KYBERNETIKA — VOLUME 47 (2011), NUMBER 2, PAGES 207–221

NONCOOPERATIVE GAMES WITH NONCOMPACT JOINT STRATEGIES SETS: INCREASING BEST RESPONSES AND APPROXIMATION TO EQUILIBRIUM POINTS

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In this paper conditions proposed in Flores-Hernández and Montes-de-Oca [3] which permit to obtain monotone minimizers of unbounded optimization problems on Euclidean spaces are adapted in suitable versions to study noncooperative games on Euclidean spaces with noncompact sets of feasible joint strategies in order to obtain increasing optimal best responses for each player. Moreover, in this noncompact framework an algorithm to approximate the equilibrium points for noncooperative games is supplied.

Keywords: monotone maximizer in an optimization problem, noncooperative game, supermodular game, increasing optimal best response for each player, equilibrium point

Classification: 91A10

1. INTRODUCTION

This paper deals with noncooperative games (see [2, 4, 10]) on Euclidean spaces with noncompact sets of feasible joint strategies and with a finite set of players in order to obtain increasing optimal best responses for each player. To meet this goal, the theory on the monotone minimizers for a certain class of minimization problems obtained in [3] is applied in a dual maximization version.

An interesting implication of this monotonicity is that it allows to construct algorithms, which generate a monotone sequence of strategies that converges to an equilibrium point. In fact, in the noncompact framework of this article an algorithm to approximate equilibrium points for noncooperative games is supplied.

The main antecedents for obtaining increasing optimal best responses for each player at noncooperative games are [5, 9, 10, 11] and [12]. In [9] and [10] *compact* sets of feasible joint strategies are assumed. On the other hand, in [5, 11] and [12] there are considered *compact* sets of feasible strategies for each player given strategies of the other players, and the sets of feasible joint strategies as the Cartesian product of these sets. In all these works, certain supermodularity conditions on the payoff functions are taken into account.

The novel contribution here is to remove the compactness in the sets of feasible joint strategies and the hypothesis of supermodularity on the payoff functions. The main assumption to meet this goal is to consider that the payoff functions in the games taken into account are sup-compact and superadditive. Also, observe that with this extension, in particular, it is possible to consider a noncompact version of the two queues in tandem model (see [12]) in order to obtain increasing optimal best responses for each player (see Example 3.4 below).

Furthermore, with the same hypothesis mentioned two paragraphs before, [9] and [10] include two algorithms to approximate the equilibrium points (Round–Robin optimization and simultaneous optimization). [1] and [12] give only one algorithm of a Round–Robin optimization kind requiring also ascending sets of feasible strategies for each player given strategies of the other players.

The contribution here is to present a version of the Round–Robin optimization algorithm with minimal compactness conditions requiring the sets of feasible strategies for each player given strategies of the other players to be decreasing or increasing.

The paper is organized as follows. Section 2 provides basic concepts and results on lattices and the conditions that guarantee the existence of monotone maximizers. In Section 3 the theory on noncooperative games and the existence of increasing optimal best responses for each player are presented. Finally, in Section 4 the algorithm to approximate equilibrium points for noncooperative games is given.

2. MONOTONE MAXIMIZERS IN OPTIMIZATION PROBLEMS

2.1. Terminology and some results of lattice theory

This section contains concepts and results of the lattice theory (see [10]) applied to a Euclidean space, for instance, $\mathbb{R}^n$, where n is a positive integer. For such a space, the partial order $\preceq$ defined componentwise will be used, i. e., if x and y are vectors, then the inequality $x \preceq y$ is understood as $x_i \leq y_i$, for all i (where $\leq$ is the usual order in $\mathbb{R}$). Moreover, $x \wedge y := \inf\{x, y\} = (\inf\{x_1, y_1\}, \ldots, \inf\{x_n, y_n\})$ and $x \vee y := \sup\{x, y\} = (\sup\{x_1, y_1\}, \ldots, \sup\{x_n, y_n\})$. Besides, $[x, \infty) := \{z = (z_1, \ldots, z_n) \in \mathbb{R}^n : x_i \leq z_i < \infty, \text{for all } i\}$ (obviously, if $n = 1$, this is the common notation of a closed-open interval in $\mathbb{R}$).

Following the notation given in [3], let Γ be a fixed subset of $\mathbb{R}^n$. Let Θ be a subset of Γ. $\hat{\gamma}$ is an *upper (lower) bound* for Θ if $\hat{\gamma} \in \Gamma$ and $\theta \preceq \hat{\gamma}$ ($\hat{\gamma} \preceq \theta$) for each $\theta \in \Theta$. $\hat{\gamma}$ is the *greatest (least) element* of Θ if $\hat{\gamma}$ is an upper (lower) bound for Θ and $\hat{\gamma} \in \Theta$. The *supremum* (*infimum*) of Θ is the least upper bound (greatest lower bound), when the set of upper (lower) bounds of Θ has a least (greatest) element. It is denoted by $\sup \Theta$ ($\inf \Theta$). The notation $\sup_\Gamma \Theta$ ($\inf_\Gamma \Theta$) is used as well if the set Γ is not clear from the context.

Γ is said to be a *lattice* if $\gamma_1 \wedge \gamma_2$ and $\gamma_1 \vee \gamma_2 \in \Gamma$, for all $\gamma_1, \gamma_2 \in \Gamma$.

Let Γ be a lattice and let Θ be a subset of Γ. Θ is a *sublattice* of Γ if Θ contains $\theta \wedge \theta'$ and $\theta \vee \theta'$ (with respect to Γ), for all $\theta, \theta' \in \Theta$. For a lattice Γ, $\mathcal{L}(\Gamma)$ denotes the set of all nonempty sublattices of Γ.

Some of the following results have already been used in [3]. Now, in this paper there are included some convenient versions of these results (and others not related to [3]), necessary for taking into account maximization problems instead of minimization ones.

Let Γ_1 and Γ_2 be sets and let Θ be a subset of $\Gamma_1 \times \Gamma_2$. The *section* of Θ in $\gamma_2 \in \Gamma_2$ is $\Theta_{\gamma_2} = \{\gamma_1 : \gamma_1 \in \Gamma_1, (\gamma_1, \gamma_2) \in \Theta\}$ and the *projection* of Θ on Γ_2 is $\Pi_{\Gamma_2}\Theta = \{\gamma_2 : \gamma_2 \in \Gamma_2, \Theta_{\gamma_2} \neq \emptyset\}$.

