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COMPATIBILITY AND CENTRAL ELEMENTS IN PSEUDO-EFFECT ALGEBRAS

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An equivalent definition of compatibility in pseudo-effect algebras is given, and its relationships with central elements are investigated. Furthermore, pseudo-MV-algebras are characterized among pseudo-effect algebras by means of compatibility.

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1. INTRODUCTION

Effect algebras (alias D-posets) have been independently introduced in 1994 by D. J. Foulis and M. K. Bennett in [3] and by F. Chovanec and F. Kôpka in [15] for modelling unsharp measurement in a quantum mechanical system. They are a generalization of many structures which arise in Quantum Physics [8] and in Mathematical Economics [4, 13], in particular they are a generalization of orthomodular posets and MV-algebras.

G. Georgescu and A. Iorgulescu in [14] introduced the concept of a pesudo-MV-algebra, which is a non-commutative generalization of an MV-algebra, and A. Dvurečenskij and T. Vetterlein in [9] introduced the more general structure of a pseudo-effect algebra, which is a non-commutative generalization of an effect algebra. The investigation of these structures is motivated by quantum mechanical experiments. For a study see for example [9, 16, 18].

In this paper we investigate compatibility in pseudo-effect algebras. The aim is to generalize results found by Riečanová in [17] for effect algebras. To this end we give a definition of compatible elements in a pseudo-effect algebra which is a direct generalization of the one given in [17, Def. 2.3]. After deriving several consequences from our definition, we show that it is equivalent to the one given in [12, §3, p. 267].

In the final section we establish the relationship between central elements and compatibility, making use also of some results from [1]. We also give a characterization of pseudo-MV-algebras as those lattice pseudo-effect algebras in which all elements are pairwise compatible, thus extending a well-known result of effect algebras to the non-commutative setting.

2. PRELIMINARIES

Definition 2.1. A partial algebra $(E, +, 0, 1)$, where $+$ is a partial binary operation and 0, 1 are constants, is called a *pseudo-effect-algebra* if, for all a, b, $c \in E$, the following properties hold:

(P1) The sums $a + b$ and $(a + b) + c$ exist if and only if $b + c$ and $a + (b + c)$ exist, and in this case $(a + b) + c = a + (b + c)$.

(P2) For any $a \in E$, there exist exactly one $d \in E$ and exactly one $e \in E$ such that $a + d = e + a = 1$.

(P3) If $a + b$ exists, there are d, $e \in E$ such that $a + b = d + a = b + e$.

(P4) If $1 + a$ or $a + 1$ exists, then $a = 0$.

We note that, if $+$ is commutative, then E becomes an effect algebra.

If we define $a \leq b$ if and only if there exists $c \in E$ such that $a + c = b$, then $\leq$ is a partial ordering on E such that $0 \leq a \leq 1$ for any $a \in E$. If E is a lattice with respect to this order, then we say that E is a *lattice pseudo-effect algebra* or a *pseudo-D-lattice*.
If E is a pseudo-effect algebra, we can define two partial binary operations on E such that, for $a, b \in E$, a/b is defined if and only if $b\backslash a$ is defined if and only if $a \leq b$, and in this case we have $(b\backslash a) + a = a + (a/b) = b$. In particular, we set ${}^{\perp}a = 1\backslash a$ and $a^{\perp} = a/1$.

In the sequel, we denote by E a pseudo-effect algebra and by L a pseudo-D-lattice. If $a, b \in E$, we write $a \perp b$ to mean that the sum $a + b$ is defined. Moreover, we put $[a, b] = \{\, c \in E \mid a \leq c \text{ and } c \leq b \,\}$.

The following properties of pseudo-effect algebras will be used (for the proofs we refer to [2, 7, 10, 18]):

Proposition 2.2. For every $a, b, c \in E$, we have:

(i) $a \perp b$ if and only if $a \leq {}^{\perp}b$ if and only if $b \leq a^{\perp}$.

(ii) If $a \leq b$, then $b\backslash(a/b) = (b\backslash a)/b = a$. In particular (for $b = 1$) we have ${}^{\perp}(a^{\perp}) = ({}^{\perp}a)^{\perp} = a$.

(iii) If $a \perp c$ and $b \perp c$, then $a + c = b + c$ implies $a = b$; similarly, if $c \perp a$ and $c \perp b$, then $c + a = c + b$ implies $a = b$.

(iv) If $a \leq b$, then $b \perp c$ implies $a + c \leq b + c$ and $c \perp b$ implies $c + a \leq c + b$.

