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# QUANTUM LOGICS AND BIVARIABLE FUNCTIONS

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New approach to characterization of orthomodular lattices by means of special types of bivariable functions $G$ is suggested. Under special marginal conditions a bivariable function $G$ can operate as, for example, infimum measure, supremum measure or symmetric difference measure for two elements of an orthomodular lattice.

*Keywords:* finite atomistic quantum logic, orthomodular lattice, conditional state, s-map, d-map, bivariable functions, modeling infimum measure, supremum measure, simultaneous measurements

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## 1. INTRODUCTION

To model noncompatible events, a quantum logic was chosen among various algebraic structures as the suitable one. This paper deals with a characterization of a center in various types of quantum logics by means of special bivariable functions defined on them. Any quantum logic can be described as a union of blocks (a block in a given quantum logic $\mathcal{L}$ is the maximal Boolean subalgebra of $\mathcal{L}$) [16]. Center $C(\mathcal{L})$ of a quantum logic $\mathcal{L}$ is its Boolean subalgebra of elements compatible with all other elements of $L$. Each quantum logic $\mathcal{L}$ has a center that can be taken as a common part of its blocks. In this paper three types of quantum logics are studied:

(T1) a quantum logic $\mathcal{L}$ as a horizontal sum of $k$ maximal Boolean algebras (blocks);

(T2) a quantum logic $\mathcal{L}$ created from two blocks with non trivial center;

(T3) a quantum logic $\mathcal{L}$ with nontrivial center as a union of $k$ blocks $\mathcal{B}_i, i \leq k$, where $\mathcal{B}_i \cap \mathcal{B}_j \subset C(\mathcal{L})$ for $i \neq j$.

## 2. PRELIMINARIES

This section is devoted to basic notions of quantum logics theory. For more information see [3, 5, 15, 16, 17].

**Definition 2.1.** Let $L$ be a lattice (a nonempty set endowed with a partial ordering $\leq$, the lattice operations supremum $\vee$ and infimum $\wedge$) with the greatest element $1_L$ and the smallest element $0_L$. Let $\perp: L \to L$ be a unary operation on $L$ with the following properties:

1. for any $a \in L$ there is a unique $a^{\perp} \in L$ such that $(a^{\perp})^{\perp} = a$ and $a \vee a^{\perp} = 1_L$;
2. if $a, b \in L$ and $a \leq b$ then $b^{\perp} \leq a^{\perp}$;
3. if $a, b \in L$ and $a \leq b$ then $b = a \vee (a^{\perp} \wedge b)$ (orthomodular law).

Then $\mathcal{L} = (L, 0_L, 1_L, \vee, \wedge, \perp)$ is said to be an orthomodular lattice.

**Definition 2.2.** Let $\mathcal{L}$ be an orthomodular lattice. Then the elements $a, b \in L$ are:

1. *orthogonal* $(a \perp b)$ if $a \leq b^{\perp}$;
2. *compatible* $(a \leftrightarrow b)$ if $a = (a \wedge b) \vee (a \wedge b^{\perp})$ and $b = (a \wedge b) \vee (a^{\perp} \wedge b)$.

**Definition 2.3.** Let $\mathcal{L}$ be an orthomodular lattice. Then the element $a \in L - \{0_L\}$ is called *an atom of* $L$ if the following statement is true: $b < a$ implies $b = 0_L$.

**Definition 2.4.** Let $\mathcal{L}$ be an orthomodular lattice. $\mathcal{L}$ is called *atomistic* if each non-zero element from $L$ is supremum of atoms of $L$. If the set of all atoms of $L$ is finite, then it is said to be *finite*

**Definition 2.5.** Let $\mathcal{L}$ be a quantum logic. A finite orthogonal partition of $1_L$ is a system $\{a_i\}_{i=1}^{k}, k \in N$ of mutually orthogonal elements of $L$ such that $\bigvee_{i=1}^{k} a_i = 1_L$.