Lemma 2.1. (Topkis [10], Lemma 2.2.3 (a)) Suppose that Γ_1 and Γ_2 are lattices and Θ is a sublattice of $\Gamma_1 \times \Gamma_2$. Then the section Θ_{γ_2} of Θ at each $\gamma_2 \in \Gamma_2$ is a sublattice of Γ_1.

Let Θ be a sublattice of a lattice Γ. Θ is a *subcomplete* sublattice of Γ if for each nonempty subset Ψ of Θ, $\sup \Psi$ and $\inf \Psi$ exist and are contained in Θ. In fact, a lattice in which every nonempty subset has a supremum and an infimum is *complete.*

Let Γ be a lattice. Let Θ and Υ be subsets of Γ. Θ is *lower* than Υ, written $\Theta \sqsubseteq \Upsilon$, if $\theta \wedge \upsilon \in \Theta$ and $\theta \vee \upsilon \in \Upsilon$ for all $\theta \in \Theta$ and $\upsilon \in \Upsilon$.

Let Γ be a lattice. Let Z be a nonempty subset of $\mathbb{R}^m$, where m is a positive integer. For $x \in Z$, let $\Gamma(x)$ be a nonempty sublattice of Γ. It is said that the correspondence $x \to \Gamma(x)$ is *ascending* if $x \to \Gamma(x)$ is increasing with respect to the relation $\sqsubseteq$, i.e., $\Gamma(x) \sqsubseteq \Gamma(y)$, for $x \preceq y$ in Z.

Lemma 2.2. (Topkis [10], Theorem 2.4.5 (a)) Suppose that Γ_1 and Γ_2 are lattices. If Θ is a sublattice of $\Gamma_1 \times \Gamma_2$, then the section Θ_{γ_2} of Θ at $\gamma_2 \in \Gamma_2$ is ascending in γ_2 on the projection $\prod_{\Gamma_2} \Theta$ of Θ on Γ_2.

Throughout this section let X and A be fixed nonempty Borel subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. For each $x \in X$, let $A(x)$ be a nonempty (measurable) subset of A (i.e., $x \to A(x)$ is a correspondence from X to A). Suppose that $\mathbb{K} := \{(x, a) : x \in X,\ a \in A(x)\}$ is a measurable subset of $X \times A$.

A correspondence $x \to A(x)$ from X to R^m is called *lower hemicontinuous* if for any sequence $\{x^k\}$ in X with a limit point x' in X and any a' in $A(x')$ there exists a sequence $\{a^k\}$ with a^k in $A(x^k)$ for each k and having a' as the limit point.

A function $W : \mathbb{K} \to \mathbb{R}$ is *superadditive* (it has isotone or increasing differences) on $\mathbb{K}$ if $W(y, a) + W(x, b) \leq W(y, b) + W(x, a)$ for all $x \preceq y$ in X and $a \preceq b$, with $a, b \in A(x) \cap A(y)$. W is called *subadditive* (it has antitone or decreasing differences) on $\mathbb{K}$ if $-W$ is superadditive on $\mathbb{K}$.

Let $\mathbb{K}$ be a lattice. A function $\omega : \mathbb{K} \to \mathbb{R}$ is *supermodular* on $\mathbb{K}$ if $\omega(k) + \omega(k') \leq \omega(k \vee k') + \omega(k \wedge k')$, for each k and $k' \in \mathbb{K}$. w is called *submodular* if $-w$ is supermodular.

Lemma 2.3. Let V, W and ω be functions from $\mathbb{K}$ to $\mathbb{R}$.

a) If V and W are superadditive functions, then $V + W$ is superadditive.

b) Suppose that $\mathbb{K}$ is a lattice. If v and w are supermodular on $\mathbb{K}$, then $v + w$ is supermodular on $\mathbb{K}$ as well.

c) Let $\mathbb{K}$ be a lattice. If W is superadditive on $\mathbb{K}$, then W is supermodular on $\mathbb{K}$.

d) Let $\mathbb{K}$ be a lattice. If $\omega(\cdot, \cdot)$ is supermodular, then $\omega(x, \cdot)$ is also supermodular, for each $x \in X$.

e) Let $\mathbb{K}$ be a lattice. If W is a superadditive, increasing, and real-valued function on K, and h is a convex, increasing, and real-valued function on the real line, then $h \circ W$ is a superadditive function on $\mathbb{K}$.

f) If $\overline{W} : \mathbb{R}^n \to \mathbb{R}$ is twice differentiable, then $\overline{W}$ is superadditive if and only if $\frac{\partial^2 \overline{W}(x)}{\partial x_i \partial x_j} \geq 0$ for all distinct i and j and all x.

P r o o f. a) is proved in [3]. The proof of b) is a direct consequence of the definition of a supermodular function. c) is a consequence of Theorem 10.12 in [7] considering Z as $\mathbb{K}$ in the corresponding proof. d) follows from Lemma 2.4 c) in [3] since $-w$ is a submodular function. e) is a consequence of the dual results for supermodular functions which appear in Table I and Theorem 3.1 of [8]. Finally, f) is a result that appears in [10], p. 42. □

Let $G : \mathbb{K} \to \mathbb{R}$ be a function, which is measurable and bounded above (for instance, a nonpositive one), and consider the following maximization problem:

$$\max_{a \in A(x)} G(x,a), \tag{1}$$

$x \in X$ (recall that $x \to A(x)$ is a correspondence from X to A). Also, for each $x \in X$, define $A^*(x)$ by

$$A^*(x) := \left\{ a \in A(x) : G(x,a) = \max_{a^* \in A(x)} G(x,a^*) \right\}.$$

Assumption 2.4. a) G is an upper semicontinuous (u.s.c.) function on $\mathbb{K}$.
b) G is sup-compact on $\mathbb{K}$, that is, for every $x \in X$ and $\bar{s} \in \mathbb{R}$, the set $A_{\bar{s}}(x) := \{a \in A(x) : G(x,a) \geq \bar{s}\}$ is compact.

Lemma 2.5. (Rieder [6], Theorem 4.1) Assumption 2.4 implies that there exists a measurable function $g : X \to A$ such that $g(x) \in A^*(x)$, $x \in X$, i. e. g is a maximizer for (1). In particular, observe that $A^*(x) \neq \emptyset$, for every $x \in X$.

Lemma 2.6. Assumption 2.4 implies that $A^*(x)$ is a compact set, for each $x \in X$.