(v) If $a \leq b \leq c$, then $b\backslash a \leq c\backslash a$ and $a/b \leq a/c$.

(vi) If $a \leq b \leq c$, then $c\backslash b \leq c\backslash a$ and $b/c \leq a/c$.

(vii) If $a \leq b \leq c$, then $(c\backslash a)\backslash(b\backslash a) = c\backslash b$ and $(a/b)/(a/c) = b/c$.

(viii) If $a \leq b \leq c$, then $(c\backslash b)/(c\backslash a) = b\backslash a$ and $(a/c)\backslash(b/c) = a/b$.

(ix) If $a \leq b$, then $b\backslash a = {}^{\perp}b/{}^{\perp}a$ and $a/b = a^{\perp}\backslash b^{\perp}$.

(x) If $a \perp b$ and $a \vee b$ exists, then $a \vee b \leq a + b$.

(xi) If $a \leq b$, $a \leq c$ and $b \wedge c$ exists, then $(b\backslash a) \wedge (c\backslash a)$ exists and equals $(b \wedge c)\backslash a$.

(xii) If $a \leq b$, $a \leq c$ and $b \wedge c$ exists, then $(a/b) \wedge (a/c)$ exists and equals $a/(b \wedge c)$.

Proposition 2.3. Let $a, b \in E$, and suppose that $a \wedge b$ exists. Then $(a \wedge b)^{\perp} = a^{\perp} \vee b^{\perp}$ and ${}^{\perp}(a \wedge b) = {}^{\perp}a \vee {}^{\perp}b$.

Similarly, if $a \vee b$ exists, then $(a \vee b)^{\perp} = a^{\perp} \wedge b^{\perp}$ and ${}^{\perp}(a \vee b) = {}^{\perp}a \wedge {}^{\perp}b$.

Proposition 2.4. Let $a, b \in E$ such that $a \vee b$ exists. For every $c \in E$ with $a \vee b \leq c$, we have $c\backslash(a \vee b) = (c\backslash a) \wedge (c\backslash b)$ and $(a \vee b)/c = (a/c) \wedge (b/c)$.

Proposition 2.5. Let $a, b, c \in E$ such that $a \leq c$ and $b \leq c$. If $(c\backslash a) \vee (c\backslash b)$ or $(a/c) \vee (b/c)$ exists, then $a \wedge b$ exists, and we have $c\backslash(a \wedge b) = (c\backslash a) \vee (c\backslash b)$ or, respectively, $(a \wedge b)/c = (a/c) \vee (b/c)$.

Sharp elements in pseudo-effect algebras are defined in the same way as in the commutative case (see [1, Def. 3.1]).

Definition 2.6. We say that $p \in E$ is *sharp* if $p \wedge p^{\perp} = 0$.

Proposition 2.7. An element $p \in E$ is sharp if and only if $p \wedge {}^{\perp}p = 0$.

Proof. See [1, Prop. 3.2]. □

A key role in this paper is played by central elements. We recall the definition of central element as given in [6, Def. 2.1].

Definition 2.8. We say that $p \in E$ is *central* if there exists an isomorphism $f\colon E \to [0, p] \times [0, p^{\perp}]$ such that

- $f(p) = (p, 0)$;
- for every $a \in E$, if $f(a) = (a_1, a_2)$, then $a = a_1 + a_2$.

The set of all central elements of E is called the *centre* of E, and denoted by $C(E)$.

We need a number of facts about central elements, which we summarize in the proposition below. The reader is referred to [1, 6] for the proofs.

Proposition 2.9. Let $p \in E$ be central. The following hold:

(i) p is sharp.

(ii) For every $a \in E$, both $a \vee p$ and $a \wedge p$ exist.

(iii) $p^{\perp}$ also is central, and ${}^{\perp}p = p^{\perp}$.

(iv) If $a \leq p$, $b \leq p$, and $a \perp b$, then $a + b \leq p$.

(v) For every $a \in E$, we have $a = (a \wedge p) + (a \wedge p^{\perp}) = (a \wedge {}^{\perp}p) + (a \wedge p)$.

In the sequel we will also make use of the following characterizations of central elements.

Theorem 2.10. Any $p \in E$ is central if and only if for every $a \in E$, both $a \wedge p$ and $a \wedge p^{\perp}$ exist and we have

$$a = (a \wedge p) \vee (a \wedge p^{\perp}).$$

Proof. See [1, Theor. 3.17 and Def. 3.6]. □

In a pseudo-D-lattice, central elements can also be characterized as follows.

