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# ASSOCIATIVE $n$–DIMENSIONAL COPULAS

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The associativity of $n$-dimensional copulas in the sense of Post is studied. These copulas are shown to be just $n$-ary extensions of associative 2-dimensional copulas with special constraints, thus they solve an open problem of R. Mesiar posed during the International Conference FSTA 2010 in Liptovský Ján, Slovakia.

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## 1. INTRODUCTION

Copulas were introduced by Sklar [13] to capture the stochastic dependence structure of random variables. Recall that for $n \geq 2$, a function $C\colon [0,1]^n \to [0,1]$ is called an $n$-dimensional copula ($n$-copula, for short) whenever it is a restriction of an $n$-dimensional distribution function with all univariate margins uniformly distributed on $[0,1]$. Hence an $n$-copula is characterized by the properties:

(C1) $C(x_1, \ldots, x_n) = x_i$ whenever $\forall j \neq i$, $x_j = 1$;

(C2) $C(x_1, \ldots, x_n) = 0$ whenever $0 \in \{x_1, \ldots, x_n\}$;

(C3) the $n$-increasing property, i.e., $\forall\, \mathbf{x}, \mathbf{y} \in [0,1]^n$, $x_i \leq y_i$, $i = 1, \ldots, n$, it holds

$$\sum_{J \subset \{1,\ldots,n\}} (-1)^{|J|} C\left(u_1^J, \ldots, u_n^J\right) \geq 0, \text{ where } u_i^J = \begin{cases} x_i, & \text{if } i \in J, \\ y_i, & \text{if } i \notin J. \end{cases} \tag{1}$$

By the Sklar theorem [13], for any $n$-dimensional random vector $Z = (X_1, \ldots, X_n)$ there is an $n$-copula $C\colon [0,1]^n \to [0,1]$ such that for each $(z_1, \ldots, z_n) \in \mathbb{R}^n$

$$F_Z(z_1, \ldots, z_n) = C\left(F_{X_1}(z_1), \ldots, F_{X_n}(z_n)\right),$$

where $F_Z$, $F_{X_1}$, $\ldots$, $F_{X_n}$ are distribution functions of the corresponding random vectors.

There are two distinguished functions which are $n$-copulas for each $n \geq 2$: the so-called *minimum* $n$-copula $M$ and the *product* $n$-copula $\Pi$, given by

$$\begin{aligned} M(x_1, \dots, x_n) &= \min\{x_1, \dots, x_n\}, \\ \Pi(x_1, \dots, x_n) &= \prod_{i=1}^{n} x_i. \end{aligned}$$

The minimum $n$-copula $M$ describes the comonotone dependence of random variables $X_1, \dots, X_n$ and the product $n$-copula $\Pi$ describes their independence. For more details we recommend monographs [4, 11].
For each $n$-copula $C$ it holds

$$W \leq C \leq M,$$

where $W$ is the so-called Fréchet-Hoeffding lower bound, given by

$$W(x_1, \dots, x_n) = \max\left\{0, \sum_{i=1}^{n} x_i - (n-1)\right\}.$$

It is a well-known fact that this function is a copula only for $n = 2$, and in that case describes the countermonotone dependence of random variables $X_1$ and $X_2$.

All the three basic 2-copulas (copulas, for short) $M$, $\Pi$ and $W$ are associative, i. e., for all $x_1$, $x_2$, $x_3 \in [0, 1]$ they satisfy the property

$$C\left(C(x_1, x_2), x_3\right) = C\left(x_1, C(x_2, x_3)\right). \tag{2}$$

Associativity as an algebraic property was originally introduced for binary functions only, see formula (2). Recently, based on ideas of Post [12], Couceiro [1] has studied the associativity of $n$-ary functions. Subsequently, during the open problem session at FSTA 2010, R. Mesiar has posed the problem of representation of associative $n$-copulas, see [8]. Recall that for $n = 2$ this problem was solved in seventies by Ling [6] and Moynihan [10].