In this paper we will deal only with finite orthogonal partitions.

**Definition 2.6.** Let $\mathcal{L}$ be an orthomodular lattice. A map $m : L \to [0,1]$ satisfying conditions:

1. $m(0_L) = 0$ and $m(1_L) = 1$
2. if $a \perp b; a, b \in L$ then $m(a \vee b) = m(a) + m(b)$

is called a state on $L$.

Let us recall the existence of orthomodular lattices without any state [4, 14].

**Definition 2.7.** A quantum logic is an orthomodular lattice with at least one state.

Multidimensional states play an important role in this paper. So we introduce some essential definitions and properties. For more details see [6, 7, 13].

**Definition 2.8.** Let $\mathcal{L}$ be a quantum logic and $L_0 = L - \{0_L\}$. A map $m : L \times L_0 \to [0,1]$ satisfying conditions:

1. for all $a \in L_0$ $m(.|a)$ is a state on $L$;
2. for all $a \in L_0$ $m(a|a) = 1$;
3. if $\{a_i\}_{i=1}^{n}, n \in N$ are mutually orthogonal elements of $L_0$, then for each $b \in L$

$$m\Big(b|\bigvee_{i=1}^{n} a_i\Big) = \sum_{i=1}^{n} m\Big(a_i|\bigvee_{j=1}^{n} a_j\Big) \cdot m(b|a_i)$$

is called a conditional state on $L$.

**Definition 2.9.** Let $\mathcal{L}$ be an orthomodular lattice. An s-map on $L$ (a simultaneous measurement map) is a map $p : L^2 \to [0,1]$ satisfying the following conditions:

(s1) $p(1_L, 1_L) = 1$;

(s2) if $a \perp b$ for some $a, b \in L$, then $p(a,b) = 0$;

(s3) if $a \perp b$ for some $a, b \in L$, then for each $c \in L$

$$p(a \vee b, c) = p(a,c) + p(b,c) \quad \text{and} \quad p(c, a \vee b) = p(c,a) + p(c,b).$$

**Definition 2.10.** Let $\mathcal{L}$ be a quantum logic. A d-map on $L$ (a difference map) is a map $d : L^2 \to [0,1]$ satisfying the following conditions:

(d1) $d(1_L, 1_L) = 0$ and $d(0_L, 1_L) = d(1_L, 0_L) = 1$;

(d2) if $a \perp b$ then $d(a,b) = d(a, 0_L) + d(0_L, b)$;

(d3) if $a \perp b$ then for each $c \in L$

$$d(a \vee b, c) = d(a,c) + d(b,c) - d(0_L, c),$$

$$d(c, a \vee b) = d(c,a) + d(c,b) - d(c, 0_L).$$

**Lemma 2.11.** Let $\mathcal{L}$ be a quantum logic and let $p$ be an s-map on $L$. Then

(1) if $a \leftrightarrow b, a, b \in L$ then $p(a,b) = p(b,a)$,

(2) for any $a \in L : p(a, 1_L) = p(a,a)$,

(3) for any $a \in L : p(a^\perp, a^\perp) = 1 - p(a,a)$.

**Lemma 2.12.** Let $\mathcal{L}$ be a quantum logic and let $d$ be a d-map on $L$. Then

(1) if $a \leftrightarrow b, a, b \in L$ then $d(a,b) = d(b,a)$,

(2) for any $a \in L : d(a^\perp, a) = 1$,

(3) for any $a \in L : d(a,a) = 0$.

**Lemma 2.13.** Let $\mathcal{L}$ be a quantum logic, let $p$ be an s-map on $L$ and let $d$ be a d-map on $L$. Then

(1) a map $\nu_p : L \to [0,1], \nu_p(a) = p(a,a)$ is a state on $L$,

(2) a map $\mu_d : L \to [0,1], \mu_d(a) = d(a, 0_L)$ is a state on $L$.