Proposition 2.11. Let $p \in L$. The following are equivalent:

(a) p is central.

(b) For every $a \in L$ we have $a = (a \wedge p) + (a \wedge p^{\perp}) = (a \wedge p^{\perp}) + (a \wedge p)$.

(c) p is sharp and, for every $a \in L$, we have $a \backslash (a \wedge p^{\perp}) \leq p$ and $a \backslash (a \wedge p) \leq p^{\perp}$.

(d) p is sharp and, for every $a \in L$, we have $(a \wedge p)/a \leq p^{\perp}$ and $(a \wedge p^{\perp})/a \leq p$.

Proof. See [1, Prop. 3.18]. □

3. COMPATIBLE ELEMENTS

We adopt a definition of compatible elements which closely resembles the one given in [17] for the commutative case. Some equivalents of this definition are presented, too.

We also show that our definition of compatibility agrees with the one which is found in the literature (see [5] or [12]).

Definition 3.1. We say that $a, b \in E$ are *compatible*, and write $a \leftrightarrow b$, if there exist $u, v \in E$ such that:

(C1) $a, b \in [u, v]$;

(C2) $a \backslash u = v \backslash b$;

(C3) $v \backslash a = b \backslash u$.

It is apparent that, for every $a, b \in E$, one has $a \leftrightarrow b$ if and only if $b \leftrightarrow a$.

Proposition 3.2. Let $a, b \in E$.

(i) If a and b are comparable, then $a \leftrightarrow b$.

(ii) If $a \perp b$, $b \perp a$ and $a + b = b + a$, then $a \leftrightarrow b$.

(iii) Suppose that both $a \vee b$ and $a \wedge b$ exist. If $(a \vee b) \backslash b = a \backslash (a \wedge b)$ and $(a \vee b) \backslash a = b \backslash (a \wedge b)$, then $a \leftrightarrow b$.

Proof.

(i) Suppose $a \leq b$ and let $u = a$, $v = b$. Trivially $a, b \in [u, v]$. Moreover $a\backslash u = 0 = v\backslash b$ and $v\backslash a = b\backslash a = b\backslash u$.

(ii) Let $u = 0$, $v = a+b = b+a$. Clearly $a, b \in [u, v]$. Moreover $a\backslash u = a = (a+b)\backslash b = v\backslash b$ and $v\backslash a = (b+a)\backslash a = b = b\backslash u$.

(iii) Let $u = a \wedge b$ and $v = a \vee b$. Clearly $a, b \in [u, v]$. Moreover $a\backslash u = a\backslash(a \wedge b) = (a \vee b)\backslash b = v\backslash b$ and $v\backslash a = (a \vee b)\backslash a = b\backslash(a \wedge b) = b\backslash u$.

□

In the commutative case the definition of compatible elements is simpler, because (C1) and (C2) imply (C3). We are going to see that, in general, this is not true.

Example 3.3. Let $E = \{0, a, b, c, 1\}$ with $a+b = b+c = c+a = 1$, while the sums $b+a$, $c+b$ and $a+c$ are not defined (see the remark at the end of §2 in [2]).

Taking $u = 0$ and $v = 1$, we have that (C1) and (C2) are satisifed but (C3) does not hold.

Proof. It is obvious that $a, b \in [u, v]$, i.e. (C1) is satisfied. Moreover $a\backslash u = a = 1\backslash b = v\backslash b$, so that (C2) is satisfied, too. On the other hand $v\backslash a = 1\backslash a = c \neq b = b\backslash u$, hence (C3) does not hold. □

In order to characterize compatible elements, we first establish some preliminary facts which will be used in the sequel.

Lemma 3.4. *Let $a, b, c \in E$. If $a \leq b \leq c$, then $a + (b/c) = (b\backslash a)/c$ and $c\backslash(a/b) = (c\backslash b) + a$.*

Proof. Note that $b/c \leq b^{\perp} \leq a^{\perp}$ and $c\backslash b \leq {}^{\perp}b \leq {}^{\perp}a$, so that the sums are defined.

We have $(b\backslash a) + a + (b/c) = b + (b/c) = c$ whence $a + (b/c) = (b\backslash a)/c$. Similarly for the other equality. □

Corollary 3.5. *If $u \leq b \leq v$ then:*

(i) $b/v = u/\big((b\backslash u)/v\big)$;

(ii) $u/b = \big((v\backslash b) + u\big)/v$;

(iii) $v\backslash b = \big(v\backslash(u/b)\big)\backslash u$;

(iv) $b\backslash u = v\backslash\big(u + (b/v)\big)$.

