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LATTICE EFFECT ALGEBRAS DENSELY EMBEDDABLE INTO COMPLETE ONES

Zdenka Riečanová

An effect algebraic partial binary operation $\oplus$ defined on the underlying set E uniquely introduces partial order, but not conversely. We show that if on a MacNeille completion $\widehat{E}$ of E there exists an effect algebraic partial binary operation $\widehat{\oplus}$ then $\widehat{\oplus}$ need not be an extension of $\oplus$. Moreover, for an Archimedean atomic lattice effect algebra E we give a necessary and sufficient condition for that $\widehat{\oplus}$ existing on $\widehat{E}$ is an extension of $\oplus$ defined on E. Further we show that such $\widehat{\oplus}$ extending $\oplus$ exists at most one.

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1. INTRODUCTION, BASIC DEFINITIONS AND FACTS

Lattice effect algebras generalize orthomodular lattices including noncompatible pairs of elements [10] and MV-algebras including unsharp elements [1]. Effect algebras were introduced by D. Foulis and M. K. Bennet [3] as a generalization of the Hilbert space effects (i. e., self-adjoint operators between zero and identity operator on a Hilbert space representing unsharp measurements in quantum mechanics). They may have importance in the investigation of the phenomenon of uncertainty.

Definition 1.1. A partial algebra $(E; \oplus, 0, 1)$ is called an *effect algebra* if 0, 1 are two distinct elements and $\oplus$ is a partially defined binary operation on E which satisfy the following conditions for any $x, y, z \in E$:

(Ei) $x \oplus y = y \oplus x$ if $x \oplus y$ is defined,

(Eii) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ if one side is defined,

(Eiii) for every $x \in E$ there exists a unique $y \in E$ such that $x \oplus y = 1$ (we put $x' = y$, a supplement of x),

(Eiv) if $1 \oplus x$ is defined then $x = 0$.

We often denote the effect algebra $(E; \oplus, 0, 1)$ briefly by E. On every effect algebra E the partial order $\leq$ and a partial binary operation $\ominus$ can be introduced as follows:

$$x \le y \text{ and } y \ominus x = z \text{ iff } x \oplus z \text{ is defined and } x \oplus z = y\,.$$

If E with the defined partial order is a lattice (a complete lattice) then $(E; \oplus, 0, 1)$ is called a *lattice effect algebra* (*a complete lattice effect algebra*).

Definition 1.2. Let E be an effect algebra. Then $Q \subseteq E$ is called a *sub-effect algebra* of E if

(i) $1 \in Q$

(ii) if out of elements $x, y, z \in E$ with $x \oplus y = z$ two are in Q, then $x, y, z \in Q$.

If E is a lattice effect algebra and Q is a sub-lattice and a sub-effect algebra of E then Q is called a *sub-lattice effect algebra* of E.

Note that a sub-effect algebra Q (sub-lattice effect algebra Q) of an effect algebra E (of a lattice effect algebra E) with inherited operation $\oplus$ is an effect algebra (lattice effect algebra) in its own right.

Important sub-lattice effect algebras of a lattice effect algebra E are

(i) $S(E) = \{x \in E \mid x \wedge x' = 0\}$ a *set of all sharp elements of* E (see [5], [6]), which is an orthomodular lattice (see [7]).

(ii) Maximal subsets of pairwise compatible elements of E called *blocks* of E (see [19]), which are in fact maximal sub-MV-algebras of E. Here, x, $y \in E$ are called *compatible* ($x \leftrightarrow y$ for short) if $x \vee y = x \oplus (y \ominus (x \wedge y))$ (see [11] and [2]).

(iii) The *center of compatibility* $B(E)$ of E, $B(E) = \bigcap\{M \subseteq E \mid M \text{ is a block of } E\} = \{x \in E \mid x \leftrightarrow y \text{ for every } y \in E\}$ which is in fact an MV-algebra (MV-effect algebra).

(iv) The *center* $C(E) = \{x \in E \mid y = (y \wedge x) \vee (y \wedge x') \text{ for all } y \in E\}$ of E which is a Boolean algebra (see [4]). In every lattice effect algebra it holds $C(E) = B(E) \cap S(E)$ (see [15] and [17]).