Moreover if $L$ is a Boolean algebra, then an s-map represents a measure of infimum, i.e $p(a,b) = \nu_p(a \wedge b)$ and a d-map represents measure of symmetric difference of two elements, i. e. $d(a,b) = \mu_d(a \bigtriangleup b)$, where $a \bigtriangleup b = (a \wedge b^\perp) \vee (a^\perp \wedge b)$.

**Lemma 2.14.** Let $\mathcal{L}$ be a quantum logic and let $p$ be an s-map on $L$. Then a map $d_p : L^2 \to [0,1]$, $d_p(a,b) = p(a,b^\perp) + p(a^\perp, b)$ is a d-map.

A special type of a d-map from Lemma 2.14 is said to be a d-map induced by an s-map $p$. It is clear, that $d_p(a, 0_L) = p(a,a)$ for any $a \in L$.

An orthogonal partition of unit $1_L$ gives the whole information about the experiment that it creates. Indeed, let $\{a_1, \ldots, a_n\}$ be an orthogonal partition of unit $1_L$. Knowing values that a state $m$ admits at each $a_i$, in fact we have information about the values of $m$ on the whole Boolean algebra given by the partition, where $a_i, i = 1, 2, \ldots, n$ are its atoms.

The problem arises when the experiment organization leads to two partitions of $1_L$, $\{a_i\}_{i=1}^n$ a $\{b_j\}_{j=1}^k$, but we are not able to get information about the set of elements $\{a_i \wedge b_j\}_{ij}^{n\times k}$ that need not be, in general, a partition of $1_L$ (while a refinement of a partition always exists in a Boolean algebra).

In that case we have

$$\bigvee_{i=1}^{n} \bigvee_{j=1}^{k} a_i \wedge b_j < 1_L\,,$$

which means that $\{a_i \wedge b_j\}_{i,j}$ is not a partition of unit $1_L$.

Despite this fact the number

$$\sum_{i=1}^{n} \sum_{j=1}^{k} m(a_i \wedge b_j)$$

can be anywhere in $[0,1]$. Especially if $\sum_{i=1}^{n} \sum_{j=1}^{k} m(a_i \wedge b_j) = 1$, virtually we are in a Boolean algebra.

To eliminate such lack of information about $a_i \wedge b_j$ we can use information about $a_i$ conditioned by $b_j$, $i = 1, \ldots, n$, $j = 1, \ldots, k$, or probability (relative frequency) of event $b_j$ (i.e. $m(b_j)$) and $a_i$ under condition $b_j$ (i.e. $m(a_i|b_j)$, where $i = 1, \ldots, n$, $j = 1, \ldots, k$ and $m(\cdot|\cdot)$ is a conditional state defined in [7]. It was proved in [6] that a function $p(u,v) = m(u)m(v|u)$ behaves for compatible elements as a measure of intersection ($p(u,v) = m(u \wedge v)$), while for noncompatible elements $p(u,v) \neq p(v,u)$ can occur. Function $p$ on a quantum logics is an s-map (Definition 2.9) and its properties are studied in [1, 2, 6, 8, 9, 10, 11].

Let $p$ be an $s$-map on $L$. Then for any two finite orthogonal partitions $A, B$ of unit $1_L$ there is a probability distribution matrix

$$P(B) = \begin{pmatrix} m(b_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & m(b_k) \end{pmatrix}$$

and conditional probability $P(A|B)$ distribution matrix

$$P(A|B) = \begin{pmatrix} m(a_1|b_1) & \cdots & m(a_1|b_k) \\ \vdots & \ddots & \vdots \\ m(a_n|b_1) & \cdots & m(a_n|b_k) \end{pmatrix}.$$

Then "a common effect" matrix $P(A,B)$ of two orthogonal partitions of unit $1_L$ can be expressed as product of matrices $P(A|B)\cdot P(B)$. A matrix $P(A,B)$ can have the following expression by means of an $s$-map [6]:

$$P(A,B) = \begin{pmatrix} p(a_1,b_1) & \cdots & p(a_1,b_k) \\ \vdots & \ddots & \vdots \\ p(a_n,b_1) & \cdots & p(a_n,b_k) \end{pmatrix},$$

where $p(a_i,b_j) = m(b_j)m(a_i|b_j)$, $i=1,\dots,n$ a $j=1,\dots,k$.