Proof.

(i) By Lemma 3.4, we have $u + (b/v) = (b\backslash u)/v$; hence

$$b/v = u/\big(u + (b/v)\big) = u/\big((b\backslash u)/v\big).$$

(ii) By Lemma 3.4, we have $v\backslash(u/b) = (v\backslash b) + u$; hence, applying Proposition 2.2(ii),

$$u/b = \big(v\backslash(u/b)\big)/v = \big((v\backslash b) + u\big)/v.$$

(iii) Similar to (i).

(iv) Similar to (ii).

□

Now we are ready to characterize compatible elements in pseudo-effect algebras.

Proposition 3.6. Let $a, b \in E$. The following are equivalent:

(a) $a \leftrightarrow b$.

(b) There exist $u, v \in E$, with $a, b \in [u, v]$, such that $a\backslash u = v\backslash b$ and $u/a = b/v$.

(c) There exist $u, v \in E$, with $a, b \in [u, v]$, such that $a/v = u/b$ and $v\backslash a = b\backslash u$.

(d) There exist $u, v \in E$, with $a, b \in [u, v]$, such that $a/v = u/b$ and $u/a = b/v$.

(e) There exist $r, s \in E$, with $r \leq b$ and $s \leq a$, such that $s/a = r/b$, $r \perp a$, $s \perp b$ and $r + a = s + b$.

Proof.

(a)⇒(b) Let $r = v\backslash a = b\backslash u$ and $s = a\backslash u = v\backslash b$, where u and v satisfy (C1), (C2) and (C3). Applying Corollary 3.5(ii), we have

$$u/a = \big((v\backslash a) + u\big)/v = \big((b\backslash u) + u\big)/v = b/v.$$

(b)⇒(d) is proved in a similary way, as well as (d)⇒(c) and (c)⇒(a).

(a)⇒(e) Let $r = v\backslash a = b\backslash u$ and $s = a\backslash u = v\backslash b$, where u and v satisfy (C1), (C2) and (C3). Clearly $r \leq b$ and $s \leq a$. Moreover we have $r + a = (v\backslash a) + a = v = (v\backslash b) + b = s + b$ and, applying Proposition 2.2(ii), $s/a = (a\backslash u)/a = u = (b\backslash u)/b = r/b$.

(e)⇒(d) Let $u = s/a = r/b$ and $v = r + a = s + b$, where r and s satisfy (e). Clearly $a, b \in [u, v]$. Moreover, applying Proposition 2.2(ii), we have $a\backslash u = a\backslash(s/a) = s = (s + b)\backslash b = v\backslash b$ and $v\backslash a = (r + a)\backslash a = r = b\backslash(r/b) = b\backslash u$.

□

Corollary 3.7. Given $a, b \in E$, we have $a \leftrightarrow b$ if and only if $a^{\perp} \leftrightarrow b^{\perp}$.

P r o o f. Suppose $a \leftrightarrow b$, and let u and v satisfy condition (d) of the previous proposition. Then $a^\perp, b^\perp \in [v^\perp, u^\perp]$; moreover, by Proposition 2.2(ix), we have $a^\perp \backslash v^\perp = a/v = u/b = u^\perp \backslash b^\perp$ and $u^\perp \backslash a^\perp = u/a = b/v = b^\perp \backslash v^\perp$.

The reverse implication is proved in a similar way. □

The previous proposition allows us to show the equivalence between our definition of compatible elements and the one given in [12, §3, p. 267].

Proposition 3.8. Let $a, b \in E$. Then $a \leftrightarrow b$ if and only if

$$\begin{aligned}&\exists a_1, b_1, c \in E: \quad a_1 \perp b, \quad b_1 \perp a, \quad a_1 \perp c, \quad b_1 \perp c,\\ &a_1 + b = b_1 + a, \qquad a = a_1 + c, \qquad b = b_1 + c.\end{aligned} \tag{1}$$

P r o o f. We show that (1) is equivalent to condition 3.6(e).

Suppose that 3.6(e) hold. Let $a_1 = s$, $b_1 = r$ and $c = s/a = r/b$. Then $a_1 \perp b$, $b_1 \perp a$ and $a_1 + b = b_1 + a$. Moreover $a = s + (s/a) = a_1 + c$ and $b = r + (r/b) = b_1 + c$.