For an element x of an effect algebra E we write $\mathrm{ord}(x) = \infty$ if $nx = x \oplus x \oplus \cdots \oplus x$ (n-times) exists for every positive integer n and we write $\mathrm{ord}(x) = n_x$ if n_x is the greatest positive integer such that $n_x x$ exists in E. An effect algebra E is *Archimedean* if $\mathrm{ord}(x) < \infty$ for all $x \in E$, $x \neq 0$.

A minimal nonzero element of an effect algebra E is called an *atom* and E is called *atomic* if under every nonzero element of E there is an atom. Properties of the set of all atoms in a lattice effect algebra E are in several cases substantial for the algebraic structure of E. For instance, the "Isomorphism theorem based on atoms" for Archimedean atomic lattice effect algebras can be proved [13]. Further, the atomicity of the center $C(E)$ of E gives us the possibility to decompose E into subdirect product (resp. direct product for complete E) of irreducible effect algebras in the case when supremum of all atoms of the center equals 1. Recently M. Kalina [8] proved that this is not true in general and we give here a necessary and sufficient conditions for that. Moreover, if a lattice effect algebra E is complete then its important sub-lattice effect algebras $S(E)$, blocks, $C(E)$ and $B(E)$ are complete sub-lattice effect algebras of E. However, not every effect algebra can be embedded as a dense sub-effect algebra into a complete one (see [16]). We are going to prove some statements about extensions of $\oplus$-operation on an Archimedean atomic lattice effect algebra $(E;\oplus,0,1)$ onto the MacNeille completion $\widehat{E}=\mathcal{MC}(E)$ of its underlying ordered set E. In [16] it was proved that there exists a $\widehat{\oplus}$-operation on $\widehat{E}=\mathcal{MC}(E)$ such that its restriction $\widehat{\oplus}_{/E}$ onto E coincides with $\oplus$ on E iff E is strongly D-continuous. Here strongly D-continuity of E means that, for every $U,Q\subseteq E$ such that $u\le q$ for all $u\in U$, $q\in Q$ holds:

$$\bigwedge\nolimits_E\{q\ominus u \mid q\in Q, u\in U\}=0 \quad \text{iff} \quad \begin{array}{l} a\le b \text{ for all } a,b\in E \text{ with } a\le q,\ u\le b\\ \text{for all } u\in U,\ q\in Q. \end{array}$$

2. EXTENSIONS OF EFFECT ALGEBRAIC OPERATIONS ONTO COMPLETIONS OF THEIR UNDERLYING SETS

Every effect algebra $(E;\oplus,0,1)$ is in fact a bounded poset or lattice since the $\oplus$-operation induces uniquely partial order on E at which 0 is the smallest and 1 the greatest element of E. The converse is not true: The different operations $\oplus_1$ and $\oplus_2$ on a set E with $0,1\in E$ may induce the same partial order on E.

Example 2.1. The lattice effect algebras $E_1=\{0,a,b,a\oplus b=1\}$ and $E_2=\{0,a,b,2a=2b=1\}$ have the underlying set the same lattice $\widetilde{E}=\{0,a,b,1=a\vee b\}$.

For a poset P and its subposet $Q\subseteq P$ we denote, for all $X\subseteq Q$, by $\bigvee_Q X$ the join of the subset X in the poset Q whenever it exists.

We say that a finite system $F=(x_k)_{k=1}^n$ of not necessarily different elements of an effect algebra $(E;\oplus,0,1)$ is *orthogonal* if $x_1\oplus x_2\oplus\cdots\oplus x_n$ (written $\bigoplus_{k=1}^n x_k$ or $\bigoplus F$) exists in E. Here we define $x_1\oplus x_2\oplus\cdots\oplus x_n=(x_1\oplus x_2\oplus\cdots\oplus x_{n-1})\oplus x_n$ supposing that $\bigoplus_{k=1}^{n-1}x_k$ is defined and $\bigoplus_{k=1}^{n-1}x_k\le x_n'$. We also define $\bigoplus\varnothing=0$. An arbitrary system $G=(x_\kappa)_{\kappa\in H}$ of not necessarily different elements of E is called orthogonal if $\bigoplus K$ exists for every finite $K\subseteq G$. We say that for an orthogonal system $G=(x_\kappa)_{\kappa\in H}$ the element $\bigoplus G$ exists iff $\bigvee\{\bigoplus K \mid K\subseteq G \text{ is finite}\}$ exists in E and then we put $\bigoplus G=\bigvee\{\bigoplus K\mid K\subseteq G \text{ is finite}\}$. (Here we write $G_1\subseteq G$ iff there is $H_1\subseteq H$ such that $G_1=(x_\kappa)_{\kappa\in H_1}$).

