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# A T-PARTIAL ORDER OBTAINED FROM T-NORMS

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A partial order on a bounded lattice $L$ is called t-order if it is defined by means of the t-norm on $L$. It is obtained that for a t-norm on a bounded lattice $L$ the relation $a \preceq_T b$ iff $a = T(x, b)$ for some $x \in L$ is a partial order. The goal of the paper is to determine some conditions such that the new partial order induces a bounded lattice on the subset of all idempotent elements of $L$ and a complete lattice on the subset $A$ of all elements of $L$ which are the supremum of a subset of atoms.

*Keywords:* triangular norm, bounded lattice, triangular action, $\bigvee$-distributive, idempotent element

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## 1. INTRODUCTION

T-norms were introduced by Karl Menger in 1942, see [24], p. 3. Triangular norms play a significant role in many branches information science [15, 17, 19]. Several authors have studied t-norms on bounded lattices. For more detail, we refer [2, 11, 12, 13, 14, 21, 22]. H. Mitsch define a natural partial order for semigroups [20]. Here, we obtain a partial order by means of t-norms and investigate some properties of this order.

The paper is organized as follows. In Section 1, we state some definitions which are crucial for our study. In Section 2, firstly, we define a t-partial order, denoted by $\preceq_T$, on a bounded lattice $L$ by means of the t-norm on $L$. Also, in this section, we investigate some connections between the orders $\leq$ and $\preceq_T$.

The main aim of the present paper is to determine some conditions such that the new partial order induces a bounded lattice on the subset of all idempotent elements of $L$ and a complete lattice on the subset $A$ of all elements of $L$ which are the supremum of a subset of atoms. In Section 3, even if $L$ is a chain (or lattice), we show that $L$ may not be a chain (or lattice) with respect to the order $\preceq_T$ by examples. We determine a necessary condition makes $L$ a lattice with respect to the order $\preceq_T$. In Section 4, we show that $H_T$ is a complete lattice with respect to $\preceq_T$, where $H_T$ is the set of all idempotent elements of t-norm $T$. By using this idea, in Proposition 4.14, we also prove that for an integral, commutative, residuated $\ell$-monoid $M = (L, \leq, \odot)$, if $M$ is divisible, then the subset $H_T$ of all idempotent elements with respect to $\odot$ forms a Heyting algebra, and the implication in $H_T$

coincides with the implication based on $\odot$. So we obtain from this conclusion that the algebraic strong De Morgan's law is not necessary for the proof of the Theorem which is in the study of Drossos [5] (Höhle [10], Corollary 2.7). In the last section, we give some open problems.

**Definition 1.1.** (Karaçal and Khadjiev [13]) Let $L$ be a bounded lattice. A triangular norm $T$ (briefly t-norm) is a binary operation on $L$ which is commutative, associative, monotone and has neutral element 1.

$$\text{Let } T_W(x,y) = \begin{cases} x & \text{if} \quad y = 1, \\ y & \text{if} \quad x = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then $T_W$ is a t-norm on $L$. Since it holds that $T_W \leq T$ for any t-norm $T$ on $L$, $T_W$ is the smallest t-norm on $L$.

The largest t-norm on a bounded lattice $L$ is given by $T_\wedge(x,y) = x \wedge y$.

**Definition 1.2.** (De Baets and Mesiar [2]) Consider a t-norm $T$ on a bounded lattice $L$. An element $x \in L$ is called a zero divisor of $T$ if there exists $y \in L$ such that $x \wedge y \neq 0$ and $T(x,y) = 0$. The set of zero divisors of $T$ is denoted by $Z(T)$.

A t-norm $T$ is called t-norm without zero divisors if $Z(T) = \varnothing$.

**Definition 1.3.** (Casasnovas and Mayor [4]) A t-norm $T$ on $L$ is divisible if the following condition holds:

$$\forall\, x,y \in L \quad \text{with} \quad x \leq y \quad \text{there is a} \quad z \in L \quad \text{such that} \quad x = T(y,z).$$

A basic example of non-divisible t-norm on any bounded lattice $L$ is the $T_W$. Trivially, the infimum $T_\wedge$ is divisible: $x \leq y$ is equivalent to $x \wedge y = x$.

**Definition 1.4.** (Birkhoff [1]) An element $x$ of $L$ is called an atom if $x$ is a minimal element of $L \backslash \{0\}$.

Denote by $A$ the set of all elements of $L$ which are supremum of some family of atoms.

**Definition 1.5.** (Birkhoff [1]) An atomic lattice is a lattice $L$ in which every element is a join of atoms, and hence of the atoms which it contains.

**Definition 1.6.** (De Baets and Mesiar [2]) Consider a t-norm $T$ on a bounded lattice $L$. An element $x \in L$ is called an idempotent element if $T(x,x) = x$.

Follows from the definition of a t-norm it immediately that the elements 0 and 1 are idempotent elements of any t-norm. These elements will be called trivial idempotent elements further on; other idempotent elements will be called non-trivial.

Denote by $H_T$ the set of all idempotent elements of $T$.

**Definition 1.7.** (Karaçal and Khadjiev [13])

(i) A t-norm $T$ on a lattice $L$ is called $\bigvee$-distributive if

$$T(a, b_1 \vee b_2) = T(a, b_1) \vee T(a, b_2)$$

for every $a, b_1, b_2 \in L$.

(ii) A t-norm $T$ on a complete lattice $L$ is called *infinitely* $\bigvee$*-distributive* if

$$T\Big(a, \bigvee_Q b_\tau\Big) = \bigvee_Q T(a, b_\tau)$$

for every subset $\{a, b_\tau \in L, \tau \in Q\}$ of $L$.

## 2. $\preceq_T$ TRIANGULAR ORDER

A natural partial order for semigroups was defined by H. Mitsch in 1986, see [20]. In this section, we give the definition of a t-partial order obtained from t-norms and investigate its properties.

**Definition 2.1.** Let $L$ be a bounded lattice, $T$ be a t-norm on $L$. The order defined as following is called a t-order (triangular order) for t-norm $T$.

$$x \preceq_T y :\Leftrightarrow T(\ell, y) = x \text{ for some } \ell \in L.$$

**Proposition 2.2.** The binary relation $\preceq_T$ is a partial order on $L$.

Proof. Since $1 \in L$ and $T(1, x) = x$, $x \preceq_T x$ holds. Thus, the reflexivity is satisfied.

Let $x \preceq_T y$ and $y \preceq_T x$. Then, there exist $\ell_1, \ell_2$ of $L$ such that $T(\ell_1, y) = x$ and $T(\ell_2, x) = y$. Hence, $x = T(\ell_1, y) \leq T(1, y) = y$; i.e, $x \leq y$. On the other hand, $y = T(\ell_2, x) \leq T(1, x) = x$; i.e, $y \leq x$. So, $x = y$. Thus, the antisymmetry is satisfied.