Conversely, assume that (1) is verified. Let $r = b_1$ and $s = a_1$. Since $b = r + c$ and $a = s + c$, we get $r \leq b$ and $s \leq a$. Furthermore $s/a = s/(s + c) = c = r/(r + c) = b$. Finally we have $r \perp a$, $s \perp b$ and $r + a = b_1 + a = a_1 + b = s + b$. □

The next result gives a characterization of compatible elements in pseudo-D-lattices, too.

Theorem 3.9. Let $a, b \in E$. Assume that both $a \vee b$ and $a \wedge b$ exist (in particular, assume that E is a pseudo-D-lattice). We have $a \leftrightarrow b$ if and only if there exists $c \in E$ such that:

$$c \leq a \wedge b \tag{2}$$

$$c \leq {}^\perp(a \vee b) \tag{3}$$

$$c + \big((a \vee b)\backslash a\big) = \big(b\backslash(a \wedge b)\big) + c \tag{4}$$

$$c + \big((a \vee b)\backslash b\big) = \big(a\backslash(a \wedge b)\big) + c. \tag{5}$$

P r o o f. Suppose $a \leftrightarrow b$, and let u and v satisfy (C1), (C2) and (C3). Applying Lemma 3.5(ii), we get

$$\begin{aligned}u/a &= \big((v\backslash a) + u\big)/v = \big((b\backslash u) + u\big)/v = b/v,\\ u/b &= \big((v\backslash b) + u\big)/v = \big((a\backslash u) + u\big)/v = a/v.\end{aligned} \tag{6}$$

Since $u \leq a \wedge b$, we may define $c = (a \wedge b)\backslash u$, and in this way (2) is satisfied. Observe further that $a \vee b \leq v$; thus, applying Proposition 2.4 and Proposition 2.2(xii), we have

$$v\backslash(a \vee b) = (v\backslash a) \wedge (v\backslash b) = (b\backslash u) \wedge (a\backslash u) = (b \wedge a)\backslash u = c, \tag{7}$$

and consequently, by Propostion 2.2(v), we also have

$$c = v\backslash(a \vee b) \leq 1\backslash(a \vee b) = {}^\perp(a \vee b),$$

so that (3) is satisfied, too.

Let $r = c + \big((a \vee b)\backslash a\big)$ and $s = c + \big((a \vee b)\backslash b\big)$ (note that, in view of (3), both these sums are defined). The definition of c implies that $a \wedge b = c + u$; hence, taking (7) and (6) in account, we obtain

$$\begin{aligned} r + a &= c + \big((a \vee b)\backslash a\big) + a = c + (a \vee b) = v \\ &= b + b/v = b + u/a = \big(b\backslash(a \wedge b)\big) + (a \wedge b) + u/a \\ &= \big(b\backslash(a \wedge b)\big) + c + u + u/a = \big(b\backslash(a \wedge b)\big) + c + a, \end{aligned}$$

and consequently, by Proposition 2.2(iii), we have $r = \big(b\backslash(a \wedge b)\big) + c$, which gives (4). In a similar way we obtain (5).

Conversely, let c satisfy (2), (3), (4) and (5). Set

$$\begin{aligned} r &= c + \big((a \vee b)\backslash a\big) = \big(b\backslash(a \wedge b)\big) + c, \\ s &= c + \big((a \vee b)\backslash b\big) = \big(a\backslash(a \wedge b)\big) + c. \end{aligned}$$

Note that, as $c \leq a \wedge b$, we have

$$r = \big(b\backslash(a \wedge b)\big) + c \leq \big(b\backslash(a \wedge b)\big) + (a \wedge b) = b$$

and, similarly, we obtain $s \leq a$. Furthermore,

$$a = \big(a\backslash(a \wedge b)\big) + (a \wedge b) = \big(a\backslash(a \wedge b)\big) + c + c/(a \wedge b) = s + c/(a \wedge b),$$

so that $s/a = c/(a \wedge b)$. Similarly one obtains $b = r + c/(a \wedge b)$, and hence $r/b = s/a$.

Finally, we have

$$\begin{aligned} r + a &= c + \big((a \vee b)\backslash a\big) + a \\ &= c + (a \vee b) = c + \big((a \vee b)\backslash b\big) + b = s + b. \end{aligned}$$

We conclude that conditon 3.6(e) is satisfied, and therefore $a \leftrightarrow b$. □

Corollary 3.10. Given $a, b \in E$ such that $a \vee b$ exists and $a \wedge b = 0$, we have $a \leftrightarrow b$ if and only if $(a \vee b)\backslash b = a$ and $(a \vee b)\backslash a = b$.