P r o o f. The proof of Lemma 2.6 is similar to the proof of Lemma 2.6 in [3] using the fact that G is u.s.c. and sup-compact instead of l.s.c. and inf-compact, because now a maximization problem is considered instead of a minimization one. □

Lemma 2.7. (Topkis [10], Theorem 2.7.1) For each $x \in X$, suppose that $A(x)$ is a lattice and $G(x, \cdot)$ is supermodular. Then, for every $x \in X$, $A^*(x)$ is a sublattice of $A(x)$.

Lemma 2.8. (Topkis [10], Theorem 2.8.1) Suppose that Assumption 2.4 holds. If A is a lattice, $x \to A(x)$ is ascending, $A(y) \subset A(x)$ for $x \preceq y$ in X, $G(x, \cdot)$ is supermodular, for each $x \in X$, and G is superadditive on $\mathbb{K}$, then $x \to A^*(x)$ is ascending.

Remark 2.9. Lemma 2.8 is also valid when $A(x) \subset A(y)$ is considered instead of $A(y) \subset A(x)$ for $x \preceq y$ in X. To do this, it suffices to substitute the expression (2.8.2) in [10], p. 75 by

$$0 \leq f(y,b) - f(y, a \vee b) \leq f(y, a \wedge b) - f(y,a) \leq f(x, a \wedge b) - f(x,a) \leq 0.$$

2.2. Increasing maximizers of superadditive functions

In this subsection, problem (1) stated in Subsection 2.1 will be referred to.

There will be presented a result which allows to obtain increasing maximizers in unbounded optimization problems. This result extends, in the context of Euclidean spaces, a previous one obtained by Topkis [10] (see Theorem 2.8.3 in [10]). The result in question is also a direct consequence of Theorem 3.2 in [3].

Lemma 2.10. Suppose that A and $\mathbb{K}$ are lattices. If $x \to A(x)$ is ascending (in particular, for each $x \in X$, $A(x)$ is a sublattice of A), $A(y) \subset A(x)$ (or $A(x) \subset A(y)$) for $x \preceq y$ in X, G is superadditive on $\mathbb{K}$, and Assumption 2.4 holds, then for $f(x) := \sup A^*(x)$, $x \in X$, it is obtained that $f(x) \preceq f(y)$, for all $x \preceq y$. Moreover, $f(x) \in A^*(x)$ for every $x \in X$, i. e., f is a maximizer for (1).

P r o o f. The proof of Lemma 2.10 is similar to the proof of Theorem 3.2 in [3]. In fact, a maximization problem is considered here instead of a minimization one. Specifically, to obtain the proof of Lemma 2.10, it is necessary to consider the proof of Theorem 3.2 in [3] with the following changes: a) substitute a subadditive G by a superadditive one, and b) instead of applying Lemmas 2.4 b), 2.6 and 2.8, use Lemmas 2.3 c), 2.6, 2.8 (and Remark 2.9). □

Remark 2.11. For Lemma 2.10, the function $f'(x) := \inf A^*(x)$, $x \in X$, also works as an increasing maximizer for (1), using Lemma 2.3 b) of [3].

Example 2.12. Consider $X = A = \mathbb{Z}$ (where $\mathbb{Z}$ is the set of integers). Take $A(x) = [x, \infty) \cap \mathbb{Z}$, $x \in X$, and define $G(x,a) = -e^{a-x}$, $(x,a) \in \mathbb{K}$.

Lemma 2.13. Example 2.12 satisfies the assumptions of Lemma 2.10. (Therefore, $f(x) := \sup A^*(x)$, $x \in X$, is an increasing maximizer.)

P r o o f. This example is a direct consequence of Example 3.2 in [3]. In [3] in Lemma 3.2 it is proved that $-G$ is a subadditive and inf-compact function on $\mathbb{K}$; therefore, G is a superadditive and sup-compact function on $\mathbb{K}$, respectively, by definition. □

3. INCREASING OPTIMAL BEST RESPONSES FOR NONCOOPERATIVE GAMES

3.1. Noncooperative games

A *noncooperative game* (see [10]) is a triple $\{N, \mathbb{K}, \{f_i : i \in N\}\}$ consisting of a nonempty set of players N, a set $\mathbb{K}$ of feasible joint strategies, and a collection of payoff functions $\{f_i : i \in N\}$ such that the payoff function $f_i(x)$ is defined on $\mathbb{K}$

for each player $i \in N$. The set of players N is assumed to be finite and it takes the form $N = \{1, \ldots, n\}$ where $n = |N|$. The *strategy* of player i is an m_i-vector x_i. Let $m = \sum_i^n m_i$. A *joint strategy* is an $m-$vector $x = (x_1, \ldots, x_n)$ composed of the strategies x_i of each of the n players. The set of feasible joint strategies $\mathbb{K}$ is a subset of $\mathbb{R}^m$. The *payoff function* for each player $i \in N$ is a real-valued function f_i defined on $\mathbb{K}$ such that for any feasible joint strategy x, player i receives the utility $f_i(x)$.

For any joint strategy x and any player i, let x_{-i} denote the vector of strategies of all players in N except player i. For any joint strategy x, any player i, and any m_i-vector a_i, let (x_{-i}, a_i) denote the joint strategy vector with the strategy x_i of player i replaced by a_i in x and the other components of x left unchanged. Then $x = (x_{-i}, x_i)$ for any joint strategy x and any player i. The set of feasible strategies for player i given the strategies x_{-i} for the other players is denoted by

$$A_i(x_{-i}) = \{a_i : (x_{-i}, a_i) \in \mathbb{K}\};$$

that is, $A_i(x_{-i})$ is the section of $\mathbb{K}$ at x_{-i}. For any player i, let

$$X_{-i} = \{x_{-i} : A_i(x_{-i}) \neq \emptyset\}$$

be the collection of all vectors x_{-i} of strategies for players other than i such that there is some strategy a_i for player i with (x_{-i}, a_i) being a feasible joint strategy; that is, X_{-i} is the projection of $\mathbb{K}$ on the coordinates of the strategies of all players except player i. For any player i, let

$$A_i = \bigcup_{x_{-i} \in X_{-i}} A_i(x_{-i})$$

be the set of all his strategies that are a component of any feasible joint strategy; that is, A_i is the projection of $\mathbb{K}$ on the coordinates of the strategy of player i. Note that x_{-i} is in $\mathbb{R}^{m-m_i}$, (x_{-i}, a_i) is in $\mathbb{R}^m$, $A_i(x_{-i})$ is a subset of $\mathbb{R}^{m_i}$, X_{-i} is a subset of $\mathbb{R}^{m-m_i}$, and A_i is a subset of $\mathbb{R}^{m_i}$. For $x \in \mathbb{K}$, define

$$\mathbb{K}(x) = (\times_{i \in N} A_i(x_{-i})) \cap \mathbb{K},$$

where $\times_{i \in N} A_i(x_{-i})$ is the usual Cartesian product of the sets $A_i(x_{-i})$, for $i \in N$.