At the first sight that approach described above is the same as in the classical probability theory. But, in contrast with it, Bayes theorem is not, in general, true. $P(A,B) = P(B,A)^T$ does not hold, in general. An $s$-map $p$ need not give a metrics [13].

## 3. CHARACTERIZATION VIA S-MAPS AND D-MAPS

In the remaining part of this paper we will deal with a finite atomistic quantum logic $\mathcal{L}$. In this section we will deal with induced d-maps only, so we will use a notation $d$ instead of $d_p$.

Simple assertion in the next lemma will be helpful in several proofs.

**Lemma 3.1.** Let $\mathcal{L}$ be a quantum logic and let $a_1,a_2,\dots,a_n, n\in N$ be mutually orthogonal elements of $L$. Then for any $c\in L$:

$$\sum_{i=1}^{n} d(a_i,c) = d\Big(\bigvee_{i=1}^{n} a_i, c\Big) + (n-1)d(0_L,c).$$

The proof of Lemma 3.1 is just a routine application of mathematical induction to (d3) of Definition 2.10.

### 3.1. A horizontal sum of $k$ Boolean subalgebras

**Lemma 3.2.** Let $\mathcal{L}$ be a horizontal sum of blocks $\mathcal{B}_1,\mathcal{B}_2$ and let $A_i$ be the set of all atoms of the block $\mathcal{B}_i$, where $n_i = card(A_i) < \infty$, $i=1,2$. Let $p$ be an s-map on $L$ and $d$ be the d-map induced by the s-map $p$. Then

$$(1) \qquad \sum_{u\in A_1, v\in A_2} p(u,v) = 1;$$

$$(2) \qquad \sum_{u\in A_1, v\in A_2} d(u,v) = n_1+n_2-2\,.$$

Proof. Let us realize that elements of $A_i$ are mutually orthogonal and $\bigvee_{u\in A_i} u = 1_L, i=1,2$.

(1) It is enough to use a finite additivity of an s-map $p$ in each coordinate.

$$\begin{aligned} \sum_{u\in A_1, v\in A_2} p(u,v) &= \sum_{u\in A_1}\sum_{v\in A_2} p(u,v) = \sum_{u\in A_1} p\Big(u, \bigvee_{v\in A_2}\Big) \\ &= \sum_{u\in A_1} p(u,1_L) = p\Big(\bigvee_{u\in A_1} u, 1_L\Big) = p(1_L,1_L) = 1\,. \end{aligned}$$

(2) Let $d$ be the d-map induced by the s-map $p$ and let $n_i = card(A_i), i = 1, 2$. Applying Lemma 3.1 we get

$$\begin{aligned}
\sum_{u\in A_1, v\in A_2} d(u,v) &= \sum_{u\in A_1}\sum_{v\in A_2} d(u,v) \\
&= \sum_{u\in A_1}\Big(d\Big(u, \bigvee_{v\in A_2} v\Big) + (n_2-1)d(u,0_L)\Big) \\
&= \sum_{u\in A_1} d(u,1_L) + (n_2-1)\sum_{u\in A_1} d(u,0_L) \\
&= d\Big(\bigvee_{u\in A_1} u, 1_L\Big) + (n_1-1)d(0_L,1_L) + (n_2-1)d\Big(\bigvee_{u\in A_1} u, 0_L\Big) \\
&= d(1_L,1_L) + (n_1-1)d(0_L,1_L) + (n_2-1)d(1_L,0_L) \\
&= (n_1+n_2-2)\,.
\end{aligned}$$