The aim of this paper is to solve the above mentioned open problem for any fixed $n > 2$. The paper is organized as follows. In the next section, the representation of associative copulas is recalled. In Section 3 we study $n$-ary associative functions on $[0, 1]$ possessing a neutral element and we show their relationship with binary associative functions. In Section 4, we introduce a representation theorem for associative $n$-copulas, together with some examples. Finally, some concluding remarks are added.

## 2. ASSOCIATIVE 2-DIMENSIONAL COPULAS

As mentioned above, 2-dimensional copulas will be referred to as copulas only. Let $C\colon [0, 1]^2 \to [0, 1]$ be an associative copula satisfying $C(x, x) < x$ for all $x \in ]0, 1[$. Then $C$ is called an Archimedean copula. Moynihan [10] has proved the next representation theorem for Archimedean copulas.

**Theorem 2.1.** A function $C\colon [0,1]^2 \to [0,1]$ is an Archimedean copula if and only if there is a continuous strictly decreasing convex function $f\colon [0,1] \to [0,\infty]$, $f(1) = 0$, such that

$$C(x_1, x_2) = f^{(-1)}\left(f(x_1) + f(x_2)\right), \tag{3}$$

where $f^{(-1)}$ is the pseudo-inverse of $f$.

Recall that the pseudo-inverse $f^{(-1)}\colon [0,\infty] \to [0,1]$ is given by

$$f^{(-1)}(u) = f^{-1}\left(\min(f(0), u)\right).$$

The function $f$ in the above theorem is called a generator of an Archimedean copula $C$. It is unique up to a positive multiplicative constant.

Copulas $W$ and $\Pi$ are Archimedean, with generators $f_{\mathrm{W}}$ and $f_\Pi$, respectively, given by $f_{\mathrm{W}}(x) = 1 - x$ and $f_\Pi(x) = -\log x$. If we define the function $f_{(1)}\colon [0,1] \to [0,\infty]$ by $f_{(1)}(x) = \frac{1}{x} - 1$, it is also a generator and the corresponding Archimedean copula $C_{(1)}\colon [0,1]^2 \to [0,1]$ is given by

$$C_{(1)}(x_1, x_2) = \frac{x_1\, x_2}{x_1 + x_2 - x_1 x_2}$$

whenever $(x_1, x_2) \neq (0,0)$.

For a general associative copula $C$ we have the next representation theorem [4, 11].

**Theorem 2.2.** A function $C\colon [0,1]^2 \to [0,1]$ is an associative copula if and only if there is a system $(\,]a_k, b_k[\,)_{k\in\mathcal{K}}$ of pairwise disjoint open subintervals of $[0,1]$ and a system $(C_k)_{k\in\mathcal{K}}$ of Archimedean copulas such that

$$C(x_1, x_2) = \begin{cases} a_k + (b_k - a_k)\, C_k\left(\frac{x_1 - a_k}{b_k - a_k}, \frac{x_2 - a_k}{b_k - a_k}\right), & \text{if } (x_1, x_2) \in\, ]a_k, b_k[^2 \\ & \text{for some } k \in \mathcal{K}, \\ M(x_1, x_2), & \text{else.} \end{cases} \tag{4}$$

Observe that if $\mathcal{K} = \emptyset$ then $C$ in (4) is the strongest copula $M$. Archimedean copulas are linked to $\mathcal{K} = \{1\}$ and $]a_1, b_1[\, = \,]\,0,1\,[$. Copula $C$ given by (4) is called an ordinal sum copula, with notation $(\langle a_k, b_k, C_k\rangle \mid k \in \mathcal{K})$.

**Example 2.3.** Let $C = \left(\langle 0, \frac{1}{2}, \Pi\rangle\right)$. Then

$$C(x_1, x_2) = \begin{cases} 2x_1 x_2, & \text{if } (x_1, x_2) \in\, ]0, \frac{1}{2}[^2, \\ M(x_1, x_2), & \text{else.} \end{cases}$$

## 3. N-ARY ASSOCIATIVE FUNCTIONS WITH NEUTRAL ELEMENT

The associativity of $n$-ary functions was introduced by Post [12].