Note that $\mathbb{K} = \times_{i \in N} A_i$ if and only if $\mathbb{K}(x) = \mathbb{K}$ for each x in $\mathbb{K}$.

For each vector x_{-i} in X_{-i}, the set of the *best responses* for player i is defined as:

$$A_i^*(x_{-i}) := \left\{ a_i \in A_i(x_{-i}) : f_i(x_{-i}, a_i) = \max_{a_i^* \in A_i(x_{-i})} f_i(x_{-i}, a_i^*) \right\},$$

given x_{-i}. For each player i, the correspondence $x_{-i} \to A_i^*(x_{-i})$ from X_{-i} to A_i is called the *best response correspondence*.

For each feasible joint strategy x in $\mathbb{K}$ and each $a \in \mathbb{K}(x)$, define

$$G(x, a) = \sum_{i \in N} f_i(x_{-i}, a_i).$$

For each feasible joint strategy $x \in \mathbb{K}$, the set of *best joint responses* is given by:

$$A^*(x) = \left\{ a \in \mathbb{K}(x) : G(x,a) = \max_{a^* \in \mathbb{K}(x)} G(x,a^*) \right\}$$

and it consists of all feasible joint strategies such that the strategy for each player i is feasible given x_{-i}, and the sum of the payoffs to the n players is maximized given that player i receives the payoff resulting from using the strategy for player i instead of x_i in x. The correspondence $x \to A^*(x)$ from $\mathbb{K}$ to $\mathbb{K}$ is called the *best joint response correspondence.*

A noncooperative game $\{N, \mathbb{K}, \{f_i : i \in N\}\}$ is a *supermodular game* if the set $\mathbb{K}$ of feasible joint strategies is a sublattice of $\mathbb{R}^m$, the payoff function $f_i(x_{-i}, a_i)$ is supermodular in a_i on $A_i(x_{-i})$ for each x_{-i} in X_{-i} and each player i, and $f_i(\cdot,\cdot)$ is superadditive on $\mathbb{K}_i := \{(x_{-i}, a_i) : x_{-i} \in X_{-i}, a_i \in A_i(x_{-i})\}$, for each i.

Remark 3.1. **a**) In the definition of a supermodular game, $f_i(x_{-i}, a_i)$ is supermodular in a_i on $A_i(x_{-i})$ for each x_{-i} in X_{-i} and each player i, as a consequence of the fact that $f_i(\cdot,\cdot)$ is superadditive on $\mathbb{K}_i$ for each i (see Lemma 2.3 c) and d)).
b) Notice that $\mathbb{K}_i = \mathbb{K}$, for each i.

A feasible joint strategy x is an *equilibrium point* if

$$f_i(x_{-i}, a_i) \leq f_i(x)$$

for each a_i in $A_i(x_{-i})$ and each $i \in N$, that is, if x is in $\mathbb{K}$ and x_i is in $A_i^*(x_{-i})$ for each i. Given an equilibrium point, there is no feasible way for any player to strictly improve its utility if the strategies of all the other players remain unchanged.

The following well-known result characterizes the equilibrium points of a noncooperative game as the fixed points of the best joint response correspondence.

Lemma 3.2. (Topkis [10], Lemma 4.2.1) The set of all equilibrium points for a noncooperative game $(N, \mathbb{K}, \{f_i, i \in N\})$ is identical to the set of fixed points of the correspondence $x \to A^*(x)$ from $\mathbb{K}$ to $\mathbb{K}$.

In the rest of this section, the theory developed in Subsection 2.2 will be applied to a certain class of supermodular games to obtain increasing optimal best responses for each player i, on Euclidean spaces.

3.2. Increasing optimal best responses for player i

Theorem 3.3. Consider a supermodular game $\{N, \mathbb{K}, \{f_i : i \in N\}\}$ for which the set $\mathbb{K}$ of feasible joint strategies is nonempty, and for each i, A_i is a lattice on $\mathbb{R}^{m_i}$, $A_i(\cdot)$ is decreasing (or increasing), and the payoff function $f_i(\cdot,\cdot)$ is u.s.c. and sup-compact on $\mathbb{K}_i$; the last consideration means that for all $x_{-i} \in X_{-i}$ and $\bar{s} \in \mathbb{R}$, $A_{\bar{s}}(x_{-i}) := \{a_i \in A_i(x_{-i}) : f_i(x_{-i}, a_i) \geq \bar{s}\}$ is compact. Then

i) The set $A_i^*(x_{-i})$ of the best responses for each player i is a nonempty compact sublattice of R^{m_i} for each x_{-i} in X_{-i}.

ii) For each i and each x_{-i} in X_{-i}, there exists the supremum of $A_i^*(x_{-i})$, and it belongs to $A_i^*(x_{-i})$ (in fact, the supremum is called the *optimal best response* for each player i and each x_{-i} in X_{-i}).

iii) The best response correspondence $x_{-i} \to A_i^*(x_{-i})$ is ascending on X_{-i} for each player i.

iv) The optimal best response is an increasing function from X_{-i} into A_i for each player i.

Proof. $A_i^*(x_{-i}) := \{a_i \in A_i(x_{-i}) : f_i(x_{-i}, a_i) = \max_{a_i^* \in A_i(x_{-i})} f_i(x_{-i}, a_i^*)\}$ is nonempty and compact because f_i satisfies Assumption 2.4 (see Lemmas 2.5 and 2.6), for each i, by hypothesis. Because $\mathbb{K}$ is a sublattice of $\mathbb{R}^m$, its section $A_i(x_{-i})$ is a sublattice of R^{m_i} for each i and each x_{-i} in X_{-i} by Lemma 2.1. Since each payoff function $f_i(x_{-i}, a_i)$ is supermodular in a_i on $A_i(x_{-i})$, for each x_{-i} in X_{-i}, by hypothesis (see also Remark 3.1 a)), and $A_i(x_{-i})$ is a lattice of $\mathbb{R}^{m_i}$, for each i, $A_i^*(x_{-i})$ is a sublattice of $A_i(x_{-i})$, for each i, as a consequence of Lemma 2.7. Thus $A_i^*(x_{-i})$ is a compact sublattice of $A_i(x_{-i})$. Part (ii) now follows from Theorem 2.3.1 of [10].