□

**Proposition 3.3.** Let a quantum logic $\mathcal{L}$ be a horizontal sum of blocks $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k$, where $A_i$ is the set of all atoms of the block $\mathcal{B}_i$, $i = 1, 2, \ldots, k$ and let $A$ be the set of all atoms of $\mathcal{L}$, $T = card(A)$. Let $p$ be an s-map on $L$ and $d$ be the d-map induced by the s-map $p$. Then

$$S_p = \sum_{u,v\in A} p(u,v) = k^2\,,$$

$$S_d = \sum_{u,v\in A} d(u,v) = 2k(T-k)\,.$$

Proof.

$$\begin{aligned}
S_p &= \sum_{u,v\in A} p(u,v) = \sum_{i=1}^{k}\sum_{j=1}^{k}\sum_{u\in A_i, v\in A_j} p(u,v) = \sum_{i=1}^{k}\sum_{j=1}^{k} 1 = k^2, \\
S_d &= \sum_{u,v\in A} d(u,v) = \sum_{i=1}^{k}\sum_{j=1}^{k}\sum_{u\in A_i, v\in A_j} d(u,v) = \sum_{i=1}^{k}\sum_{j=1}^{k}(n_i+n_j-2) = 2k(T-k).
\end{aligned}$$

□

**Corollary 3.4.** If a quantum logic $\mathcal{L}$ is a horizontal sum of $k$ blocks, $k \in N$, where $A$ is the set of all atoms of $\mathcal{L}$ then $S_p = \sum_{u,v\in A} p(u,v) \in N$ and $S_d = \sum_{u,v\in A} d(u,v) \in N$.

From the corollary it results that if $S_p = \sum_{u,v\in A} p(u,v) \notin N$, then $\mathcal{L}$ cannot be a horizontal sum of blocks. The opposite is not true as we can see from Example 3.7 in the following subsection.

### 3.2. Quantum logic with nontrivial center consisting of two blocks

**Lemma 3.5.** Let a quantum logic $\mathcal{L}$ with a non-trivial center $C(\mathcal{L})$ consists of blocks $\mathcal{B}_i, i = 1, 2$. Let $A_i \cup C$ be the sets of all atoms of the block $\mathcal{B}_i, i = 1, 2$ and let $A$ be the set of all atoms of $L$. Let $p$ be an s-map on $L$ and let $d$ be the d-map induced by the s-map $p$. Then

$$(1) \quad \sum_{u\in A_1, v\in A_2} d(u,v) = (n_1 + n_2 - 2)p(c^\perp, c^\perp) = \sum_{u\in A_2, v\in A_1} d(u,v)\,,$$

$$(2) \quad \sum_{u\in A_1, v\in C} d(u,v) = n_1 p(c,c) + rp(c^\perp, c^\perp) = \sum_{u\in C, v\in A_1} d(u,v)\,,$$

$$(3) \quad \sum_{u,v\in C} d(u,v) = 2(r-1)p(c,c)\,,$$

where $n_i = card(A_i),\ i = 1, 2,\ r = card(C),\ c = \bigvee_{u\in C} u$ and $c^\perp = \bigvee_{u\in A_1} u = \bigvee_{u\in A_2} u$.

P r o o f. Using Lemma 3.1 and the facts $\bigvee_{u\in A_1} u = \bigvee_{u\in A_2} u = c^\perp$, $\bigvee_{u\in C} u = c$ we get

$$\begin{aligned}
(1) \sum_{u\in A_1, v\in A_2} d(u,v) &= d\Big(\bigvee_{u\in A_1} u, \bigvee_{v\in A_2} v) + (n_1 - 1)d(0_L, \bigvee_{v\in A_2)} v\Big) \\
&\quad +(n_2 - 1)d\Big(\bigvee_{u\in A_1} u, 0_L\Big) \\
&= d(c^\perp, c^\perp) + (n_1 - 1)d(0_L, c^\perp) + (n_2 - 1)d(c^\perp, 0_L) \\
&= (n_1 + n_2 - 2)d(0_L, c^\perp) \\
&= (n_1 + n_2 - 2)p(c^\perp, c^\perp)\,.
\end{aligned}$$