**Definition 3.1.** Let $n \geq 2$ and $I$ be a real interval. A function $F\colon I^n \to I$ is said to be associative whenever for all $x_1, \dots, x_n, \dots, x_{2n-1} \in I$ it holds

$$\begin{aligned} &F\left(F(x_1, \dots, x_n), x_{n+1}, \dots, x_{2n-1}\right) = F\left(x_1, F(x_2, \dots, x_{n+1}), x_{n+2}, \dots, x_{2n-1}\right) \\ &= \dots = F\left(x_1, \dots, x_{n-1}, F(x_n, \dots, x_{2n-1})\right). \end{aligned} \tag{5}$$

Evidently, for $n = 2$, formulas (5) and (2) coincide, i. e., the Post $n$-ary associativity is a concept extending the standard notion of associativity for binary functions (operations). In the next definition, we recall the notion of neutral element, see [3].

**Definition 3.2.** Let $n \geq 2$ and $I$ be a real interval. A function $F : I^n \to I$ is said to have neutral element $e \in I$ whenever $F(x_1, \dots, x_n) = x_i$ if $x_j = e$ for each $j \neq i$.

Evidently, property (C1) of $n$-copulas means that $n$-copulas have neutral element $e = 1$. We say that a function $F$ is an $n$-ary extension of a binary function $G$ if it holds

$$F(x_1, \dots, x_n) = G(G(\dots G(G(x_1, x_2), x_3) \dots), x_{n-1}), x_n)$$

for all $n$-tuples in $I^n$.

**Example 3.3.**

(i) Define a mapping $F \colon \mathbb{R}^3 \to \mathbb{R}$ by $F(x_1, x_2, x_3) = x_1 - x_2 + x_3$. Then $F$ is a ternary associative function. Observe that there is no binary associative function whose ternary extension coincides with $F$. Moreover, $F$ has no neutral element.

(ii) Let $C \colon [0, 1]^3 \to [0, 1]$ be given by $C(x_1, x_2, x_3) = x_1 \min\{x_2, x_3\}$. Then $e = 1$ is neutral element of $C$, but $C$ is not associative. Note that $C$ is a ternary copula.

**Theorem 3.4.** Consider $n \geq 2$. Let $I$ be a real interval and $e \in I$. Then the following claims are equivalent:

(i) A mapping $F \colon I^n \to I$ is associative function with neutral element $e$.

(ii) There is a binary associative function $G \colon I^2 \to I$ with neutral element $e$ whose $n$-ary extension is $F$.

*Proof.* If $n = 2$, the claim is trivial. Suppose that $n > 2$.

- (i) $\Leftarrow$ (ii) The proof is trivial.
- (i) $\Rightarrow$ (ii) Define a function $G \colon I^2 \to I$ by $G(x_1, x_2) = F(x_1, x_2, e, \dots, e)$. Then $G(x_1, e) = F(x_1, e, \dots, e) = x_1$ and $G(e, x_2) = F(e, x_2, e, \dots, e) = x_2$, i. e., $e$ is a neutral element of $G$. Moreover, for any $x_1, x_2, x_3 \in I$ it holds

$$G(G(x_1, x_2), x_3) = F(F(x_1, x_2, e, \dots, e), x_3, e, \dots, e)$$
$$= F(x_1, x_2, F(\underbrace{e, \dots, e,}_{(n-2)\text{-times}}\ x_3, e), \underbrace{e, \dots, e}_{(n-3)\text{-times}}) = F(x_1, x_2, x_3, \underbrace{e, \dots, e}_{(n-3)\text{-times}}),$$

and

$$G(x_1, G(x_2, x_3)) = F(x_1, F(x_2, x_3, e, \dots, e), e, \dots, e)$$
$$= F(x_1, x_2, F(x_3, \underbrace{e, \dots, e}_{(n-1)\text{-times}}), \underbrace{e, \dots, e}_{(n-3)\text{-times}}) = F(x_1, x_2, x_3, \underbrace{e, \dots, e}_{(n-3)\text{-times}}),$$