It is well known that any partial ordered set P can be embedded into a complete lattice $\widehat{P}=\mathcal{MC}(P)$ called a *MacNeille completion* (or *completion by cuts*). It has been shown (see [26]) that the MacNeille completion of P (up to isomorphism unique over P) is any complete lattice $\widehat{P}$ into which P can be supremum-densely and infimum-densely embedded (i.e., for every element $x\in\widehat{P}$ there exist $Q,S\subseteq P$ such that $x=\bigvee_{\widehat{P}}\varphi(Q)=\bigwedge_{\widehat{P}}\varphi(S)$, where $\varphi: P\to\widehat{P}$ is the embedding). We usually identify P with $\varphi(P)\subseteq\widehat{P}$. In this sense $\widehat{P}$ inherits all infima and suprema existing in P.

Definition 2.2. Let $(E;\oplus_E,0_E,1_E)$ and $(F;\oplus_F,0_F,1_F)$ be effect algebras. A bijective map $\varphi: E\to F$ is called an *isomorphism* if

(i) $\varphi(1_E)=1_F$,

(ii) for all $a,b\in E$: $a\le_E b'$ iff $\varphi(a)\le_F \big(\varphi(b)\big)'$ in which case $\varphi(a\oplus_E b)=\varphi(a)\oplus_F\varphi(b)$.

We write $E \cong F$. Sometimes we identify E with $F = \varphi(E)$. If $\varphi : E \to F$ is an injection with properties (i) and (ii) then φ is called an *embedding*. We say that E is *densely embeddable* into F if there is an embedding $\varphi : E \to F$ of effect algebras such that to each $x \in F$, $x \neq 0$ there exists $y \in E$, $y \neq 0$ with $\varphi(y) \leq x$. Then $\varphi(E)$ is called a dense sub-effect algebra of F.

Remark 2.3. Note that for an effect algebra $(E; \oplus, 0, 1)$ an extension of $\oplus$-operation onto $\widehat{E} = \mathcal{MC}(E)$ exists iff E *is a dense sub-effect algebra of* $\widehat{E}$ (equivalently, an extension $\widehat{\oplus}$ onto $\widehat{E} = \mathcal{MC}(E)$ exists iff E *can be densely embedded into* $\widehat{E}$). This follows from the fact that in such a case E is a supremum-dense sub-effect algebra of the complete lattice effect algebra $\widehat{E}$, and conversely.

Theorem 2.4. Let $(E; \vee, \wedge, ', 0, 1)$ be an orthomodular lattice and let $E^* = \mathcal{MC}(E)$ be a MacNeille completion of E. Then

(i) There exists a unique $\oplus$-operation on E such that $(E; \oplus, 0, 1)$ is a lattice effect algebra in which partial order coincides with partial order of the orthomodular lattice E.

(ii) E^* is an orthomodular lattice iff there exists a unique $\oplus^*$-operation on E^* such that $(E^*; \oplus^*, 0, 1)$ is a complete lattice effect algebra and $\oplus^*_{/E} = \oplus$.

Proof. (i) Let $(E; \oplus, 0, 1)$ be a lattice effect algebra in which partial order coincides with partial order of the orthomodular lattice E. Then for $x, y \in E$, $x \oplus y$ exists iff $x \leq y'$, in which case $x \oplus y = (x \vee y) \oplus (x \wedge y) = x \vee y$, since $x \wedge y \leq y' \wedge y = 0$. Conversely, for every orthomodular lattice E the operation $\oplus$ defined by $x \oplus y = x \vee y$ iff $x \leq y'$ satisfies axioms of an effect algebra (see [2]).

(ii) This follows by (i) and the fact that E is a sub-lattice of E^*. Moreover, E^* is an orthomodular lattice iff for the effect algebra $(E; \oplus, 0, 1)$ derived from the orthomodular lattice E there exists an extension $\oplus^*$ on E^* such that $(E^*; \oplus^*, 0, 1)$ is a complete lattice effect algebra (see [16, Theorem 6.5]). □

Recall that a lattice effect algebra with a unique block is called an *MV-effect algebra*.