$$\begin{aligned}
(2) \sum_{u\in A_1, v\in C} d(u,v) &= d\Big(\bigvee_{u\in A_1} u, \bigvee_{v\in C} v\Big) + (n_1 - 1)d\Big(0_L, \bigvee_{v\in C} v\Big) \\
&\quad + (r-1)d\Big(\bigvee_{u\in A_1} u, 0_L\Big) \\
&= d(c^\perp, c) + (n_1 - 1)d(0_L, c) + (r-1)d(c^\perp, 0_L) \\
&= 1 + (n_1 - 1)p(c,c) + (r-1)p(c^\perp, c^\perp) \\
&= p(c,c) + p(c^\perp, c^\perp) + (n_1 - 1)p(c,c) + (r-1)p(c^\perp, c^\perp) \\
&= n_1 p(c,c) + rp(c^\perp, c^\perp)\,.
\end{aligned}$$

The property (3) can be proved by analogy.

□

**Proposition 3.6.** Let a quantum logic $\mathcal{L}$ with a non-trivial center $C(\mathcal{L})$ consist of blocks $\mathcal{B}_i\ i = 1, ..., k$. Let $A_i \cup C$ be the sets of all atoms of the block $\mathcal{B}_i\ i = 1, ..., k$

and let $A$ be the set of all atoms of $L$. Let $p$ be an s-map on $L$ and let $d$ be the d-map induced by the s-map $p$. Then

$$\sum_{u,v\in A} d(u,v) = 2(n_1+n_2+r-1)p(c,c)+4(n_1+n_2+r-2)p(c^\perp,c^\perp),$$

where $n_i = card(A_i), i = 1,2, r = card(C), c = \bigvee_{u\in C} u$ and $c^\perp = \bigvee_{u\in A_1} u = \bigvee_{u\in A_2} u$.

P r o o f.

$$\begin{aligned}
\sum_{u,v\in A} d(u,v) &= \sum_{u,v\in A_1} d(u,v) + \sum_{u,v\in A_2} d(u,v) + \sum_{u,v\in C} d(u,v) \\
&\quad +2\sum_{u\in A_1, v\in A_2} d(u,v) + 2\sum_{u\in A_1, v\in C} d(u,v) + 2\sum_{u\in A_2, v\in C} d(u,v) \\
&= 2(n_1-1)p(c^\perp,c^\perp) + 2(n_2-1)p(c^\perp,c^\perp) + 2(r-1)p(c,c) \\
&\quad +(n_1+n_2-2)p(c^\perp,c^\perp) + 2(n_1p(c,c)+rp(c^\perp,c^\perp)) \\
&\quad +2(n_2p(c,c)+rp(c^\perp,c^\perp)) \\
&= 2(n_1+n_2+r-1)p(c,c)+4(n_1+n_2+r-2)p(c^\perp,c^\perp)
\end{aligned}$$

□

**Example 3.7.** Let $\mathcal{L}$ be a quantum logic. Let $A = \{a_1, a_2, c\}$ and $B = \{b_1, b_2, c\}$ be two orthogonal partitions of unit of $L$ ($\mathcal{L} = \mathcal{B}_A \cup \mathcal{B}_B$). So it is a special case of the previous proposition, where $n_1 = 2, n_2 = 2, r = 1$. Then

$$\begin{aligned}
S_d &= 2(n_1+n_2+r-1)p(c,c)+4(n_1+n_2+r-2)p(c^\perp,c^\perp) \\
&= 8p(c,c)+12p(c^\perp,c^\perp) \\
&= 12-4p(c,c)\,.
\end{aligned}$$

So $S_d \in [8,12]$. If we choose $p(c,c) \in \{0, 0.25, 0.5, 0.75, 1\}$, then characteristic number $S_d$ is in $N$ even the quantum logic $\mathcal{L}$ is not a horizontal sum of two blocks.