which proves the associativity of $G$. From this proof it is also obvious that if $n=3$, then $F(x_1,x_2,x_3)=G(G(x_1,x_2),x_3)$. For $n>3$, $G(G(x_1,x_2),x_3) = F(x_1,x_2,x_3,\underbrace{e,\dots,e}_{(n-3)\text{-times}})$ and similarly we can show that
$G(G(G(x_1,x_2),x_3),x_4)=F(x_1,x_2,x_3,x_4,\underbrace{e,\dots,e}_{(n-4)\text{-times}})$.
By induction on $n$ it can be proved that for any $n>2$,

$$G(G(\dots G(G(x_1,x_2),\dots),x_{n-1}),x_n)=F(x_1,\dots,x_n).$$

□

Theorem 3.4 shows that under the neutral element existence, the associativity of $n$-ary functions is classically related to the associativity of binary functions.

## 4. ON THE STRUCTURE OF ASSOCIATIVE N-DIMENSIONAL COPULAS

Based on Theorems 2.1, 2.2, 3.4 and recent results on ordinal sum structure of $n$-copulas proved by Mesiar and Sempi [9], we have the next result.

**Corollary 4.1.** Let $n \ge 2$. A function $C\colon [0,1]^n \to [0,1]$ is an associative $n$-copula if and only if there is a system $(\,]a_k,b_k[\,)_{k\in\mathcal{K}}$ of pairwise disjoint open subintervals of $]\,0,1\,[$, and a system $(C_k)_{k\in\mathcal{K}}$ of associative $n$-copulas satisfying the diagonal inequality $C_k(x,\dots,x)<x$ for all $x\in]\,0,1\,[$ and $k\in\mathcal{K}$ such that

$$C(x_1,\dots,x_n)=\begin{cases} a_k+(b_k-a_k)\,\mathrm{C}_k\left(\frac{\min\{x_1,b_k\}-a_k}{b_k-a_k},\dots,\frac{\min\{x_n,b_k\}-a_k}{b_k-a_k}\right), \\ \qquad \text{if } \min\{x_1,\dots,x_n\}\in]a_k,b_k[ \quad \text{for some } k\in\mathcal{K}, \\ M(x_1,\dots,x_n), \qquad\qquad \text{else.}\end{cases} \tag{6}$$

To complete the representation of associative $n$-copulas, the characterization of such copulas satisfying the diagonal inequality is necessary.

**Theorem 4.2.** Let $n\ge 2$. A function $C\colon[0,1]^n\to[0,1]$ is an associative $n$-copula satisfying the diagonal inequality $C(x,\dots,x)<x$ for all $x\in]\,0,1\,[$ if and only if there is a generator $f$ whose pseudo-inverse $f^{(-1)}$ is an $(n-2)$-times differentiable function with derivatives alternating the sign, such that $(-1)^n\,\frac{\mathbf{d}^{n-2}f^{(-1)}}{\mathbf{d}\,x^{n-2}}$ is a convex function, and

$$C(x_1,\dots,x_n)=f^{(-1)}\left(\sum_{i=1}^{n} f(x_i)\right). \tag{7}$$

P r o o f . The sufficiency of conditions follows from [7].
By Theorem3.4, $C$ is an $n$-ary extension of an associative copula $G$. Suppose that $G(x_0,x_0)=x_0$ for some $x_0\in]\,0,1\,[$. Then

$$C(x_0,\dots,x_0)=G(G(\dots G(G(x_0,x_0),\dots),x_0),x_0)=x_0,$$

which violates the diagonal inequality satisfied by $C$. Therefore $G$ also satisfies the diagonal inequality, i. e., $G(x,x) < x$ for all $x \in ]\,0,1\,[$. By Theorem 1, formula (7) is satisfied for some generator $f$. Moreover, $C$ given by (7) is $n$-increasing and hence, according to the results of McNeil and Nešlehová in [7], the required properties of $f$ are necessary. □