Let $x \preceq_T y$ and $y \preceq_T z$. Then, there exist $\ell_1, \ell_2$ of $L$ such that $T(\ell_1, y) = x$ and $T(\ell_2, z) = y$. For $T(\ell_1, \ell_2)$ of $L$, $T(T(\ell_1, \ell_2), z) = T(\ell_1, T(\ell_2, z)) = T(\ell_1, y) = x$. Thus, $x \preceq_T z$. This means that the relation $\preceq_T$ satisfies the transitivity. So, we have that $\preceq_T$ is a partial order on $L$.

□

**Proposition 2.3.** If $(x, y) \in \preceq_T$, then $(x, y) \in \leq$ .

Proof. Let $(x, y) \in \preceq_T$. Then, there exists an element $\ell$ of $L$ such that $x = T(\ell, y) \leq T(1, y) = y$. Thus $(x, y) \in \leq$. □

**Remark 2.4.** (i) If $(x, y) \in \leq$, then $(x, y) \in \preceq_T$ may not be true. For example, let $L = \{0, a, b, c, 1\}$ and consider the order $\leq$ on $L$ as follows:

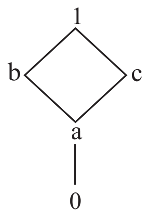


**Fig. 1.** The order $\leq$ on $L$.

Being $T = T_W$, we can see $a \leq b$ but $a \not\preceq_{T_W} b$. Indeed;

if $a \preceq_{T_W} b$, then there exists an element $\ell$ of $L$ such that $T_W(\ell, b) = a$.

If $\ell = 0$, then $a = 0$, which is a contradiction. If $\ell = a, b$ or $c$, then $T_W(\ell, b) = 0 = a$. This is a contradiction. If $\ell = 1$, then $T_W(1, b) = b = a$, which is not possible. Therefore, there doesn't exist any element $\ell$ of $L$ satisfying $T_W(\ell, b) = a$. Thus, $a \not\preceq_{T_W} b$. Here, the order $\preceq_{T_W}$ on $L$ is as follows:

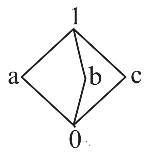


**Fig. 2.** The order $\preceq_{T_W}$ on $L$.

(ii) Let $L$ be a bounded lattice and $T$ be a t-norm on $L$. By Definition 1.3, it is easily shown that $\preceq_T$ is equal to $\leq$ if and only if $T$ is a divisible t-norm on $L$.

## 3. SOME PROPERTIES OF THE PARTIALLY ORDERED SET $(L, \preceq_T)$

In Section 2, we show that $(L, \preceq_T)$ is a partially ordered set. In this section, we give some examples for t-norms such that $(L, \preceq_T)$ is a lattice or not.

**Remark 3.1.** Even if $(L, \leq)$ is a chain, $(L, \preceq_T)$ may not be a chain. For example, consider the lattice $L = [0, 1]$ and the nilpotent minimum t-norm $T^{nM}$ defined by $T^{nM}(x, y) = \begin{cases} 0 & x + y \leq 1, \\ \min(x, y) & \text{otherwise,} \end{cases}$ [16].

$1/3 \leq 1/2$, but $1/2$ and $1/3$ aren't comparable with respect to the relation $\preceq_{T^{nM}}$ on $[0, 1]$. Indeed; if $1/2 \preceq_{T^{nM}} 1/3$, by Proposition 2.3, then $1/2 \leq 1/3$ which is a contradiction. Therefore, $1/2 \not\preceq_{T^{nM}} 1/3$.

On the other hand, if $1/3 \preceq_{T^{nM}} 1/2$, then there exists an element $\ell \in [0, 1]$ such that $T^{nM}(\ell, 1/2) = 1/3$. Since $T^{nM}(\ell, 1/2) = 1/3$, $\ell + 1/2 > 1$; i.e, $\ell > 1/2$. Thus, $T^{nM}(\ell, 1/2) = \min(\ell, 1/2) = 1/2$, which is impossible. Therefore, $1/3 \not\preceq_{T^{nM}} 1/2$ as required. Hence, $1/2$ isn't comparable with $1/3$ according to the relation $\preceq_{T^{nM}}$ on $[0, 1]$.

**Remark 3.2.** Let $L$ be a bounded lattice. Consider a t-norm $T$ on $L$. For $X \subseteq L$, we denote the set of the upper bounds of $X$ with respect to $\preceq_T$ on $L$ by $\overline{X}_T$. Also, we denote the set of the lower bounds of $X$ with respect to $\preceq_T$ on $L$ by $\underline{X}_T$. Being $X = \{a, b\}$, since $T(a, b) \preceq_T a$ and $T(a, b) \preceq_T b$, we have that $T(a, b) \in \underline{\{a, b\}}_T$. Thus, $\underline{\{a, b\}}_T \neq \emptyset$. Since $T(a, 1) = a$ and $T(b, 1) = b$, $a \preceq_T 1$ and $b \preceq_T 1$. Thus, $1 \in \overline{\{a, b\}}_T$, so we obtain that $\overline{\{a, b\}}_T \neq \emptyset$. Also,0 is the smallest element and 1 is the greatest element with respect to $\preceq_T$. If there exist the greatest element of the lower bounds and the least element of the upper bounds with respect to $\preceq_T$, respectively, we will denote by $\bigwedge_T$ and $\bigvee_T$.
$L$ may not be a lattice with respect to the order $\preceq_T$. The following example illustrates that.

**Example 3.3.** Let $L = [0, 1]$ and $T^{nM}$ be the nilpotent minimum t-norm on $[0, 1]$. Then, $(L, \preceq_{T^{nM}})$ is a meet-semilattice, but not a join-semilattice.

If $x \preceq_{T^{nM}} y$ or $y \preceq_{T^{nM}} x$, then $x \wedge_{T^{nM}} y$ is equal to $x$ or $y$. Let $x \not\preceq_{T^{nM}} y$ and $y \not\preceq_{T^{nM}} x$. It must be $x+y \leq 1$. Otherwise, if $x+y > 1$, then $T^{nM}(x, y) = \min(x, y)$. Thus, if $T^{nM}(x, y) = x$ or $y$, then we have that $x \preceq_{T^{nM}} y$ or $y \preceq_{T^{nM}} x$, which are contradictions.