P r o o f. Indeed, if c satisfies (2), we must have $c = 0$. □

4. CENTRAL ELEMENTS

In this section we will see how central elements and compatibility are related.

First we establish some facts which are of independent interest.

Proposition 4.1. If $p \in E$ is central then, for every $a \in E$:

(i) $a \wedge p$, $a \wedge {}^{\perp}p$ and $a \wedge p^{\perp}$, as well as $a \vee p$, $a \vee {}^{\perp}p$ and $a \vee p^{\perp}$ exist;

(ii) $a\backslash(a \wedge p) = a \wedge {}^{\perp}p$;

(iii) $(a \wedge p)/a = a \wedge p^{\perp}$;

(iv) $(a \vee p)\backslash a = {}^{\perp}a \wedge p$;

(v) $a/(a \vee p) = a^{\perp} \wedge p$.

Proof. Let $a \in E$. Then (i) follows from Proposition 2.9(ii) and (iii). Now, by Proposition 2.9(v), we have $a = (a \wedge {}^{\perp}p) + (a \wedge p) = (a \wedge p) + (a \wedge p^{\perp})$, hence $a\backslash(a \wedge p) = a \wedge {}^{\perp}p$ and $(a \wedge p)/a = a \wedge p^{\perp}$.

Moreover, taking into account Proposition 2.9(iii), it follows from (ii), applying Proposition 2.2(ix), Proposition 2.3 and Proposition 2.2(ii), that

$$a/(a \vee p) = a^{\perp}\backslash(a^{\perp} \wedge p^{\perp}) = a^{\perp} \wedge ({}^{\perp}p)^{\perp} = a^{\perp} \wedge p.$$

Similarly one gets that $(a \vee p)\backslash a = {}^{\perp}a \wedge p$. □

Lemma 4.2. If $p \in E$ is central then, for every $a \in E$, ($a \wedge p^{\perp}$, $a \wedge {}^{\perp}p$ and $a \vee p$ exists, and) we have

$$p + (a \wedge p^{\perp}) = (a \wedge {}^{\perp}p) + p = a \vee p.$$

Proof. In view of Proposition 4.1(i), all suprema and infima used below in this proof are defined.

Let $a \in E$. In [1, Lemma 3.13], it has been proved that

$$p + (a \wedge p^{\perp}) = (a \wedge {}^{\perp}p) + p = (a \wedge {}^{\perp}p) \vee p.$$

Hence it suffices to show that

$$(a \wedge {}^{\perp}p) \vee p = a \vee p. \tag{8}$$

By Proposition 2.4, Proposition 2.9(iii) and Proposition 4.1(iii), we get

$$\begin{aligned} (a \vee p)\backslash\big((a \wedge {}^{\perp}p) \vee p\big) &= \big((a \vee p)\backslash(a \wedge {}^{\perp}p)\big) \wedge \big((a \vee p)\backslash p\big) \\ &= \big((a \vee p)\backslash(a \wedge p^{\perp})\big) \wedge \big((a \vee p)\backslash p\big) \\ &= \Big((a \vee p)\backslash\big((a \wedge p)/a\big)\Big) \wedge \big((a \vee p)\backslash p\big). \end{aligned} \tag{9}$$

Now, observe that by Proposition 4.1(iv) we have $(a \vee p)\backslash a = {}^{\perp}a \wedge p$. Thus, from (9), applying Lemma 3.4 (with $a \wedge p$ in place of a, a in place of b and $a \vee b$ in place of c) and Proposition 2.9(iv), we obtain

$$\begin{aligned} (a \vee p)\backslash\big((a \wedge {}^{\perp}p) \vee p\big) &= \Big((a \vee p)\backslash\big((a \wedge p)/a\big)\Big) \wedge \big((a \vee p)\backslash p\big) \\ &= \Big(\big((a \vee p)\backslash a\big) + (a \wedge p)\Big) \wedge \big((a \vee p)\backslash p\big) \\ &= \big(({}^{\perp}a \wedge p) + (a \wedge p)\big) \wedge \big((a \vee p)\backslash p\big) \\ &\leq p \wedge \big((a \vee p)\backslash p\big) \leq p \wedge {}^{\perp}p. \end{aligned}$$

Since, by Proposition 2.9(i) and (iii), $p \wedge {}^{\perp}p = 0$, we conclude that $(a \vee p)\backslash\big((a \wedge {}^{\perp}p) \vee p\big) = 0$, and (8) follows. □

