} \sum_{i\in\mathbb{N}} f_{A_i}(x) m \circ h_M^R(\mathrm{d}x) \\ &= \sum_{i\in\mathbb{N}} \int_{X_M} f_{A_i}(x) m \circ h_M^R(\mathrm{d}x) = \sum_{i\in\mathbb{N}} P_\Xi^m(A_i). \end{aligned}$$

□

We will use the notation $\mathcal{P}(\Xi) := \{P_\Xi^m : m \in \mathcal{S}_\sigma(M)\}$. Then $\mathcal{P}(\Xi)$ is a set of probability measures on $(\Omega, \mathcal{A})$.

Definition 4.6. Let $(Z, \mathcal{F})$ and $(Y, \mathcal{G})$ be measurable spaces, Ξ be a $(Z, \mathcal{F})$-generalized observable on E and let $\nu : Z \times \mathcal{G} \to [0, 1]$ be a WMK with respect to $\mathcal{P}(\Xi)$. The mapping ψ from $\mathcal{P}(\Xi)$ to the set of probability measures on $(Y, \mathcal{G})$ defined by

$$\psi(P_\Xi^m)(G) := \int_Z \nu(z, G) P_\Xi^m(\mathrm{d}z)$$

will be called the *smearing* of the generalized observable Ξ with respect to ν.

If, in addition, there is a $(Y, \mathcal{G})$-generalized observable Θ on E such that

$$P_\Theta^m(G) = \psi(P_\Xi^m)(G), G \in \mathcal{G},$$

we say that Θ is *defined by the smearing of* Ξ *with respect to* ν.

Lemma 4.7. The smearing of $\tilde{\xi}$ coincides with the smearing of ξ.

P r o o f. By Definition 4.6 and (4), $\psi(P^m_{\tilde{\xi}}(G)) = \int_Z \nu(z,G)P^m_{\tilde{\xi}}(\mathrm{d}z) =$
$= \int_Z \nu(z,G)m(\xi(\mathrm{d}z))$. □

Theorem 4.8. Every generalized observable Ξ is defined by a smearing of a basic observable h^R_M.

P r o o f. Let Ξ be a $(\Omega,\mathcal{A})$-generalized observable on E. There is a block M such that for every $A \in \mathcal{A}$, the range of $\Xi(A)$ belongs to M. By Theorem 4.1, for every $A \in \mathcal{A}$, there is a measurable function $f_A : X_M \to [0,1]$ such that $\Xi(A)(B) = h^R_M(f_A^{-1}(B))$ for every $B \in \mathcal{B}(\mathbb{R})$. Then we have for every $m \in \mathcal{S}_\sigma(M)$,

$$\begin{aligned} P^m_\Xi(A) &= \int_{\mathbb{R}} tm \circ \Xi(A)(\mathrm{d}t) \\ &= \int_{\mathbb{R}} tm \circ h^R_M(f_A^{-1}(\mathrm{d}t)) \\ &= \int_{X_M} f_A(x) m \circ h^R_M(\mathrm{d}x). \end{aligned}$$

Defining $\nu(x,A) = f_A(x)$, it is easy to check that ν is a WMK with respect to $\{m \circ h^R_M : m \in \mathcal{S}_\sigma(M)\}$, and we obtain that Ξ is defined by a smearing of h^R_M (equivalently, of $\widetilde{h}^R_M$) with respect to the WMK ν. □

Our last theorem shows that if there is a faithful state on E, then the system of generalized observables is closed under smearings.

Theorem 4.9. Let $(Z,\mathcal{F})$ and $(Y,\mathcal{G})$ be measurable spaces, Ξ be a $(Z,\mathcal{F})$-generalized observable with ranges in M, and let $\mu : Z \times \mathcal{G} \to [0,1]$ be a WMK. If there is a faithful state in $\mathcal{S}_\sigma(M)$, then the smearing of Ξ with respect to μ defines a generalized observable.

P r o o f. By Theorem 4.8, Ξ is defined by a smearing of h^R_M with respect to a WMK $\nu : X_M \times \mathcal{F} \to [0,1]$, so that $P^m_\Xi(F) = \int_{X_M} \nu(x,F) m \circ h^R_M(\mathrm{d}x)$, $m \in \mathcal{S}_\sigma(M)$. Then $\psi(P^m_\Xi)(G) = \int_Z \mu(z,G)P^m_\Xi(\mathrm{d}z) = \int_{X_M} \lambda(x,G)h^R_M(\mathrm{d}x)$, where $\lambda : X_M \times \mathcal{G} \to [0,1]$ is a composition of μ and ν. Since λ is a WMK, the function $f_G : X_M \to [0,1]$, $f_G(x) := \lambda(x,G)$ is measurable for every $G \in \mathcal{G}$, and it is easy to see that $\Theta(G) := h^R_M \circ f_G^{-1}$ is a generalized observable.

Moreover,

$$\begin{aligned} P^m_\Theta(G) &= \int_{\mathbb{R}} tm \circ h^R_M(f_G^{-1}(\mathrm{d}t)) = \int_{X_M} f_G(x) m \circ h^R_M(\mathrm{d}t) \\ &= \int_Z \mu(z,G) P^m_\Xi(\mathrm{d}z) = \psi(P^m_\Xi)(G). \end{aligned}$$

□

5. CONCLUDING REMARKS

We have proved that on a lattice effect algebra, every observable is defined by a smearing of a sharp observable, but further conditions are needed to ensure that a smearing of an observable defines an observable. In order to close the set of compatible observables under smearings, we need to introduce generalized observables. It might be instructive to compare this situation with the algebra of Hilbert space effects.

Let H be a complex, separable Hilbert space, and $\mathcal{E}(H)$ denote the effect algebra of Hilbert space effects, i.e., operators between 0 and I in the usual ordering of self-adjoint operators. The set $\mathcal{E}(H)$ can be organized into an effect algebra, and the effect algebra ordering coincides with the original one. In this ordering, $\mathcal{E}(H)$ is far from being a lattice (see, e.g., [8, § 1.11]).

Recall that observables on $\mathcal{E}(H)$ coincide with normalized positive operator-valued measures (POVM). The sharp observables are projection valued measures (PVM), which correspond to self-adjoint operators. If the range of a POVM consists of commuting effects, it is contained in a maximal Abelian von Neumann subalgebra $\mathcal{R}$, which can be described as the set of all measurable functions of a sharp observable (or the corresponding self-adjoint operator), and $\mathcal{E}(H) \cap \mathcal{R}$ can be organized into a σ-MV-algebra. The corresponding tribe $\mathcal{T}$ in the LS-representation contains all the $Sh(\mathcal{T})$-measurable functions. It follows that every generalized observable on $\mathcal{E}(H) \cap \mathcal{R}$ is an observable and the set of observables with range in $\mathcal{R}$ is closed under smearings.

In fact, the last statement is true for the set of all observables on $\mathcal{E}(H)$: every smearing of an observable is again an observable. Moreover, the set of σ-additive states on $\mathcal{E}(H)$ is rich enough so that every POVM is determined by the set of its probability distributions in all states.

On the other hand, it is well known ([13, 15], that a POVM is defined by a smearing of a sharp observable iff it has commutative range. But most of POVMs have non-commutative ranges.

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