Let $x + y \leq 1$. It is clear that $0 \in \underline{\{x, y\}}_{T^{nM}}$. We claim that $x \wedge_{T^{nM}} y = 0$. Let $k \in \underline{\{x, y\}}_{T^{nM}} \setminus \{0\}$. Then, $k \preceq_{T^{nM}} x$ and $k \preceq_{T^{nM}} y$. There exist two elements $\ell_1, \ell_2$ of $[0, 1]$ such that $k = T^{nM}(x, \ell_1) = T^{nM}(y, \ell_2)$. Since $0 \neq k = T^{nM}(x, \ell_1) = T^{nM}(y, \ell_2)$ and $T^{nM}(x, \ell_1) \neq 0$, we have that $x+\ell_1 > 1$ and $T^{nM}(x, \ell_1) = \min(x, \ell_1)$. If $T^{nM}(x, \ell_1) = \min(x, \ell_1) = x$, then $k = x \preceq_{T^{nM}} y$. Since $x$ and $y$ aren't comparable, this is a contradiction. Then, we have that $T^{nM}(x, \ell_1) = \ell_1 = k$. Since $x+\ell_1 > 1$ and $x + y \leq 1$, we obtain that $k = \ell_1 > 1 - x \geq y$. Since $0 \neq k = T^{nM}(y, \ell_2)$, $k = T^{nM}(y, \ell_2) = \min(y, \ell_2)$. If $k = T^{nM}(y, \ell_2) = \min(y, \ell_2) = y$, then this contradicts the fact that $x$ and $y$ aren't comparable. Then $k = T^{nM}(y, \ell_2) = \ell_2$. Since $k = \ell_1 = \ell_2 > 1 - x \geq y$, $T^{nM}(y, k) = \min(y, k) = y$. This is a contradiction since $x$ and $y$ aren't comparable. So, $k = 0$. Hence, $x \wedge_{T^{nM}} y$ exists.

Now, let us show that $(L, \preceq_{T^{nM}})$ is not a join-semilattice. By Remark 3.1, we know that 1/2 and 1/3 aren't comparable. Let $k \in \overline{\{1/2, 1/3\}}_{T^{nM}}$. Then, $1/2 \preceq_{T^{nM}} k$ and $1/3 \preceq_{T^{nM}} k$. Thus, there exist $\ell_1$ and $\ell_2$ such that $1/2 = T^{nM}(k, \ell_1)$ and $1/3 = T^{nM}(k, \ell_2)$. It follows from $1/2 = \min(k, \ell_1)$ and $1/3 = \min(k, \ell_2)$ that $k + \ell_1 > 1$ and $k + \ell_2 > 1$. By Remark 3.1, we have that $\ell_1 = 1/2$ and $\ell_2 = 1/3$. Therefore, since $1/2 = T^{nM}(k, 1/2)$ and $1/3 = T^{nM}(k, 1/3)$, $k + 1/2 > 1$ and $k + 1/3 > 1$. Hence, we have that $k > 2/3 > 1/2$, and so $\overline{\{1/2, 1/3\}}_{T^{nM}} \subseteq (2/3, 1]$.

Now, we want to show that $\overline{\{1/2, 1/3\}}_{T^{nM}} = (2/3, 1]$. If $x \in (2/3, 1]$, then $x > 2/3$. Now, let us prove that $x \succeq_{T^{nM}} 1/2$ and $x \succeq_{T^{nM}} 1/3$. We investigate whether there exist two elements $x_1$ and $x_2$ such that $1/2 = T^{nM}(x, x_1)$ and $1/3 = T^{nM}(x, x_2)$. If we choose $x_1 = 1/2$ and $x_2 = 1/3$, then we have that $1/2 = T^{nM}(x, 1/2)$ and $1/3 = T^{nM}(x, 1/3)$. Therefore, we obtain that $x \succeq_{T^{nM}} 1/2$ and $x \succeq_{T^{nM}} 1/3$. Since there doesn't exist the least element of $\overline{\{1/2, 1/3\}}_{T^{nM}} = (2/3, 1]$ with respect to $\preceq_{T^{nM}}$, $([0, 1], \preceq_{T^{nM}})$ is not a join-semilattice.

**Proposition 3.4.** Let $L$ be a bounded lattice and $T$ be a t-norm on $L$. If $T = T_W$, then for arbitrary $a \in L\backslash\{0,1\}$ it holds that $a \wedge_T b = 0$ and $a \vee_T b = 1$ for every $b \in L\backslash\{0,1,a\}$. Thus, $(L, \preceq_{T_W})$ is a lattice.

Proof. For every $a, b \neq 0, 1$ and $a \neq b$, since $T_W(a,b) = 0$ and for all $k \in L$, $T_W(a,k) \neq b$ and $T_W(b,k) \neq a$, $a$ and $b$ are not comparable with respect to $\preceq_{T_W}$.

We claim that for arbitrary $a \in L\backslash\{0,1\}$ it satisfies $a \wedge_{T_W} b = 0$ for every $b \in L\backslash\{0,1,a\}$.

If $a \wedge_{T_W} b = x \neq 0$, then $x \preceq_{T_W} a$ and $x \preceq_{T_W} b$. Thus, there exists $x_1 \in L\backslash\{0\}$ such that $0 \neq x = T_W(a, x_1)$. If $x = a$, then this is a contradiction since $a$ and $b$ aren't comparable with respect to $\preceq_{T_W}$. If $T_W(a, x_1) = 1$ or $x_1$, then we obtain that $a = 1$. This contradicts to the choice of $a$. Hence, $a \wedge_{T_W} b = 0$.

Similarly, let us show that for arbitrary $a \in L\backslash\{0,1\}$, $a \vee_{T_W} b = 1$ for every $b \in L\backslash\{0,1,a\}$. Let $a \vee_{T_W} b = x$. Then $a \preceq_{T_W} x$ and $b \preceq_{T_W} x$, and so there exist $x_1, x_2 \in L\backslash\{0\}$ such that $T_W(x, x_1) = a$ and $T_W(x, x_2) = b$. If $x = a$, then $T_W(a, x_2) = b$ which is a contradiction since $a$ and $b$ aren't comparable with respect to $\preceq_{T_W}$. Then, $x_1 = a$, so it must be $x = 1$. Therefore, $a \vee_{T_W} b = 1$. Finally, we have that $(L, \preceq_{T_W})$ is a lattice.

Now, we give an example such that $(L, \preceq_T)$ is a lattice and $T \neq T_W$. □

**Example 3.5.** Consider the t-norm

$$T(x,y) = \begin{cases} 0 & (x,y) \in (0,1/2)^2, \\ \min(x,y) & \text{otherwise}, \end{cases}$$

on $[0,1]$ ([16], p. 18, 1.24 Example). Then $([0,1], \preceq_T)$ is a lattice.

Choose that $x$ and $y$ aren't comparable with respect to $\preceq_T$. Otherwise, it is trivial. We want to show that $x \vee_T y = 1/2$.