Lemma 2.5. Let $(E; \oplus, 0, 1)$ be an Archimedean atomic MV-effect algebra. Let $\widehat{E} = \mathcal{MC}(E)$ be a MacNeille completion of E and let us identify E with $\varphi(E)$ (where $\varphi : E \to \widehat{E}$ is the embedding). Then

(i) There exists a unique $\widehat{\oplus}$-operation on $\widehat{E}$ making $\widehat{E}$ a complete MV-effect algebra $(\widehat{E}; \widehat{\oplus}, 0, 1)$.

(ii) The restriction $\widehat{\oplus}_{/E}$ coincides with $\oplus$ on E and E is a sub-MV-effect algebra of $\widehat{E}$.

Proof. (i) Since $\widehat{E}$ is a complete atomic MV-effect algebra, it is isomorphic to a direct product of finite chains. Since $\widehat{\oplus}$ on the direct product is defined coordinatewise, we obtain that this operation on $\widehat{E}$ is unique.

(ii) By [18, Theorem 3.4], E is a sub-MV-effect algebra of $\widehat{E}$ (see also [22, Theorem 3.1]). Hence the restriction $\widehat{\oplus}_{/E}$ coincides with $\oplus$ on E. □

Definition 2.6. A *direct product* $\prod\{E_\kappa \mid \kappa \in H\}$ of effect algebras E_κ is a cartesian product with $\oplus$, 0, 1 defined "coordinatewise", i. e., $(a_\kappa)_{\kappa\in H} \oplus (b_\kappa)_{\kappa\in H}$ exists iff $a_\kappa \oplus_\kappa b_\kappa$ is defined for each $\kappa \in H$ and then $(a_\kappa)_{\kappa\in H} \oplus (b_\kappa)_{\kappa\in H} = \big(a_\kappa \oplus_\kappa b_\kappa\big)_{\kappa\in H}$. Moreover, $0 = (0_\kappa)_{\kappa\in H}$, $1 = (1_\kappa)_{\kappa\in H}$.

A *subdirect product* of a family $\{E_\kappa \mid \kappa \in H\}$ of lattice effect algebras is a sublattice-effect algebra Q of the direct product $\prod\{E_\kappa \mid \kappa \in H\}$ such that each restriction of the natural projection pr_{κ_i} to Q is onto E_{κ_i}.

Proposition 2.7. There is an Archimedean atomic lattice effect algebra $(E;\oplus,0,1)$ such that there are infinitely many different operations $\widehat{\oplus}_n$ on a MacNeille completion $\widehat{E} = \mathcal{MC}(E)$ of E at which $(\widehat{E};\widehat{\oplus}_n,0,1)$ are mutually non-isomorphic.

Example 2.8. Let $E_k^{(1)} \simeq E_1$, $k = 1,2,\dots,n$; $E_k^{(2)} \simeq E_2$, $k = n+1, n+2, \dots$ where E_1, E_2 are those from Example 2.1. Let

$$\widehat{E}^{(n)} \cong \left(\prod_{k=1}^{n} E_k^{(1)}\right) \times \left(\prod_{k=n+1}^{\infty} E_k^{(2)}\right) \cong B_n \times M_n.$$

Here $(\widehat{E}^{(n)};\widehat{\oplus}_n,0,1)$, where $0 = (0_k)_{k=1}^{\infty}$, $1 = (1_k)_{k=1}^{\infty}$ and $x \in \widehat{E}^{(n)}$ iff $x = (x_k)_{k=1}^{\infty}$ with $x_k \in \widehat{E}_k^{(1)}$ for $k = 1,2,\dots,n$ and $x_k \in \widehat{E}_k^{(2)}$ for $k = n+1, n+2, \dots$ are mutually non-isomorphic complete distributive lattice effect algebras. Nevertheless the underlying complete lattices $\widehat{E}^{(n)}$ are isomorphic to the complete lattice $\widehat{E} \cong \prod_{k=1}^{\infty} E_k$ where $E_k = \widetilde{E}$ from Example 2.1, $k = 1,2,\dots$. Moreover, $B_n = \prod_{k=1}^{n} E_k^{(1)}$, $n = 1,2,\dots$ are complete atomic Boolean algebras with $2n$ atoms and $M_n = \prod_{k=n+1}^{\infty} E_k^{(2)}$, $n = 1,2,\dots$ are complete atomic lattice effect algebras with infinitely many blocks.