**Example 4.3.**

(i) As already mentioned, the product $n$-copula $\Pi$ is associative for any $n \geq 2$. Evidently, $\Pi(x, \dots, x) = x^n < x$ whenever $x \in ]\,0,1\,[$. As the generator $f_\Pi$ of the copula $\Pi$ is given by $f_\Pi(x) = -\log x$, it holds $f_\Pi^{(-1)}(x) = f_\Pi^{-1}(x) = e^{-x}$, hence for any $k$, $\frac{\mathbf{d}^k f_\Pi^{-1}(x)}{\mathbf{d}\, x^k} = (-1)^k e^{-x}$. Derivatives alternate the sign and for any $n \geq 2$, $(-1)^n \frac{\mathbf{d}^{n-2} f_\Pi^{(-1)}(x)}{\mathbf{d}\, x^{n-2}} = e^{-x}$ is a convex function.

(ii) A similar result can be shown for the generator $f_{(1)}$ introduced in Section 2, given by $f_{(1)}(x) = \frac{1}{x} - 1$. It holds $f_{(1)}^{(-1)}(x) = f_{(1)}^{-1}(x) = (1+x),^{-1}$ which implies that
$(-1)^n \frac{\mathbf{d}^{n-2} f_\Pi^{(-1)}(x)}{\mathbf{d}\, x^{n-2}} = (n-2)!\,(1+x)^{-n+1}$ is convex. The corresponding $n$-copula $C_{(1)}$ is given by $C_{(1)}(x) = \left(\sum_{i=1}^n \frac{1}{x_i} - (n-1)\right)^{-1}$.

(iii) The weakest associative $n$-copula is the Clayton copula $C_{(-\frac{1}{n-1})}$ generated by the generator $f_{(-\frac{1}{n-1})} \colon [0,1] \to [0,\infty]$, $f_{(-\frac{1}{n-1})} = 1 - x^{\frac{1}{n-1}}$. The corresponding pseudo-inverse $f_{(-\frac{1}{n-1})}^{(-1)} \colon [0,\infty] \to [0,1]$ is given by

$$f_{(-\frac{1}{n-1})}^{(-1)}(x) = \begin{cases} (1-x)^{n-1}, & \text{if } x \leq 1, \\ 0, & \text{if } x > 1. \end{cases}$$

Then $(-1)^n \frac{\mathbf{d}^{n-2} f_{(-\frac{1}{n-1})}^{(-1)}(x)}{\mathbf{d}\, x^{n-2}} = (n-1)!\, \max\{1-x, 0\}$ is convex but not differentiable. For more details we recommend [7].

(iv) The function $C \colon [0,1]^n \to [0,1]$ given by

$$\mathrm{C}(x_1, \dots, x_n) = \begin{cases} 2^{n-1} \prod\limits_{i=1}^{n} \min\left\{x_i, \frac{1}{2}\right\}, & \text{if } \min\{x_1, \dots, x_n\} < \frac{1}{2}, \\ M(x_1, \dots, x_n), & \text{else,} \end{cases} \tag{8}$$

is an $n$-ary extension of the ordinal sum copula $\left(\langle 0, \frac{1}{2}, \Pi\rangle\right)$ introduced in Example 2.3. As $n$-ary function $\Pi$ is an associative $n$-copula for each $n \geq 2$, our function $C$ given by (8) is also an associative $n$-copula for each $n \geq 2$.

## 5. CONCLUDING REMARKS

We have solved the Problem 2.1 posed in [8], showing that associative $n$-copulas are just $n$-ary extensions of appropriate associative copulas. Based on Theorem 3.4, similar results can be formulated for the representation of continuous $n$-ary triangular norms or triangular conorms [5], and also for $n$-ary uninorms [2] continuous up to the case when $\{0,1\} \subseteq \{x_1, \dots, x_n\}$.

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