Proposition 4.3. If $p \in E$ is central then, for every $a \in E$, ($a \wedge p$ exists and) we have

$$p = (a \wedge p) + (a^{\perp} \wedge p) = ({}^{\perp}a \wedge p) + (a \wedge p).$$

P r o o f. It suffices to show that

$$\forall a \in E \qquad p = ({}^{\perp}a \wedge p) + (a \wedge p). \tag{10}$$

Indeed, given $a \in E$, by (10) and Proposition 2.2(ii) we have

$$p = \big({}^{\perp}(a^{\perp}) \wedge p\big) + \big(a^{\perp} \wedge p\big) = (a \wedge p) + (a^{\perp} \wedge p).$$

Now we prove (10). Fix $a \in E$. Since by Proposition 2.9(iii) $p^{\perp}$ is central and ${}^{\perp}p = p^{\perp}$, we can apply Lemma 4.2, and we obtain

$$p^{\perp} + ({}^{\perp}a \wedge p) = {}^{\perp}a \vee p^{\perp}. \tag{11}$$

Finally, by Proposition 2.2(ix), Proposition 2.3, Proposition 2.9(iii) and (11), we have

$$\begin{aligned} p\backslash(a \wedge p) &= {}^{\perp}p/{}^{\perp}(a \wedge p) = {}^{\perp}p/({}^{\perp}a \vee {}^{\perp}p) \\ &= p^{\perp}/({}^{\perp}a \vee p^{\perp}) = p^{\perp}/\big(p^{\perp} + ({}^{\perp}a \wedge p)\big) = {}^{\perp}a \wedge p, \end{aligned}$$

and therefore $p = ({}^{\perp}a \wedge p) + (a \wedge p)$. □

The following result will help in characterizing central elements by means of compatibility.

Proposition 4.4. If $p \in E$ is central then, for every $a \in E$, both $a \vee p$ and $a \wedge p$ exist, and the following hold:

$$(a \vee p)\backslash p = a\backslash(a \wedge p), \tag{12}$$

$$(a \vee p)\backslash a = p\backslash(a \wedge p). \tag{13}$$

P r o o f. As already observed, for every $a \in E$, both $a \vee p$ and $a \wedge p$ exist. Now fix $a \in E$. By Proposition 4.1(ii), we have $a\backslash(a \wedge p) = a \wedge {}^{\perp}p$. On the other hand, by Lemma 4.2, $(a \wedge {}^{\perp}p) + p = a \vee p$, so that $a \wedge {}^{\perp}p = (a \vee p)\backslash p$. Hence (12) follows.

Since by Proposition 4.1(iv) we have $(a \vee p)\backslash a = {}^{\perp}a \wedge p$, to prove (13) it suffices to show that

$${}^{\perp}a \wedge p = p\backslash(a \wedge p),$$

but this follows from Proposition 4.3. □

In the next theorem, compatibility is used to characterize central elements of a pseudo-D-lattice, thus generalizing a result of [17].

Lemma 4.5. Given $c, d \in L$, we have $(c \vee d)\backslash d = c\backslash(c \wedge d)$ if and only if $(c \wedge d)/d = c/(c \vee d)$.

P r o o f. Suppose that $(c \vee d)\backslash d = c\backslash(c \wedge d)$. Applying Corollary 3.5(i) and Proposition 2.2(ii), we have

$$\begin{aligned} c/(c \vee d) &= (c \wedge d)/\big((c\backslash(c \wedge d))/(c \vee d)\big) \\ &= (c \wedge d)/\big(((c \vee d)\backslash d)/(c \vee d)\big) = (c \wedge d)/d. \end{aligned}$$

Conversely, suppose that $c/(c \vee d) = (c \wedge d)/d$. Applying Corollary 3.5(iii) and Proposition 2.2(ii), we have

$$\begin{aligned} (c \vee d)\backslash d &= \big((c \vee d)\backslash((c \wedge d)/d)\big)\backslash(c \wedge d) \\ &= \big((c \vee d)\backslash(c/(c \vee d))\big)\backslash(c \wedge d) = c\backslash(c \wedge d). \end{aligned}$$

□

Theorem 4.6. Let $p \in L$ be sharp. The following are equivalent:

(a) p is central.

(b) For every $a \in L$, $(a \vee p)\backslash p = a\backslash(a \wedge p)$ and $(a \vee p)\backslash a = p\backslash(a \wedge p)$.

(c) For every $a \in L$, $(a \vee p)\backslash p = a\backslash(a \wedge p)$ and $(a \wedge p)/a = p/(a \vee p)$.