Assume now that $E^* = \prod_{k=1}^{\infty} E_k^{(1)}$. Clearly E^* is a complete atomic Boolean algebra. Set $E = \{x \in E^* \mid x \text{ or } x' \text{ is finite}\}$ hence $x \in E$ iff x or x' is a join of a finite set of atoms of E^*. Then E is a sub-lattice effect algebra of E^* (even a Boolean sub-algebra of E^*) with $\oplus$-operation $x \oplus y = x \vee y$ iff $x \wedge y = 0$ in the Boolean algebra E. Hence E is not a sub-lattice effect algebra of any $\widehat{E}^{(n)}$, since $\widehat{\oplus}_{n/E}$ does not coincide with $\oplus$ on E, $n = 1,2,\dots$.

Theorem 2.9. Let $(E;\oplus,0,1)$ be an Archimedean atomic lattice effect algebra and let $E^* = \mathcal{MC}(E)$ be a MacNeille completion of a lattice E. Let there exist a $\oplus^*$-operation on E^* making $(E^*;\oplus^*,0,1)$ a complete lattice effect algebra. The following conditions are equivalent:

(i) For every atom a of E, $\mathrm{ord}(a)$ in E equals $\mathrm{ord}(a)$ in E^* at which for every positive integer $k \leq \mathrm{ord}(a)$

$$\underbrace{a \oplus^* a \oplus^* \cdots \oplus^* a}_{k-\text{times}} = \underbrace{a \oplus a \oplus \cdots \oplus a}_{k-\text{times}}$$

and for every pair $a, b \in A_E$: $a \leftrightarrow b$ in E iff $a \leftrightarrow b$ in E^*.

(ii) The restriction $\oplus^*_{/E}$ of $\oplus^*$ onto E coincides with $\oplus$ on E (equivalently E is a sub-lattice effect algebra of E^*).

In this case for any maximal orthogonal set $A \subseteq A_E$ there are unique atomic blocks M_A of E and M_A^* of E^* with $A \subseteq M_A \cap M_A^*$ and $M_A^* = \mathcal{MC}(M_A)$.

Proof. (i) $\Longrightarrow$ (ii): Let A_E and A_{E^*} be sets of atoms of E and E^* respectively. Since E is supremum-dense in E^*, we obtain that $A_E = A_{E^*}$. It follows by [12] that to every maximal set of pairwise compatible atoms $A \subseteq A_E = A_{E^*}$ there exist unique blocks M_A of E and M_A^* of E^* with A as a common set of atoms. Hence $A \subseteq M_A$ and $A \subseteq M_A^*$. Let us show that $M_A \subseteq M_A^*$. For that assume $x \in M_A$. Then by [21, Theorem 3.3] there exist a set $\{a_\kappa \mid \kappa \in \mathcal{H}\} \subseteq A$ and positive integers $k_\kappa \leq \mathrm{ord}(a_\kappa)$, $\kappa \in \mathcal{H}$ such that

$$\begin{aligned} x &= \bigoplus\nolimits_{M_A}\{k_\kappa a_\kappa \mid \kappa \in \mathcal{H}\} = \bigvee\nolimits_{M_A}\{k_\kappa a_\kappa \mid \kappa \in \mathcal{H}\} = \bigvee\nolimits_{E}\{k_\kappa a_\kappa \mid \kappa \in \mathcal{H}\} \\ &= \bigvee\nolimits_{E^*}\{k_\kappa a_\kappa \mid \kappa \in \mathcal{H}\} = \bigvee\nolimits_{M_A^*}\{k_\kappa a_\kappa \mid \kappa \in \mathcal{H}\} = \bigoplus\nolimits_{M_A^*}\{k_\kappa a_\kappa \mid \kappa \in \mathcal{H}\} \in M_A^* \end{aligned}$$

since $k_\kappa a_\kappa \in M_A \cap M_A^*$ for all $k_\kappa \leq \mathrm{ord}(a_\kappa)$, $\kappa \in \mathcal{H}$, M_A is a bifull sub-lattice of E (see [14]), E^* inherits all infima and suprema existing in E and M_A^* is a complete sub-lattice of E^* (see [20, Theorem 2.8]). This proves that $M_A \subseteq M_A^*$.