If $x \notin (0,1/2)$ or $y \notin (0,1/2)$, then $T(x,y) = \min(x,y)$. This contradicts that $x$ and $y$ aren't comparable with respect to $\preceq_T$. Therefore, $x, y \in (0,1/2)$.

Now, let us show that $1/2 \in \overline{\{x,y\}}_T$. From $T(x,1/2) = \min(x,1/2) = x$, we have that $x \preceq_T 1/2$. Similarly, $y \preceq_T 1/2$ holds. Therefore, we obtain that $1/2 \in \overline{\{x,y\}}_T$. Choose arbitrary $k \in \overline{\{x,y\}}_T$. Then, $x \preceq_T k$ and $y \preceq_T k$. There exist $\ell_1, \ell_2 \in [0,1]$ satisfying that

$$x = T(k, \ell_1) \quad \text{and} \quad y = T(k, \ell_2).$$

Then, $0 \neq x = T(k, \ell_1) = \min(k, \ell_1)$. Since $x$ and $y$ aren't comparable, the case $x = k$ is not possible. Then, $\ell_1 = x \in (0,1/2)$. Hence, $k \notin (0,1/2)$ and $k \neq 0$; i.e, $k \geq 1/2$. Since $1/2 = T(k,1/2) = \min(k,1/2)$, we have that $k \succeq_T 1/2$, and so $x \vee_T y = 1/2$.

Choose arbitrary $k \in \underline{\{x,y\}}_T$. Since $k \preceq_T x$ and $k \preceq_T y$, there exist $x_1, y_1 \in [0,1]$ such that $k = T(x, x_1)$ and $k = T(y, y_1)$. Since $x$ and $y$ aren't comparable with respect to $\preceq_T$, it is not possible $k = x$ or $k = y$. Therefore $k = x_1$ and $k = y_1$. It follows from $k = T(x,k)$ that $k \leq x < 1/2$. Therefore, $x, k \in (0,1/2)$ implies $k = T(x,k) = 0$. So, $x \wedge_T y = 0$. Hence, $([0,1], \preceq_T)$ is a lattice.

**Proposition 3.6.** Let $L$ be a lattice and $T$ be any t-norm on $L$. If $a \preceq_T b$ for $a, b \in L$, then $T(a,c) \preceq_T T(b,c)$ for every $c \in L$.

Proof. Let $a \preceq_T b$ for $a, b \in L$. Then, there exists $x \in L$ such that $T(x, b) = a$. Since $T(a, c) = T(T(x, b), c) = T(x, T(b, c))$, $T(a, c) \preceq_T T(b, c)$. So, this shows that the monotonicity holds. □

**Corollary 3.7.** Let $L$ be a lattice and $T$ be any t-norm on $L$. If $(L, \preceq_T)$ is a lattice, then $T : (L, \preceq_T)^2 \to (L, \preceq_T)$ is a t-norm.

**Proposition 3.8.** Let $(L, \leq)$ be a bounded lattice and $T$ be a t-norm on $L$. If $(L, \preceq_T)$ is a chain, then $T$ is a divisible t-norm; i.e, $\leq = \preceq_T$.

Proof. For $a, b \in L$, let $a < b$ and $a \not\preceq_T b$. Since $(L, \preceq_T)$ is a chain, $b \prec_T a$, and so $b < a$ by Proposition 2.3. This is a contradiction. Therefore, $\preceq_T = \leq$. □

## 4. SOME DETERMINATIONS ON SETS $H_T$ AND $A$

**Proposition 4.1.** Let $L = [0, 1]$ and $T$ be any t-norm on $L$. Then, $(H_T, \preceq_T)$ is a chain.

Proof. Let $a$ and $b$ be two idempotent elements on $[0, 1]$. Since $a$ and $b$ are two elements of $[0, 1]$, we have that $a \leq b$ or $b \leq a$. Suppose that $a \leq b$. Since $a = T(a, a) \leq T(a, b) \leq T(a, 1) = a$, $T(a, b) = a$. So, $a \preceq_T b$. If $b \leq a$, then similarly we obtain that $b \preceq_T a$. Hence, $(H_T, \preceq_T)$ is a chain. □

**Theorem 4.2.** Let $L$ be a complete lattice and $T$ be any t-norm on $L$. Then $a \wedge_T b = T(a, b)$ for every $a, b \in H_T$. If $T$ is an infinitely $\bigvee$-distributive t-norm, then $\bigvee_T \{a_\tau | \tau \in Q\} = \bigvee \{a_\tau | \tau \in Q\}$ for every $\{a_\tau | \tau \in Q\} \subseteq H_T$ and $(H_T, \preceq_T)$ is a complete lattice.

Proof. Since $T(a, b) \preceq_T a$ and $T(a, b) \preceq_T b$, $T(a, b) \in \underline{\{a, b\}}_T$. Let $\ell \in \underline{\{a, b\}}_T$ be arbitrary. This implies that $\ell \preceq_T a$ and $\ell \preceq_T b$. In that case, there exist two elements $a_1, b_1 \in L$ such that

$$\ell = T(a, a_1) = T(b, b_1).$$

Also, $T(\ell, b) = T(T(b, b_1), b) = T(T(b, b), b_1) = T(b, b_1) = \ell$ and similarly $T(\ell, a) = T(T(a, a_1), a) = T(T(a, a), a_1) = T(a, a_1) = \ell$ holds. Since $\ell = T(\ell, a) \preceq_T T(b, a) = T(a, b)$ by monotonicity of $T$ on $(L, \preceq_T)$, we have that $T(a, b)$ is the greatest element of the lower bounds of $\{a, b\}$ with respect to $\preceq_T$ and so $T(a, b) = a \wedge_T b$.

For every $Q$, let us show that $\bigvee_T \{a_\tau | \tau \in Q\} = \bigvee \{a_\tau | \tau \in Q\}$, where $\{a_\tau | \tau \in Q\} \subseteq H_T$. Since, for every $\tau \in Q$, $a_\tau = T(a_\tau, a_\tau) \leq T(\bigvee_{\tau \in Q} a_\tau, \bigvee_{\tau \in Q} a_\tau)$, $\bigvee_{\tau \in Q} a_\tau \leq T(\bigvee_{\tau \in Q} a_\tau, \bigvee_{\tau \in Q} a_\tau) \leq \bigvee_{\tau \in Q} a_\tau$. Then, we have that for every $\tau \in Q$, $\bigvee_{\tau \in Q} a_\tau \in H_T$.