Because $\mathbb{K}$ is a sublattice of $\mathbb{R}^m$, the section $A_i(x_{-i})$ is ascending in x_{-i} on the projection X_{-i} for each i by Lemma 2.2. Parts (iii) and (iv) follow from Lemma 2.10 considering that $A = A_i$, $\mathbb{K} = \mathbb{K}_i$ (see Remarks 2.11 and 3.1 b)), for each i. □

3.3. Example

Example 3.4. Two queues in tandem (see [12]).
Consider two single-server queues in tandem. Each server has *iid* (independent and identically distributed) exponential service times, with nonnegative rates μ_1 and μ_2, respectively. Assume $\mu \geq \mu_1 \vee \mu_2$. Server 1 has an infinite source of input jobs, and there is a finite buffer between server 1 and server 2. The throughput of the system is $\mu_1 \wedge \mu_2$; and the expected number of jobs in the buffer is equal to $\mu_1/(\mu_2 - \mu_1)$ if $\mu_1 < \mu_2$, and equal to $M > 0$ if it is otherwise (see Example 2.4, p. 455 in [12]). For $i = 1, 2$, let $p_i(\mu_1 \wedge \mu_2)$ be the profit function for service i, and $c_i(\mu_i)$ be the operating cost function. Suppose that it is also an inventory cost function $g(\cdot)$ for the jobs in the buffer.

The two servers (players) maximize the following payoff functions respectively:

$$f_1(\mu_1, \mu_2) := p_1(\mu_1 \wedge \mu_2) - c_1(\mu_1) - g\left(\frac{\mu_1}{\mu_2 - \mu_1}\right),$$

and

$$f_2(\mu_1, \mu_2) := p_2(\mu_1 \wedge \mu_2) - c_2(\mu_2) - g\left(\frac{\mu_1}{\mu_2 - \mu_1}\right),$$

with

$$A_1(\mu_2) = \begin{cases} \{\mu_1 : 0 \leq \mu_1 < \mu_2\}, & \mu_2 \neq \mu \\ \{\mu_1 : 0 \leq \mu_1 \leq \mu_2\}, & \mu_2 = \mu, 0 \end{cases}.$$

$$A_2(\mu_1) = \begin{cases} \{\mu_2 : \mu_1 < \mu_2 \leq \mu\}, & \mu_1 \neq 0 \\ \{\mu_2 : \mu_1 \leq \mu_2 \leq \mu\}, & \mu_1 = 0 \end{cases}.$$

Lemma 3.5. Let $g(\cdot)$ be an increasing, convex, bounded, twice differentiable and real-valued function on $\mathbb{R}^2$, let $p_i(\cdot)$ be an increasing, continuous, convex and bounded function on the real line, and let $c_i(\cdot)$ be a continuous and bounded function, for $i = 1, 2$. Then the game $(N, \mathbb{K}, \{f_i : i \in N\})$ of Example 3.4 is a supermodular one, and the optimal best response for player i is increasing.

Proof. It is not difficult to observe that $A_1(\cdot)$ is increasing (i.e. $A_1(x) \subset A_1(y)$ for $x \leq y$), $A_2(\cdot)$ is decreasing (i.e. $A_2(y) \subset A_2(x)$ for $x \leq y$), and that A_i (for $i = 1, 2$) and $\mathbb{K}$ are lattices.

It is possible to verify that $p_i(\mu_1 \wedge \mu_2)$ for $i = 1, 2$ is superadditive applying Lemma 2.3 e), due to $W(\mu_1, \mu_2) := \mu_1 \wedge \mu_2$ is an increasing, superadditive and real-valued function on $\mathbb{K}_i = \{(\mu_{-i}, \mu_i) : \mu_{-i} \geq 0, \mu_i \in A_i(\mu_{-i})\}$ for $i = 1, 2$, and using the fact that $p_i(\cdot)$ is an increasing and convex function, for $i = 1, 2$. Also, Lemma 2.3 f) implies that $g\left(\frac{\mu_1}{\mu_2 - \mu_1}\right)$ is a superadditive function. Thus, f_i is superadditive on $\mathbb{K}_i$ for each i, as a consequence of the above and of applying Lemma 2.3 a).

By Lemma 2.3 b), $f_1(\cdot, \mu_2)$ and $f_2(\mu_1, \cdot)$ are trivially supermodular, for all $\mu_2 \in A_2(\mu_1)$ and all $\mu_1 \in A_1(\mu_2)$, respectively; it is also due to Remark 3.1 a).

Thus, the game described here is a supermodular one.

Furthermore, observe that f_i is a continuous function on $\mathbb{K}_i$, for each i, and it is sup-compact on $\mathbb{K}_i$ due to the continuity and boundedness of f_i for $i = 1, 2$. Thus, the hypotheses of Theorem 3.3 are satisfied, concluding so that Example 3.4 has an increasing optimal best response for player i. □

Remark 3.6. Observe that the sets $A_1(\mu_2)$, $\mu_2 \neq \mu$ and $A_2(\mu_1)$, $\mu_1 \neq 0$ are noncompact. The set $\mathbb{K}$ is noncompact as well.

4. ALGORITHMS TO APPROXIMATE AN EQUILIBRIUM POINT

The following algorithm corresponds to the iterative decision-making process by which the n players take turns with each player successively maximizing that player's own payoff function with respect to its own feasible strategies while the strategies of the other $n - 1$ players are held fixed; that is, each individual player proceeds in a Round–Robin fashion (see [10] pp. 185–188) to update his own strategy by selecting a best response.

4.1. Round–Robin optimization

The following algorithm will be used under the assumptions of Theorem 4.2 below in order to generate an infinite sequence of feasible joint strategies; besides, a stopping rule is provided.

Algorithm 4.1. Given a noncooperative game $(N, \mathbb{K}, \{f_i : i \in N\})$, proceed as follows:

a) If $\mathbb{K}$ contains its infimum, $\inf \mathbb{K}$, set $x^{0,0} = \inf(\mathbb{K})$. Otherwise, stop.

b) Given $x^{k,i}$ in $\mathbb{K}$ for any nonnegative integers k and i with $i < n$, let $x^{k,i+1} = (x^{k,i}_{-(i+1)}, a^{k,i+1}_{i+1})$ where $a^{k,i+1}_{i+1}$ is the infimum of the set $A^*_{i+1}(x^{k,i}_{-(i+1)})$ if such an element exists. Otherwise, stop.

c) Increment i by 1. If $i = n$ then, to set $x^{k+1,0} = x^{k,n}$, increment k by 1, and take $i = 0$; $x^{k,n}$ has been generated for some k. Return to step b) and continue.