Now let $y \in M_A^*$. Then again by [21, Theorem 3.3] there exist $\{b_\beta \mid \beta \in \mathcal{B}\} \subseteq A$ and positive integers $l_\beta \leq \mathrm{ord}(b_\beta)$, $\beta \in \mathcal{B}$ such that

$$y = \bigoplus_{M_A^*}\{l_\beta b_\beta \mid \beta \in \mathcal{B}\} = \bigvee_{M_A^*}\{l_\beta b_\beta \mid \beta \in \mathcal{B}\}$$

which proves that M_A is supremum-dense in M_A^*, as $l_\beta b_\beta \in M_A$ for all $\beta \in \mathcal{B}$.

Since $1 \in M_A \cap M_A^*$ we obtain that

$$\begin{aligned} 1 &= \bigoplus\nolimits_{M_A}\{n_a a \mid a \in A\} = \underbrace{\bigoplus\nolimits_{M_A}\{k_\kappa a_\kappa \mid \kappa \in \mathcal{H}\}}_{x} \oplus \\ &\quad \underbrace{\bigoplus\nolimits_{M_A}\left(\{(n_{a_\kappa} - k_\kappa)a_\kappa \mid \kappa \in \mathcal{H}\} \cup \{n_a a \mid a \in A, a \neq a_\kappa \text{ for every } \kappa \in \mathcal{H}\}\right)}_{x' \in M_A} \\ &= \bigoplus\nolimits_{M_A^*}\{n_a a \mid a \in A\} = \underbrace{\bigoplus\nolimits_{M_A^*}\{k_\kappa a_\kappa \mid \kappa \in \mathcal{H}\}}_{x} \oplus^* \\ &\quad \underbrace{\bigoplus\nolimits_{M_A^*}\left(\{(n_{a_\kappa} - k_\kappa)a_\kappa \mid \kappa \in \mathcal{H}\} \cup \{n_a a \mid a \in A, a \neq a_\kappa \text{ for every } \kappa \in \mathcal{H}\}\right)}_{x'^* \in M_A^*}. \end{aligned}$$

Thus, by axiom (Eiii) of effect algebras, we obtain that

$$x' = \bigoplus\nolimits_{M_A}\left(\{(n_{a_\kappa} - k_\kappa)a_\kappa \mid \kappa \in \mathcal{H}\} \cup \{n_a a \mid a \in A, a \neq a_\kappa \text{ for every } \kappa \in \mathcal{H}\}\right)$$

and

$$x'^* = \bigoplus\nolimits_{M_A^*}\left(\{(n_{a_\kappa} - k_\kappa)a_\kappa \mid \kappa \in \mathcal{H}\} \cup \{n_a a \mid a \in A, a \neq a_\kappa \text{ for every } \kappa \in \mathcal{H}\}\right).$$

As above, we get that $x' = x'^*$.

Thus by de Morgan laws for supplementation on M_A^* we obtain that M_A is also infimum-dense in M_A^*. This proves that $M_A^* = \mathcal{MC}(M_A)$ is a MacNeille completion of a M_A.

Assume now that $x, y \in E$ with $x \oplus^* y$ defined in E^*. Then $x \leftrightarrow y$ in E^* and hence by [9] there exists an atomic block M^* of E^* such that $\{x, y, x \oplus y\} \subseteq M^*$. Now, by [12] we obtain that there exists a maximal pairwise compatible set $A \subseteq A_E = A_{E^*}$ such that $A \subseteq M^*$ and an atomic block block M of E such that $A \subseteq M$. As we have proved above, $M \subseteq M^* = \mathcal{MC}(M)$. Since $x, y \in M^* \cap E = M$ we obtain by Lemma 2.5 that M is a sub-effect algebra of M^* and hence $x \oplus^* y = x \oplus y$. Thus, we have proved that the restriction $\oplus^*_{/E}$ onto E coincides with $\oplus$ on E. Consequently, E is a sub-lattice effect algebra of E^* because we have also $0, 1 \in E$ and for any $x, x' \in E$ the equalities $1 = x \oplus x' = x \oplus^* x'$ holds, as we have just proved above.