Moreover, $\bigvee_{\tau \in Q} a_\tau \in \overline{\{a_\tau | \tau \in Q\}}_T$. Indeed, since $a_\tau = T(a_\tau, a_\tau) \leq T(\bigvee_{\tau \in Q} a_\tau, a_\tau) \leq a_\tau$; i.e., $T(\bigvee_{\tau \in Q} a_\tau, a_\tau) = a_\tau$, and so $a_\tau \preceq_T \bigvee_{\tau \in Q} a_\tau$. Thus, $\bigvee_{\tau \in Q} a_\tau \in \overline{\{a_\tau | \tau \in Q\}}_T$. Let $k$ be an arbitrary element of the set $\overline{\{a_\tau | \tau \in Q\}}_T$. Then, $a_\tau \preceq_T k$. So, there exists the subset $\{b_\tau | \tau \in Q\} \subseteq L$ such that $a_\tau = T(k, b_\tau)$. Since $T$ is an infinitely $\bigvee$-distributive t-norm, $\bigvee_{\tau\in Q} a_\tau = \bigvee_{\tau\in Q} T(k, b_\tau) = T(k, \bigvee_{\tau\in Q} b_\tau)$. Therefore, $\bigvee_{\tau\in Q} a_\tau \preceq_T k$. This shows that $\bigvee_T\{a_\tau|\tau \in Q\} = \bigvee\{a_\tau|\tau \in Q\}$. By $0 \in H_T$, $(H_T, \preceq_T)$ is a complete lattice. □

The following corollary follows from the proof of Theorem 4.2.

**Corollary 4.3.** If $L$ is a bounded lattice and $T$ is a $\bigvee$-distributive t-norm on $L$, then it is easily obtained that $a \wedge_T b = T(a, b)$ and $a \vee_T b = a \vee b$ for $a, b \in H_T$. Thus, $(H_T, \preceq_T)$ is a bounded lattice.

**Corollary 4.4.** If $T$ is an infinitely $\bigvee$-distributive t-norm on $L$, then $(H_T \preceq_T)$ is a Heyting algebra (see [7], p. 273, for the definition of Heyting algebra).

Proof. For $a \in H_T$ and $\{b_\tau|\tau \in Q\} \subseteq H_T$, since $T$ is an infinitely $\bigvee$-distributive t-norm, $T(a, \bigvee_{\tau\in Q} b_\tau) = \bigvee_{\tau\in Q} T(a, b_\tau)$. By Theorem 4.2, since $T(a, \bigvee_{\tau\in Q} b_\tau) = a \wedge_T (\bigvee_{\tau\in Q} b_\tau)$ and $\bigvee_{\tau\in Q} T(a, b_\tau) = \bigvee_{\tau\in Q}(a \wedge_T b_\tau)$, we obtain that $a \wedge_T (\bigvee_{\tau\in Q} b_\tau) = \bigvee_{\tau\in Q}(a \wedge_T b_\tau)$. Thus $(H_T, \preceq_T)$ is a Heyting algebra. □

**Corollary 4.5.** Let $L$ be a complete lattice, $T$ be an infinitely $\bigvee$-distributive and divisible t-norm. Then, $(H_T, \leq)$ is a Heyting algebra.

Proof. Since $T$ is a divisible t-norm, we have that $\leq = \preceq_T$. For any $a, b \in H_T$, $a \wedge b \preceq_T a$ and $a \wedge b \preceq_T b$, whence $a \wedge b \preceq_T a \wedge_T b$. On the other hand, since $a \wedge_T b = T(a, b) \leq a \wedge b$, it holds that $T(a, b) = a \wedge b$, for $a, b \in H_T$. □

**Corollary 4.6.** Let $L$ be a bounded lattice and $T$ be a $\bigvee$-distributive and divisible t-norm. Then, $(H_T, \leq)$ is a distributive lattice.

Let $L$ be a complete lattice, $T$ be a t-norm on $L$ and $L_1 \subseteq L$. The notation $T \downarrow L_1$ will be used for the restriction of $T$ to $L_1$.

**Proposition 4.7.** Let $(L, \leq)$ be a bounded lattice and $T$ be a t-norm on $L$. Then $T \downarrow H_T$ is a t-norm on $(H_T, \preceq_T)$.

Proof. For every $a, b \in H_T$, we must show that $T(a, b) \in H_T$. Making use associativity of t-norm T, we have that

$$\begin{aligned}
T(T(a,b),T(a,b)) &= T(T(T(a,b),a),b) \\
&= T(T(T(b,a),a),b) = T(T(b,T(a,a)),b) \\
&= T(T(b,a),b) = T(T(a,b),b) \\
&= T(a,T(b,b)) = T(a,b).
\end{aligned}$$

Then, $T(a, b)$ is an element of $H_T$. From Proposition 3.6 it is obvious that the monotonicity of $T$ with respect to $\preceq_T$ and associativity. Since $1 \in H_T$, we have that $T(x, 1) = x$ for every $x \in H_T$. Therefore, $T \downarrow H_T$ is a t-norm on $H_T$. □

**Proposition 4.8.** Let $T$ be a t-norm on a bounded lattice $L$ and $K \subseteq L$ be a lattice with respect to the order on $L$. If $x \wedge_T y = T(x,y)$ for every $x, y \in K$, then $T \downarrow K = \wedge$. Especially, if $K = L$, then $T = \wedge$.

Proof. Since, for every $x \in K$, $x \wedge_T x = x = T(x,x)$, we obtain that $x \wedge y = T(x \wedge y, x \wedge y) \leq T(x,y) \leq x \wedge y$, for every $x, y \in K$. Therefore, $T(x,y) = x \wedge y$. □

**Corollary 4.9.** (Mitsch [20]) Let $L = [0,1]$ and $T$ be any t-norm on $L$. Then $(H_T, \preceq_T) = (H_T, \leq)$

**Definition 4.10.** (Wang [25], Hájek [9]) A triple $(L, \leq, \odot)$ is called an integral residuated $\ell$-monoid if and only if the following three conditions are satisfied:

(i) $(L, \leq, \vee, \wedge, 0, 1)$ is a lattice, where $\vee, \wedge, 0, 1$, respectively, stand for the join operation on $L$, the meet operation on $L$, the bottom element of $L$ and the top element of $L$ and $0 \neq 1$.

(ii) $(L, \odot)$ is a monoid with the identity 1.

(iii) There exists a binary operation $\rightarrow$ on $L$ fulfilling the adjunction property;

$$(AD) \quad x \odot y \leq z \Leftrightarrow x \leq y \rightarrow z, \quad \forall x, y, z \in L.$$

In an integral residuated $\ell$-monoid, the adjunction property $(AD)$ determines $\rightarrow$, uniquely, and $\rightarrow$ is called residuum operation on $L$. An integral residuated $\ell$-monoid $(L, \leq, \odot)$ is called an integral commutative residuated $\ell$-monoid (i.e, a residuated lattice) if $(L, \odot)$ is a commutative monoid.

**Definition 4.11.** (Höhle [10]) Let $M = (L, \leq, \odot)$ be any commutative, integral, residuated $\ell$-monoid. $M$ satisfies the algebraic strong De Morgan's law if for all $x, y \in M$ it holds that $(x \rightarrow y) \vee (y \rightarrow x) = 1$.