Theorem 4.2 establishes monotonicity and stops when applied to a supermodular game with certain characteristics. Part (a) of Theorem 4.2 shows that the sequence $x^{k,i}$ generated by Algorithm 4.1 is increasing in k and i. This reduces the problem of finding $x^{k,i+1}$ given $x^{k,i}$ for $i < n$ in step b) of Algorithm 4.1 from a maximization problem over $A_{i+1}(x^{k,i}_{-(i+1)})$ to a maximization problem over $A_{i+1}(x^{k,i}_{-(i+1)}) \cap [x^{k,i}_{i+1}, \infty)$.

Theorem 4.2. Consider a supermodular game $(N, \mathbb{K}, \{f_i : i \in N\})$ for which the set $\mathbb{K}$ of feasible joint strategies is nonempty such that $S = \inf \mathbb{K} \in \mathbb{K}$, and for each i, the set A_i is a lattice on $\mathbb{R}^{m_i}$, the set $A_i(S_{-i})$ is compact if the set $A_i(\cdot)$ is decreasing (where S_{-i} denotes the infimum of $\mathbb{K}$ except the ith component), and the set A_i is compact if $A_i(\cdot)$ is increasing, and the payoff function $f_i(x_{-i}, a_i)$ is u.s.c. and sup-compact on $\mathbb{K}_i$. Then

a) Algorithm 4.1 never stops at step a) or step b) and this generates an infinite sequence $x^{k,i}$ that is increasing in k and i for $k = 0, 1, \dots$ and $i = 0, \dots, n$. Hence, there exists a feasible joint strategy x' in $\mathbb{K}$ such that $\lim_{k\to\infty} x^{k,i} = x'$ for $i = 0, \dots, n$.

b) If a feasible joint strategy appears n successive times in the sequence $\{x^{k,i} : k \geq 0, 1 \leq i \leq n\}$ generated by step b) of Algorithm 4.1, then this joint strategy is an equilibrium point.

c) If Algorithm 4.1 generates an equilibrium point at some iteration, then it generates the same equilibrium point at each subsequent iteration.

Proof. Algorithm 4.1 does not stop at step (a) because the nonempty sublattice $\mathbb{K}$ contains its infimum, and it does not stop at step (b) by part (ii) of Theorem 3.3. Therefore, steps (a) and (b) of Algorithm 4.1 proceed without stopping, and this generates an infinite sequence of feasible joint strategies.

Since $x^{0,0}$ is the infimum of the sublattice $\mathbb{K}$, $x^{0,i} \leq x^{0,i+1}$ for $i = 0, \dots, n-1$; specifically, $x^{0,i}_{i+1} \leq x^{0,i+1}_{i+1} = a^{0,i+1}_{i+1}$ for $i = 0, \dots, n-1$, where $a^{0,i+1}_{i+1}$ is the infimum of $A^*_{i+1}(x^{0,i}_{-(i+1)}) \subset A_{i+1}(x^{0,i}_{-(i+1)})$ and $x^{0,i}_{i+1} \in A_{i+1}(x^{0,i}_{-(i+1)})$. In addition, $x^{0,i} \leq x^{1,i}$ for $i = 0, \dots, n$, because $x^{1,i} \geq x^{1,0}$ (since $x^{1,i} \geq x^{1,i-1}$ for $i = 1, \dots, n$), and $x^{1,0} := x^{0,n} \geq x^{0,i}$, for $i = 0, \dots, n$. Now suppose that integers k' and i' are such that $1 < k'$, $0 \leq i' \leq n-1$, $x^{k,i} \leq x^{k,i+1}$ for all $k = 1, \dots, k'-1$ and $i = 0, \dots, n-1$. This supposition implies that $x^{k,i} \leq x^{k+1,i}$ for all $k = 1, \dots, k'-1$ and $i = 0, \dots, n$ because $x^{k+1,i} \geq x^{k+1,0} := x^{k,n} \geq x^{k,i}$ for $i = 0, \dots, n$. Suppose also that $x^{k',i} \leq x^{k',i+1}$ for $i = 0, \ldots, i'-1$. Since this supposition holds for $k' = 1$ and $i' = 1$, it suffices to show that $x^{k',i'} \leq x^{k',i'+1}$. Because $a_{i'+1}^{k'-1,i'+1}$ is the infimum of $A_{i'+1}^*(x_{-(i'+1)}^{k'-1,i'})$, $a_{i'+1}^{k',i'+1}$ is the infimum of $A_{i'+1}^*(x_{-(i'+1)}^{k',i'})$, and $x^{k'-1,i'} \leq x^{k',i'}$ by the induction hypothesis, part (iv) of Theorem 3.3 implies that $a_{i'+1}^{k'-1,i'+1} \leq a_{i'+1}^{k',i'+1}$ and hence

$$x^{k',i'} = (x_{-(i'+1)}^{k',i'}, a_{i'+1}^{k'-1,i'+1}) \leq (x_{-(i'+1)}^{k',i'}, a_{i'+1}^{k',i'+1}) = x^{k',i'+1}.$$

Therefore, $x^{k,i}$ is an infinite sequence increasing in k and i, for $k \geq 0$ and $i = 0, \ldots, n$; specifically, $a_i^{k,i} \leq a_i^{k+1,i}$ for $k \geq 0$ and $i = 0, \ldots, n$.

This result and the fact that $A_i(\cdot)$ is decreasing for some i, by hypothesis, implies that

$$a_i^{k,i} \in A_i^*(x_{-i}^{k,i-1}) \subset A_i(x_{-i}^{k,i-1}) \subset \ldots \subset A_i(x_{-i}^{0,i-1}) \subset A_i(x_{-i}^{0,0}),$$

for $k \geq 0$. Then, as $a_i^{k,i} \in A_i(x_{-i}^{0,0)})$ is an increasing sequence in k, for each i fixed, and $A_i(x_{-i}^{0,0})$ is a compact set, by hypothesis, $a_i^{k,i}$ converges to some $x_i' \in A_i(x_{-i}^{0,0})$, for this kind of i.

Now, if $A_i(\cdot)$ is increasing for the rest of i, the result obtained implies that

$$a_i^{k,i} \in A_i^*(x_{-i}^{k,i-1}) \subset A_i(x_{-i}^{k,i-1}) \subset A_i,$$

for $k \geq 0$. Then, as $a_i^{k,i} \in A_i$ is an increasing sequence in k, for each i fixed, and A_i is a compact set, by hypothesis, $a_i^{k,i}$ converges to some $x_i' \in A_i$, for these i.