(ii) $\Longrightarrow$ (i): This is trivial. □

Corollary 2.10. Let $(E; \oplus, 0, 1)$ be an Archimedean atomic lattice effect algebra and let $E^* = \mathcal{MC}(E)$. Then there exists at most one $\oplus^*$-operation on E^* such that $(E^*; \oplus^*, 0, 1)$ is a complete lattice effect algebra and the restriction $\oplus^*_{/E}$ of $\oplus^*$ onto E coincides with $\oplus$ on E.

Proof. Let $\oplus_1^*$ and $\oplus_2^*$ be such that make E^* a complete lattice effect algebra at which $\oplus^*_{1/E}$ and $\oplus^*_{2/E}$ coincide with $\oplus$ on E. Set $E_1^* = E_2^* = E^*$ and, for simplicity, let us use symbols E_1^* for complete lattice effect algebra $(E_1^*; \oplus_1^*, 0, 1)$ and E_2^* for $(E_2^*; \oplus_2^*, 0, 1)$. Since the effect algebra E is a sub-lattice effect algebra of E_1^* as well as of E_2^*, we obtain that for any $x \in E$ the supplements x' in E, E_1^* and E_2^* coincide. Further $A_E = A_{E_1^*} = A_{E_2^*} \subseteq E$. Thus for any $y \in E^*$ there exists an orthogonal set $A_y = \{a_\kappa \mid \kappa \in \mathcal{H}\} \subseteq A_E$ and positive integers $k_\kappa \le \operatorname{ord}(a_\kappa)$, $\kappa \in \mathcal{H}$ such that

$$y \;=\; \bigvee_{E^*}\{k_\kappa a_\kappa \mid \kappa \in \mathcal{H}\} = \bigoplus_{E_1^*}\{k_\kappa a_\kappa \mid \kappa \in \mathcal{H}\} = \bigoplus_{E_2^*}\{k_\kappa a_\kappa \mid \kappa \in \mathcal{H}\},$$

which gives $y' = \bigwedge_{E^*}\{(k_\kappa a_\kappa)' \mid \kappa \in \mathcal{H}\}$. Hence y' in E_1^* and E_2^* coincides. It follows that for $y, z \in E^*$ there exists $y \oplus_1^* z$ iff $y \oplus_2^* z$ exists iff $z \le y'$. Let $A_z = \{c_\alpha \mid \alpha \in \Lambda\} \subseteq A_E$ and $l_\alpha \le \operatorname{ord}(c_\alpha)$, $\alpha \in \Lambda$ be such that $z = \bigvee_{E^*}\{l_\alpha c_\alpha \mid \alpha \in \Lambda\}$. Then $A_y \cup A_z \subseteq A \subseteq A_E$ for some maximal orthogonal set A of atoms and hence by Theorem 2.9 there are unique blocks M of E, M_1^* of E_1^* and M_2^* of E_2^* such that $A \subseteq M \cap M_1^* \cap M_2^*$. Moreover by Theorem 2.9 we have $M_1^* = M_2^* = \mathcal{MC}(M)$, which by Lemma 2.5 implies that $\oplus^*_{1/M_1^*} = \oplus^*_{2/M_2^*}$. Since $x, y \in M_1^* \cap M_2^*$ we obtain that $x \oplus_1^* y = x \oplus_2^* y \in M_1^* \cap M_2^*$. This proves that $\oplus_1^* = \oplus_2^*$ on E^*. □

Note that in [13] the necessary and sufficient conditions for isomorphism of two Archimedean atomic lattice effect algebras are given. These conditions are based on isomorphism of their atomic blocks.

Finally note that if $(E; \oplus, 0, 1)$ is a complete lattice effect algebra with atomic center $C(E)$ then E is isomorphic to a direct product of the family $\{[0, p] \mid p \in E$ atom of $C(E)\}$ of irreducible lattice effect algebras. This is because then $C(E)$ is a complete sublattice of E and hence then $\bigvee_{C(E)} A_{C(E)} = \bigvee_E A_{C(E)} = 1$, where $A_{C(E)} = \{p \in C(E) \mid p \text{ atom of } C(E)\}$ (see [23, Theorem 3.1]).

M. Kalina showed (see [8]) that for an Archimedean atomic lattice effect algebra E with atomic center $C(E)$ the condition $\bigvee_E A_{C(E)} = 1$ need not be satisfied. Hence the center $C(E)$ of E need not be a *bifull sub-lattice* of E (meaning that $\bigvee_{C(E)} D = \bigvee_E D$ for any $D \subseteq C(E)$ for which at least one of the elements $\bigvee_{C(E)} D$, $\bigvee_E D$ exists).