**Theorem 4.12.** (Drossos [5]) Let $M = (L, \leq, \odot)$ be an integral, commutative, residuated $\ell$-monoid. If $M$ is divisible and satisfies the algebraic strong De Morgan's law, then the subset $H_T$ of all idempotent elements with respect to $\odot$ forms a Heyting algebra, and the implication in $H_T$ coincides with the implication based on $\odot$.

**Remark 4.13.** In the study of Drossos [5] (Höhle[10], Corollary 2.7), in Theorem 4.12, the algebraic strong De Morgan's law is unnecessary for the proof of this theorem. The following Proposition 4.14 proves that the algebraic strong De Morgan's law is unnecessary for the Theorem 4.12 in the study of Drossos [5] (Höhle[10], Corollary 2.7).

**Proposition 4.14.** Let $M = (L, \leq, \odot)$ be an integral, commutative, residuated $\ell$-monoid. If $M$ is divisible, then the subset $H_T$ of all idempotent elements with respect to $\odot$ forms a Heyting algebra, and the implication in $H_T$ coincides with the implication based on $\odot$.

P r o o f. Firstly, to show that $\odot$ is a t-norm, we must show that $\odot$ satisfy the monotonicity. We obtain that

$$x \odot (x \Rightarrow y) \leq y$$

follows from $(x \Rightarrow y) \leq (x \Rightarrow y)$ by the adjunction property in the Definition 4.10(iii), similarly $x \leq (y \Rightarrow x \odot y)$ follows from $(x \odot y) \leq (x \odot y)$. Let $x, y, z \in L$ and $x \leq y$. $y \leq (z \Rightarrow (y \odot z))$, and so $x \leq (z \Rightarrow (y \odot z))$. Thus, $x \odot z \leq y \odot z$. Since $\odot$ is commutative, for every $x, y, z \in L$ we obtain that $z \odot x \leq z \odot y$. Thus $\odot$ satisfy the monotonicity and so $\odot$ is a t-norm.

Now, let us show that $\odot$ is $\bigvee$-distributive t-norm on $L$. For every $x, y, z \in L$, the inequality $(x \odot z) \vee (y \odot z) \leq (x \vee y) \odot z$ is obvious by monotonicity of $\odot$. Conversely, since $x \odot z \leq (x \odot z) \vee (y \odot z)$, $x \leq z \Rightarrow [(x \odot z) \vee (y \odot z)]$. Similarly, $y \leq z \Rightarrow [(x \odot z) \vee (y \odot z)]$ holds. Thus, $x \vee y \leq z \Rightarrow [(x \odot z) \vee (y \odot z)]$ and so $(x \vee y) \odot z \leq (x \odot z) \vee (y \odot z)$. Hence, $(x \vee y) \odot z = (x \odot z) \vee (y \odot z)$. This implies that $\odot$ is $\bigvee$-distributive t-norm on $L$.

By applying Corollary 4.3, we obtain that $x \wedge_{\odot} y = x \odot y$ for every $x, y \in H_T$. Since $M$ is divisible, by Remark 2.4(ii), $\preceq_{\odot}$ is equal to $\leq$. Thus, $x \wedge y = x \wedge_{\odot} y = x \odot y$ for every $x, y \in H_T$. □

**Proposition 4.15.** Let $T$ be a t-norm without zero divisors on $[0,1]$ and $(L, \preceq_T)$ be a lattice. Then, for every $a, b \in L \setminus \{0\}$, $a \wedge_T b \neq 0$.

P r o o f. For every $a, b \in L \setminus \{0\}$, since $T(a,b) \in \underline{\{a,b\}}_T$, $T(a,b) \preceq_T a \wedge_T b$. By Proposition 2.3, $T(a,b) \leq a \wedge_T b$. If $a \wedge_T b = 0$, then we obtain that $T(a,b) = 0$. This is a contradiction since $T$ is a t-norm without zero divisors on $[0,1]$. □

**Theorem 4.16.** Let $L$ be a complete lattice and $T$ be an infinitely $\bigvee$-distributive t-norm without zero divisors on $L$. Let

$$A = \{a \in L \mid a \text{ is supremum of some family of atoms}\}.$$

Then, $(A, \preceq_T)$ is a complete lattice. Furthermore, in this lattice $a \wedge_T b = T(a,b)$ and $T \downarrow A = \bigwedge_T$.

P r o o f. Let $a, b$ be two elements of $A$. There exist the sets $Q_1 = \{a_\tau | a_\tau \text{ is an atom}\}$ and $Q_2 = \{b_\beta | b_\beta \text{ is an atom}\}$ such that

$$a = \bigvee_{a_\tau \in Q_1} a_\tau, \quad b = \bigvee_{b_\beta \in Q_2} b_\beta.$$

Let us show that if $Q_1 \subseteq Q_2$, then $a \preceq_T b$ $\qquad (*)$

For every atoms $a_\tau, x_\beta, a_\tau \neq x_\beta$, it holds that $T(a_\tau, x_\beta) = 0$. Since $T$ is a t-norm without zero divisors, we note that for every atom $a_\tau$, $T(a_\tau, a_\tau) = a_\tau$. Therefore, we have that

$$\begin{aligned} T(a,b) &= T\left(\bigvee_{a_\tau \in Q_1} a_\tau, \bigvee_{b_\beta \in Q_2} b_\beta\right) \\ &= \left[\bigvee_{\substack{b_\beta \in Q_2 \setminus Q_1 \\ a_\tau \in Q_1}} T(a_\tau, b_\beta)\right] \vee \left[\bigvee_{\substack{b_\beta \in Q_1 \\ a_\tau \in Q_1}} T(a_\tau, b_\beta)\right] \\ &= \bigvee_{\substack{b_\beta \in Q_1 \\ a_\tau \in Q_1}} T(a_\tau, b_\beta) = \bigvee_{a_\tau \in Q_1} T(a_\tau, a_\tau) = \bigvee_{a_\tau \in Q_1} a_\tau = a. \end{aligned}$$

In this case, we have that $a \wedge_T b = a$.

Let $a, b$ be arbitrary elements of $A$. Now, we suppose that $Q_1 \not\subseteq Q_2$ and $Q_2 \not\subseteq Q_1$. There exist the sets $Q_1 = \{a_\tau | a_\tau \text{ is an atom}\}$ and $Q_2 = \{b_\beta | b_\beta \text{ is an atom}\}$ such that $a = \bigvee_{a_\tau \in Q_1} a_\tau$, $b = \bigvee_{b_\beta \in Q_2} b_\beta$.