(d) For every $a \in L$, $(a \wedge p)/p = a/(a \vee p)$ and $(a \wedge p)/a = p/(a \vee p)$.

(e) For every $a \in L$, $(a \wedge p)/p = a/(a \vee p)$ and $(a \vee p)\backslash a = p\backslash(a \wedge p)$.

(f) For every $a \in L$, $p \leftrightarrow a$.

P r o o f.

(a)⇒(b) follows by Proposition 4.4.

(b)⇒(c) follows applying Lemma 4.5; in the same way one also proves that (c)⇒(d), (d)⇒(e) and (e)⇒(b).

(b)⇒(f) follows by Proposition 3.2(iii).

(f)⇒(b) Let $a \in L$. By Theorem 3.9, there exists c satisfying (2), (3), (4) and (5) In particular we must have $c \leq p \wedge {}^{\perp}p$ and hence $c = 0$, by Proposition 2.7.

(b)⇒(a) Let $a \in L$. Since $(a \vee p)\backslash a = p\backslash(a \wedge p)$, by Lemma 4.5 (with $c = p$ and $d = a$) we also have

$$(a \wedge p)/a = p/(a \vee p) \leq p^{\perp}. \tag{14}$$

Moreover, since $(a^{\perp} \vee p)\backslash a^{\perp} = p\backslash(a^{\perp} \wedge p)$, applying Proposition 2.2(ix) and (ii) and Proposition 2.3, we get

$$\begin{aligned} (a \wedge {}^{\perp}p)/a &= (a \wedge {}^{\perp}p)^{\perp}\backslash a^{\perp} = \big(a^{\perp} \vee ({}^{\perp}p)^{\perp}\big)\backslash a^{\perp} \\ &= (a^{\perp} \vee p)\backslash a^{\perp} = p\backslash(a^{\perp} \wedge p) \leq p. \end{aligned} \tag{15}$$

From (14) and (15), applying Proposition 2.4, and Proposition 2.7, it follows that

$$\big((a \wedge {}^{\perp}p) \vee (a \wedge p)\big)/a = \big((a \wedge {}^{\perp}p)/a\big) \wedge \big((a \wedge p)\big)/a \leq p \wedge p^{\perp} = 0.$$

Hence $a = (a \wedge {}^{\perp}p) \vee (a \wedge p)$. As a was arbitrary, we conclude, by Theorem 2.10, that ${}^{\perp}p$ is central. Therefore, by Proposition 2.9(iii) and Proposition 2.2(ii), p is central, too. □

In [11], pseudo-MV-algebras are characterized as follows.

Theorem 4.7. A pseudo-D-lattice L is (identifiable with) a pseudo-MV-algebra if and only if

$$\forall a, b \in L: \quad (a \vee b)\backslash b = a \backslash (a \wedge b) \tag{16}$$

or, equivalently,

$$\forall a, b \in L: \quad b/(a \vee b) = (a \wedge b)/a. \tag{17}$$

P r o o f. See [11, Theor. 8.7 and Prop. 8.15(γ) and (δ)]. □

The above facts allow us to give another characterization of pseudo-MV-algebras.

Theorem 4.8. A pseudo-D-lattice L is (identifiable with) a pseudo-MV-algebra if and only if

$$\forall a, b \in L: \quad a \leftrightarrow b. \tag{18}$$

P r o o f. In the light of Theorem 4.7, it suffices to prove that (16)⇒(18) and (18)⇒(17).

(16)⇒(18) This follows immediately from Proposition 3.2(iii).

(18)⇒(17) Given $a, b \in L$, let $d = b/(a \vee b)$ and $e = a/(a \vee b)$. Applying Proposition 2.5, we obtain

$$d \vee e = \big(b/(a \vee b)\big) \vee \big(a/(a \vee b)\big) = (a \wedge b)/(a \vee b). \tag{19}$$

Similarly, by Proposition 2.4, we get

$$d \wedge e = \big(b/(a \vee b)\big) \wedge \big(a/(a \vee b)\big) = (a \vee b)/(a \vee b) = 0.$$

Since $d \leftrightarrow e$, we may apply Corollary 3.10; thus, by Proposition 2.2(viii) and (19), we obtain

$$\begin{aligned}(a \wedge b)/a &= \big((a \wedge b)/(a \vee b)\big)\backslash\big(a/(a \vee b)\big)\\ &= (d \vee e)\backslash e = d = b/(a \vee b).\end{aligned}$$

□

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