This occurs e. g., for every sub-lattice effect algebra E_1 of finite and cofinite elements of the direct product $E = G \times B$, where B is a complete Boolean algebra with countably many atoms and G is an irreducible Archimedean atomic (o)-continuous lattice effect algebra with infinite top element. M. Kalina constructed such lattice effect algebra G in [8].

Theorem 2.11. Let E be an Archimedean atomic lattice effect algebra with atomic center $C(E)$. The following conditions are equivalent:

(i) $\bigvee_E A_{C(E)} = 1$.

(ii) For every $a \in A_E$ there exists $p_a \in A_{C(E)}$ such that $a \leq p_a$.

(iii) For every $z \in C(E)$ it holds:

$$z = \bigvee_{C(E)} \{p \in A_{C(E)} \mid p \leq z\} = \bigvee_E \{p \in A_{C(E)} \mid p \leq z\}.$$

(iv) $C(E)$ is a bifull sub-lattice of E.

In this case E is isomorphic to a subdirect product of Archimedean atomic irreducible lattice effect algebras.

Proof.

(i) $\Longleftrightarrow$ (ii): This was proved in [25, Lemma 1].

(i) $\Longrightarrow$ (iii): Let $z \in C(E)$. Then, as $C(E) \subseteq B(E)$, we have by [7] that

$$z = z \wedge \bigvee_E A_{C(E)} = \bigvee_E \{z \wedge p \mid p \in A_{C(E)}\} = \bigvee_E \{p \in A_{C(E)} \mid p \leq z\}.$$

The last follows from the fact that $p \wedge z \in C(E)$ for all $p \in A_{C(E)}$.

(iii)$\Longrightarrow$ (iv): Let $D \in C(E)$ and let there exist $\bigvee_{C(E)} D = d \in C(E)$. Using (iii) we have that $z = \bigvee_{C(E)}\{p \in A_{C(E)} \mid p \leq z\} = \bigvee_E\{p \in A_{C(E)} \mid p \leq z\}$, for every $z \in C(E)$. Moreover, for every $p \in A_{C(E)}$, $p \leq d$ we have

$$p = p \wedge \bigvee_{C(E)} \{z \in C(E) \mid z \in D\} = \bigvee_{C(E)} \{p \wedge z \in C(E) \mid z \in D\},$$

hence there exists $z \in D$ such that $p \leq z$. Conversely, $p \in A_{C(E)}$, $p \leq z \in D$ imply that $p \leq d$. This proves that

$$\{p \in A_{C(E)} \mid p \leq d\} = \bigcup \{\{p \in A_{C(E)} \mid p \leq z\} \mid z \in D\},$$

which by (iii) gives that

$$\begin{aligned}\bigvee_{C(E)} D = d &= \bigvee_E \{p \in A_{C(E)} \mid p \leq d\} = \bigvee_E \bigcup \{\{p \in A_{C(E)} \mid p \leq z\} \mid z \in D\} \\ &= \bigvee_E \{\bigvee_E \{p \in A_{C(E)} \mid p \leq z\} \mid z \in D\} = \bigvee_E \{z \in C(E) \mid z \in D\} = \bigvee_E D.\end{aligned}$$

Since $D \subseteq C(E)$ iff $D' = \{z' \mid z \in D\} \subseteq C(E)$, we obtain that $\bigwedge_{C(E)} D = \bigwedge_E D$.

(iv) $\Longrightarrow$ (i): This is trivial.

Now, assume that (i) holds. Then from [23, Theorem 3.1] we get that E is isomorphic to a subdirect product of Archimedean atomic irreducible lattice effect algebras. □

Open Problem. Assume that $(E; \oplus, 0, 1)$ is an Archimedean atomic lattice effect algebra such that some effect-algebraic $\oplus^*$-operation onto $\widehat{E} = \mathcal{MC}(E)$ exists. Still unanswered question is whether then there exists also such $\widehat{\oplus}$-operation on $\widehat{E}$ that extends the operation $\oplus$.

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Zdenka Riečanová, Department of Mathematics, Faculty of Electrical Engineering and Information Technology, Slovak University of Technology, Ilkovičova 3, SK-812 19 Bratislava. Slovak Republic.

e-mail: zdenka.riecanova@stuba.sk