We want to show that $a \wedge_T b = \bigvee\limits_{x_\gamma \in Q_1 \cap Q_2} x_\gamma$. Using by $(*)$, we obtain that $\bigvee_{x_\gamma \in Q_1 \cap Q_2} x_\gamma \preceq_T a$, and $\bigvee_{x_\gamma \in Q_1 \cap Q_2} x_\gamma \preceq_T b$. Therefore,

$$\bigvee_{x_\gamma \in Q_1 \cap Q_2} x_\gamma \in \underline{\{a,b\}}_T.$$

Let $t \in \underline{\{a,b\}}_T$ be arbitrary. Then, $t \preceq_T a$, $t \preceq_T b$ and $t = \bigvee_{p_\zeta \in Q} p_\zeta$ for some set $Q$ of atoms. In this case, there exist $a_1$ and $b_1 \in L$ such that

$$T(a, a_1) = t \qquad \text{and} \qquad T(b, b_1) = t.$$

Therefore, $t = T(\bigvee_{a_\tau \in Q_1} a_\tau, a_1) = \bigvee_{a_\tau \in Q_1} T(a_\tau, a_1) = \bigvee_{a_\nu \in Q^*} a_\nu$, where $Q^* \subseteq Q_1$. Similarly, it holds that $t = \bigvee_{b_\mu \in Q^{**}} b_\mu$, where $Q^{**} \subseteq Q_2$.

Let $a_\alpha$ be an arbitrary element of $Q^*$. $a_\alpha \preceq_T \bigvee_{a_\nu \in Q^*} a_\nu$ is obvious by $(*)$. Also, $a_\alpha \preceq_T \bigvee_{b_\mu \in Q^{**}} b_\mu$ since $a_\alpha \preceq_T \bigvee_{a_\nu \in Q^*} a_\nu = \bigvee_{b_\mu \in Q^{**}} b_\mu$. Thus, there exists an element $x_1$ of $L$ such that $a_\alpha = T(x_1, \bigvee_{b_\mu \in Q^{**}} b_\mu) = \bigvee_{b_\mu \in Q^{**}} T(x_1, b_\mu)$. If $T(x_1, b_\mu) = 0$, for every $\mu$, then $a_\alpha = 0$, which is a contradiction. Moreover, since $a_\alpha$ is an atom, it is not possible that $a_\alpha = \bigvee_{b_\mu \in Q^{**}} b_\mu$, for $b_\mu$ which is not more than one. So, there exists $\tau_i$ such that $a_\alpha = b_{\tau_i} \in Q^{**}$. Therefore, $Q^* \subseteq Q^{**}$. Similarly, it is easy to show that $Q^{**} \subseteq Q^*$. Thus, $Q^* = Q^{**}$. Since $Q^* = Q^{**} \subseteq Q_1 \cap Q_2$, we have that $t = \bigvee_{x_\beta \in Q^*} x_\beta \preceq_T \bigvee_{x_\gamma \in Q_1 \cap Q_2} x_\gamma$ by using $(*)$. Therefore, $a \wedge_T b = \bigvee_{x_\gamma \in Q_1 \cap Q_2} x_\gamma$.

Choose arbitrary $\{a_\tau |\ \tau \in I\} \subseteq A$. For $\tau \in I$, there exists the set $Q_\tau$ of atoms such that $a_\tau = \bigvee_{x_\nu \in Q_\tau} x_\nu$. Then, let us show that $\bigvee_T a_\tau = \bigvee_{x_\beta \in \bigcup_{\tau \in I} Q_\tau} x_\beta$.

By using $(*)$, we have that $a_\tau \preceq_T \bigvee_{x_\beta \in \bigcup_{\tau \in I} Q_\tau} x_\beta$. Hence, we obtain that $\bigvee_{x_\beta \in \bigcup_{\tau \in I} Q_\tau} x_\beta \in \overline{\{a_\tau |\ \tau \in I\}}_T$.

If $s \in \overline{\{a_\tau |\ \tau \in I\}}_T$, then $a_\tau \preceq_T s$. Let $s = \bigvee_{c_\nu \in Q_*} c_\nu$. Choose arbitrary $y \in Q_\tau$. Using by $(*)$, $y \preceq_T a_\tau$. Since $a_\tau \preceq_T s$ and $y \preceq_T a_\tau$, by transitivity, we obtain that $y \preceq_T s$. Then, there exists $x_1 \in L$ such that

$$y = T(s, x_1) = T\Big(\bigvee_{c_\nu \in Q_*} c_\nu, x_1\Big) = \bigvee_{c_\nu \in Q_*} T(c_\nu, x_1).$$

Similarly, for every $y \in Q_\tau$, there exists $c_\beta \in Q_*$ such that $y = c_\beta \in Q_*$. Then, $Q_\tau \subseteq Q_*$. Since $\tau$ is arbitrary, we have that $\bigcup_{\tau\in I} Q_\tau \subseteq Q_*$.

Since $\bigvee_{x_\beta \in \bigcup_{\tau\in I} Q_\tau} x_\beta \preceq_T \bigvee_{c_\nu \in Q_*} c_\nu = s$, for every $s \in \overline{\{a_\tau | \tau \in I\}}_T$, $\bigvee_{x_\beta \in \bigcup_{\tau\in I} Q_\tau} x_\beta$ is the least element of $\overline{\{a, b\}}_T$. So, $\bigvee_T a_\tau = \bigvee_{x_\beta \in \bigcup_{\tau\in I} Q_\tau} x_\beta$.

As a conclusion, it is easily seen that $a \wedge_T b = \bigvee_{x_\beta \in Q_1 \cap Q_2} x_\beta =$
$= T(\bigvee_{a_\tau \in Q_1} a_\tau, \bigvee_{b_\beta \in Q_2} b_\beta) = T(a, b)$. Thus, for every $a \in A$, $T(a, a) = a \wedge_T a = a$. Here, it is obtained that $T \downarrow A = \bigwedge_T$.

The supremum of all elements of $A$ is the greatest element. Since the supremum on empty set is 0, $0 \in A$. Hence, $T \downarrow A$ is an infinitely $\bigvee_T$-distributive t-norm on $(A, \preceq_T)$. □

**Corollary 4.17.** If the number of atoms on $L$ is finite and $T$ is a $\bigvee$-distributive t-norm without zero divisors, then above Theorem 4.16 is again true.

**Corollary 4.18.** Let $L$ be a complete lattice and $T$ be an infinitely $\bigvee$-distributive t-norm without zero divisors on $L$. Then, $(A, \preceq_T)$ is a Boolean algebra.