So $x^{k,i}$ converges to $x' = (x_1', \ldots, x_n')$.

Part (b) follows directly from the definition of an equilibrium point.

Suppose $x^{k,i}$ is an equilibrium point and $i < n$. To establish part (c), it suffices to show that $x^{k,i+1} = x^{k,i}$, or equivalently, that $a_{i+1}^{k,i+1} = a_{i+1}^{k,i}$. Because $x^{k,i}$ is an equilibrium point, $a_{i+1}^{k,i}$ is in $A_{i+1}^*(x_{-(i+1)}^{k,i})$ by Lemma 3.2. By step (b) of Algorithm 4.1, $a_{i+1}^{k,i+1}$ is the infimum of $A_{i+1}^*(x_{-(i+1)}^{k,i})$ and so $a_{i+1}^{k,i+1} \leq a_{i+1}^{k,i}$. By part (a), $a_{i+1}^{k,i} \leq a_{i+1}^{k,i+1}$. Hence, $a_{i+1}^{k,i+1} = a_{i+1}^{k,i}$, and part (c) holds. □

Theorem 4.3 below shows that the limit point of an increasing sequence generated by Algorithm 4.1 is an equilibrium point.

Theorem 4.3. Consider a supermodular game $(N, \mathbb{K}, \{f_i : i \in N\})$ for which the set $\mathbb{K}$ of feasible joint strategies is nonempty, for each i $x_{-i} \to A_i(x_{-i})$ is a lower hemicontinuous correspondence from X_{-i} to $\mathbb{R}^{m_i}$, and the payoff function $f_i(x)$ is continuous in x on K for each i. The limit point of the increasing sequence generated by Algorithm 4.1 is an equilibrium point.

Proof. Let $\{x^{k,i}\}$ be the sequence generated by Algorithm 4.1. By part a) of Theorem 4.2, this sequence is increasing in k and i and it converges to a limit point x'. Pick any i with $1 \leq i \leq n$ and any a_i' in $A_i(x_{-i}')$. Since $\lim_{k\to\infty} x^{k,i} = x'$ and $x_{-i} \to A_i(x_{-i})$ is a lower hemicontinuous correspondence, there exists a_i^k in

$A_i(x^{k,i}_{-i})$ for $k = 0, 1, \ldots$ such that $\lim_{k\to\infty} a^k_i = a'_i$. By the construction of $x^{k,i}_i$ in step b) of Algorithm 4.1, $f_i(x^{k,i}_{-i}, a^k_i) \leq f_i(x^{k,i}_{-i}, x^{k,i}_i) = f_i(x^{k,i})$ for each k (recall that $a^k_i \in A_i(x^{k,i}_{-i})$ and

$$\begin{aligned} x^{k,i}_i &= \inf A^*_i(x^{k,i}_{-i}) \\ &= \inf \left\{ a_i \in A_i(x^{k,i}_{-i}) : f_i(x^{k,i}_{-i}, a_i) = \max_{a^*_i \in A_i(x^{k,i}_{-i})} f_i(x^{k,i}_{-i}, a^*_i) \right\} \end{aligned}$$

for each k). By the continuity of $f_i(x)$,

$$f_i(x'_{-i}, a'_i) = \lim_{k\to\infty} f_i(x^{k,i}_{-i}, a^k_i) \leq \lim_{k\to\infty} f_i(x^{k,i}) = f_i(x').$$

Hence, x' is an equilibrium point. □

Lemma 4.4. Example 3.4 has an equilibrium point and it can be approximated by an increasing sequence of feasible joint strategies applying Algorithm 4.1.

P r o o f. The conclusion of Theorem 4.2 is valid for Example 3.4 since $\inf \mathbb{K} = (0, 0) \in \mathbb{K}$, and $A_2(0) = [0, \mu]$ and $A_1 = [0, \mu]$ are both compact sets. The rest of hypotheses has already been verified in the proof of Lemma 3.5.

Furthermore, it is not difficult to observe that Example 3.4 satisfies the hypotheses of Theorem 4.3, because $x_{-i} \to A_i(x_{-i})$ is a lower hemicontinuous correspondence from X_{-i} to $\mathbb{R}^{m_i}$ for each i, and the rest has already been verified in the proof of Lemma 3.5. □

The following examples illustrate Algorithm 4.1 when the equilibrium point appears in two and in four iterations.

Example 4.5. Consider $A_i(\cdot)$, $i = 1, 2$ as in Example 3.4 with the following specific characteristics in the payoff functions f_1 and f_2:

1. Let p_i be an increasing, continuous, convex and bounded function on the real line with $p_i(0) = 0$, for $i = 1, 2$.
2. Let $c_i(\cdot)$ be an increasing, continuous and bounded function.
3. Let $g(\cdot)$ be an increasing, convex, bounded, twice differentiable and real-valued function on $\mathbb{R}^2$.

Lemma 4.6. In Example 4.5, the equilibrium point is reached in two iterations.

P r o o f. Following Algorithm 4.1, observe that $\inf \mathbb{K} = (0, 0) = \mu^{0,0} = (\mu^{0,0}_1, \mu^{0,0}_2)$, also the facts that $k = 0$ and $i = 0$ imply that $\mu^{0,1} = (a^{0,1}_1, \mu^{0,0}_{-1})$, with

$$\begin{aligned} a^{0,1}_1 &= \inf A^*_1(\mu^{0,0}_{-1}) = \inf A^*_1(\mu^{0,0}_2) = \inf A^*_1(0) \\ &= \inf \left\{ a_1 \in A_1(0) : f_1(0, a_1) = \max_{a^*_1 \in \{0\}} f_1(0, a^*_1) \right\}, \end{aligned} \qquad (2)$$

where

$$f_1(0,a_1^*) = p_1(0 \wedge a_1^*) - c_1(a_1^*) - g\left(\frac{a_1^*}{0-a_1^*}\right)$$
$$= p_1(0) - c_1(a_1^*) - g(M).$$

Then $a_1^* = 0$, $a_1^{0,1} = 0$, and $\mu^{0,1} = (0,0)$.