Proof. For $a, b, c \in A \backslash \{0\}$ we must show that $a \wedge_T (b \vee_T c) = (a \wedge_T b) \vee_T (a \wedge_T c)$. It is sufficient that $a \wedge_T (b \vee_T c) \leq (a \wedge_T b) \vee_T (a \wedge_T c)$. There exist the sets $Q_1 = \{a_\tau | a_\tau \text{ is an atom}\}$ and $Q_2 = \{b_\beta | b_\beta \text{ is an atom}\}$ such that $b = \bigvee_{a_\tau \in Q_1} a_\tau$, $c = \bigvee_{b_\beta \in Q_2} b_\beta$. Using the proof of Theorem 4.16, $a \wedge_T (b \vee_T c) = T(a, b \vee_T c)$ and $b \vee_T c = (\bigvee_{a_\tau \in Q_1} a_\tau) \vee (\bigvee_{b_\beta \in Q_2} b_\beta) = \bigvee_{x_\beta \in Q_1 \cup Q_2} x_\beta$. Thus,

$$\begin{aligned} a \wedge_T (b \vee_T c) &= T\left(a, \left(\bigvee_{a_\tau \in Q_1} a_\tau\right) \vee \left(\bigvee_{b_\beta \in Q_2} b_\beta\right)\right) \\ &= T\left(a, \bigvee_{a_\tau \in Q_1} a_\tau\right) \vee T\left(a, \bigvee_{b_\beta \in Q_2} b_\beta\right) \\ &= \left(\bigvee_{a_\tau \in Q_1} T(a, a_\tau)\right) \vee \left(\bigvee_{b_\beta \in Q_2} T(a, b_\beta)\right). \end{aligned}$$

Since the elements $a_\tau, b_\beta$ are atoms, $T(a, a_\tau) \preceq_T a \wedge_T b = T(a, b)$ and $T(a, b_\beta) \preceq_T a \wedge_T c = T(a, c)$. So, since $T(a, a_\tau)$ and $T(a, b_\beta)$ are atoms, $(\bigvee_{a_\tau \in Q_1} T(a, a_\tau)) \vee (\bigvee_{b_\beta \in Q_2} T(a, b_\beta)) = (\bigvee_{a_\tau \in Q_1} T(a, a_\tau)) \vee_T (\bigvee_{b_\beta \in Q_2} T(a, b_\beta)) \leq T(a, b) \vee_T T(a, c)$ holds, as required. Let $Q$ be the set of all atoms of $L$. Then it is obvious that the complement of $b$ is $\bigvee_{a_\tau \in Q \backslash Q_1} a_\tau$. □

**Example 4.19.** Let $K$ be a non-empty set, $|K| \geq 3$, $L_K = \{0, p_\tau(\tau \in K), b, 1\}$, $0 < p_\tau < b < 1$, $\tau \in K$, and for $\alpha \neq \beta$, $p_\alpha \wedge p_\beta = 0$. Then, there doesn't exist a $\bigvee$-distributive t-norm without zero divisors on $L_K$. Suppose that $T$ is a $\bigvee$-distributive t-norm without zero divisors on $L_K$. Thus we have that $0 \preceq_T p_\tau(\tau \in K) \preceq_T b$. It is obvious that $A = \{0, p_\tau(\tau \in K), b\}$. Since $p_\tau \wedge p_\beta = 0$ and $p_\tau \vee p_\beta = b$, $A = \{0, p_\tau(\tau \in K), b\}$ is not a Boolean algebra. It contradicts to Corollary 4.18.

Since a relatively complemented lattice $L$ of finite length is atomic (see [1]), the proof of Corollary 4.20 is obtained.

**Corollary 4.20.** If $L$ is a relatively complemented lattice of finite length and $L$ admits a $\bigvee$-distributive t-norm without zero divisors on $L$, then $(L, \leq)$ is a Boolean algebra.

**Corollary 4.21.** If $L$ is an atomic Brouwerian lattice, then $(L, \leq)$ is a Boolean algebra.

**Corollary 4.22.** Let $L$ be a Brouwerian lattice and $T$ be an infinitely $\bigvee$-distributive t-norm without zero divisors. Then

$$T \downarrow (A, \preceq_T) = \bigwedge \downarrow A.$$

**Corollary 4.23.** Let $L$ be a bounded lattice, $T$ be an infinitely $\bigvee$-distributive t-norm without zero divisors and $A(L)$ be the set of atoms. Then, there exists a subset $A$ of $L$ such that $A \approx 2^{A(L)}$ and $A$ is a Boolean algebra.

Proof. Set $A = \{a \in L | \text{a is supremum of some family of atoms}\}$. The mapping $f : A \to 2^{A(L)}$ given by $f(\bigvee_Q p_i) = \bigcup_Q \{p_i\}$ is an isomorphism from $A$ to $2^{A(L)}$. $\square$

**Corollary 4.24.** If $L$ is a distributive lattice and $|A(L)| < \infty$, then $L$ has a sublattice $A$ such that $A \approx 2^{A(L)}$ and $A$ is a Boolean algebra.

## 5. OPEN PROBLEMS

We end this paper with posing some new open problems.

In Example 3.3, we show that $(L = [0,1], \preceq_{T^{nM}})$ is a meet-semilattice, but not a join-semilattice, where $T^{nM}$ is the nilpotent minimum t-norm.

(1) Characterize the class of t-norms on $[0,1]$ which make $(L, \preceq_T)$ a semi lattice (meet-semilattice or join-semilattice).

We show that $(L, \preceq_T)$ is a lattice when $T = T_W$ and in Example 3.5, $([0,1], \preceq_T)$ is a bounded lattice.

(2) Let $L$ be a bounded lattice and $T$ be a t-norm on $L$. Give other examples that $(L, \preceq_T)$ is a lattice.

In Theorem 4.2, we show that $(H_T, \preceq_T)$ is a complete lattice and in Theorem 4.16, we show that $(A, \preceq_T)$ is a complete lattice.

(3) Let $L$ be a complete lattice, $T$ be any t-norm on $L$. Find new ways to define $B \subseteq L$ such that $(B, \preceq_T)$ is a complete lattice.

## 6. CONCLUSIONS

We have determined that some subsets of $L$ which is a lattice with respect to $\preceq_T$. $([0,1], \preceq_{T^{nM}})$ is a meet-semilattice but not a join-semilattice, where $T^{nM}$ is the nilpotent minimum t-norm. If t-norm $T$ defines as in Example 3.5, then $([0,1], \preceq_T)$ is a lattice. When $L$ is a complete lattice and $T$ is any t-norm, $(H_T, \preceq_T)$ is a complete lattice, where $(H_T, \preceq_T)$ is the set of all idempotent elements of t-norm $T$. If $L$ is a complete lattice and $T$ is an infinitely $\bigvee$-distributive t-norm without zero divisors on $L$, then the set of the elements which are supremum of any family of atoms of $L$ is a complete lattice. Furthermore, the infimum of two elements $a, b$ of this lattice is $T(a, b)$.

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