Now for $i = 1$, $\mu^{0,2} = (\mu_{-2}^{0,1}, a_2^{0,2})$, with

$$a_2^{0,2} = \inf A_2^*(\mu_{-2}^{0,1}) = \inf A_2^*(\mu_1^{0,1}) = \inf A_2^*(0)$$
$$= \inf\left\{a_2 \in A_2(0) : f_2(0,a_2) = \max_{a_2^* \in [0,\mu]} f_2(0,a_2^*)\right\}, \tag{3}$$

where

$$f_2(0,a_2^*) = p_2(0 \wedge a_2^*) - c_2(a_2^*) - g\left(\frac{0}{a_2^*-0}\right)$$
$$= p_2(0) - c_2(a_2^*) - g(0),$$

and as c_2 is an increasing function on $[0,\mu]$, the maximum of f_2 ocurrs in $a_2^* = 0$. Thus $a_2^{0,2} = 0$ and $\mu^{0,2} = (0,0)$. Then $\mu^{0,1} = (0,0) = \mu^{0,2}$ implies that $(0,0)$ is an equilibrium point, by Theorem 4.2 b) and c). □

Example 4.7. Consider $A_i(\cdot)$, $i = 1,2$ as in Example 3.4 with the following specific characteristics in the payoff functions f_1 and f_2:

1. Let p_i be a constant function on the real line, i. e. $p_i(\cdot) = c$, for $c \in \mathbb{R}$.
2. Let $c_1(\mu_1) = \left(\mu_1 - \frac{\mu}{4}\right)^2$ and $c_2(\mu_2) = \left(\mu_2 - \frac{\mu}{2}\right)^2$.
3. Let $g(\cdot)$ be a constant function on $\mathbb{R}^2$, i. e. $g(\cdot) = d$, for $d \in \mathbb{R}$.

Lemma 4.8. In Example 4.7, the equilibrium point is reached in four iterations.

P r o o f. Following Algorithm 4.1, observe that $\inf \mathbb{K} = (0,0) = \mu^{0,0} = (\mu_1^{0,0}, \mu_2^{0,0})$, also the facts that $k = 0$ and $i = 0$ imply that $\mu^{0,1} = (a_1^{0,1}, \mu_{-1}^{0,0})$, where $a_1^{0,1}$ is defined as in (2) with

$$f_1(0,a_1^*) = p_1(0) - c_1(a_1^*) - g(M)$$
$$= c - \left(a_1^* - \frac{\mu}{4}\right)^2 - d.$$

Then $a_1^* = 0$, $a_1^{0,1} = 0$, and $\mu^{0,1} = (0,0)$.

Now for $i = 1$, $\mu^{0,2} = (\mu_{-2}^{0,1}, a_2^{0,2})$, where $a_2^{0,2}$ is defined as in (3), with

$$f_2(0,a_2^*) = p_2(0) - c_2(a_2^*) - g(0)$$
$$= c - \left(a_2^* - \frac{\mu}{2}\right)^2 - d.$$

Thus, $a_2^* = \frac{\mu}{2}$, $a_2^{0,2} = \frac{\mu}{2}$, and $\mu^{0,2} = (0, \frac{\mu}{2})$.

When $i = 2 = n$, $\mu^{1,0} = \mu^{0,2} = (0, \frac{\mu}{2})$, $k = 1$ and $i = 0$. In this case, $\mu^{1,1} = (a_1^{1,1}, \mu_{-1}^{1,0})$, with

$$\begin{aligned} a_1^{1,1} &= \inf A_1^*(\mu_{-1}^{1,0}) = \inf A_1^*(\mu_2^{1,0}) = \inf A_1^*\left(\frac{\mu}{2}\right) \\ &= \inf\left\{a_1 \in A_1\left(\frac{\mu}{2}\right) : f_1\left(\frac{\mu}{2}, a_1\right) = \max_{a_1^* \in [0, \frac{\mu}{2})} f_1\left(\frac{\mu}{2}, a_1^*\right)\right\}, \end{aligned}$$

where

$$\begin{aligned} f_1\left(\frac{\mu}{2}, a_1^*\right) &= p_1\left(\frac{\mu}{2} \wedge a_1^*\right) - c_1(a_1^*) - g\left(\frac{a_1^*}{\frac{\mu}{2} - a_1^*}\right) \\ &= p_1(a_1^*) - \left(a_1^* - \frac{\mu}{4}\right)^2 - d \\ &= c - \left(a_1^* - \frac{\mu}{4}\right)^2 - d. \end{aligned}$$

Thus, $a_1^* = \frac{\mu}{4}$, $a_1^{1,1} = \frac{\mu}{4}$, and $\mu^{1,1} = (\frac{\mu}{4}, \frac{\mu}{2})$.

Now for $i = 1$, $\mu^{1,2} = (\mu_{-2}^{1,1}, a_2^{1,2})$, with

$$\begin{aligned} a_2^{1,2} &= \inf A_2^*(\mu_{-2}^{1,1}) = \inf A_2^*(\mu_1^{1,1}) = \inf A_2^*\left(\frac{\mu}{4}\right) \\ &= \inf\left\{a_2 \in A_2\left(\frac{\mu}{4}\right) : f_2\left(\frac{\mu}{4}, a_2\right) = \max_{a_2^* \in (\frac{\mu}{4}, \mu]} f_2\left(\frac{\mu}{4}, a_2^*\right)\right\}, \end{aligned}$$

where

$$\begin{aligned} f_2\left(\frac{\mu}{4}, a_2^*\right) &= p_2\left(\frac{\mu}{4} \wedge a_2^*\right) - c_2(a_2^*) - g\left(\frac{\frac{\mu}{4}}{a_2^* - \frac{\mu}{4}}\right) \\ &= p_2\left(\frac{\mu}{4}\right) - \left(a_2^* - \frac{\mu}{2}\right)^2 - d \\ &= c - \left(a_2^* - \frac{\mu}{2}\right)^2 - d. \end{aligned}$$

Then $a_2^* = \frac{\mu}{2}$, $a_2^{1,2} = \frac{\mu}{2}$, and $\mu^{1,2} = (\frac{\mu}{4}, \frac{\mu}{2})$. Thus, as $\mu^{1,1} = \left(\frac{\mu}{4}, \frac{\mu}{2}\right) = \mu^{1,2}$, then $(\frac{\mu}{4}, \frac{\mu}{2})$ is an equilibrium point, by Theorem 4.2 b) and c). □

ACKNOWLEDGEMENT

The authors wish to thank three anonymous referees for helpful comments and suggestions which aid to improve this paper.

This work was partially supported by *Fundación Sofía Kovalévskaia of the Sociedad Matemática Mexicana*.

(Received June 3, 2010)

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