### 3.3. Quantum logic with nontrivial center consisting of k blocks

**Proposition 3.8.** Let a quantum logic $\mathcal{L}$ with a non-trivial center $C(\mathcal{L})$ consists of blocks $\mathcal{B}_i$ $i = 1,\dots,k$. Let $A_i \cup C$ be the sets of all atoms of the block $\mathcal{B}_i$ $i = 1,\dots,k$ and let $A$ be the set of all atoms of $L$. Let $p$ be an s-map on $L$ and let $d$ be the d-map induced by the s-map $p$. Then

$$S_d = \sum_{u,v\in A} d(u,v) = 2(T-1)p(c,c)+2k(T-k)p(c^\perp,c^\perp),$$

where $T = card(A), c = \bigvee_{u\in C} u$ and $c^\perp = \bigvee_{u\in A_i} u$ for each $i = 1,2,\dots,k$.

Proof. Let us denote $n_i = card(A_i), i = 1, 2, \dots, k, r = card(C)$ and $t = \sum_{i=1}^{k} n_i$. Then

$$\begin{aligned}
S_d &= \sum_{u,v\in A} d(u,v) \\
&= \sum_{i=1}^{k}\sum_{j=1}^{k}\sum_{u\in A_i, v\in A_j} d(u,v) + 2\sum_{i=1}^{k}\sum_{u\in A_i, v\in C} d(u,v) + \sum_{u,v\in C} d(u,v) \\
&= \sum_{i=1}^{k}\sum_{j=1}^{k}(n_i+n_j-2)p(c^{\perp},c^{\perp}) + 2\sum_{i=1}^{k}(n_i p(c,c) + rp(c^{\perp},c^{\perp})) + 2(r-1)p(c,c) \\
&= 2k(t-k)p(c^{\perp},c^{\perp}) + 2tp(c,c) + 2krp(c^{\perp},c^{\perp}) + 2(r-1)p(c,c) \\
&= 2(t+r-1)p(c,c) + 2k(t+r-k)p(c^{\perp},c^{\perp}) \\
&= 2(T-1)p(c,c) + 2k(T-k)p(c^{\perp},c^{\perp}).
\end{aligned}$$

□

## 4. SPECIAL BIVARIABLE FUNCTIONS ON A QUANTUM LOGIC

In [12] a bivariable map on a quantum logic $\mathcal{L}$ was introduced. Under special marginal conditions it represents, for example, infimum measure, supremum measure or symmetric difference measure for two elements of $\mathcal{L}$.

### 4.1. Basic notions

**Definition 4.1.** Let $\mathcal{L}$ be a quantum logic. A map $G : L^2 \to [0,1]$ satisfying the conditions:

(G1) if $u, v \in \{0_L, 1_L\}$ then $G(u,v) \in \{0_L, 1_L\}$ and $G(0_L, 1_L) = G(1_L, 0_L)$;

(G2) if $a \perp b$ then $G(a,b) = G(a,0_L) + G(0_L,b) - G(0_L,0_L)$;

(G3) if $a \perp b$, $c \in L$ then

$$G(a\vee b, c) = G(a,c) + G(b,c) - G(0_L, c),$$
$$G(c, a\vee b) = G(c,a) + G(c,b) - G(c, 0_L);$$

is said to be a special bivariable map on $\mathcal{L}$. The set of all special bivariable maps $G$ on $\mathcal{L}$ will be denoted by $\Gamma$.

Now we recall the basic properties of special bivariable maps (see [12]).

**Lemma 4.2.** Let $\mathcal{L}$ be a quantum logic and let $G \in \Gamma$. Then

1. $G(a,b) = G(b,a)$ if $a \leftrightarrow b$ $a, b \in L$;
2. $G(1_L, 0_L) = G(a, a^{\perp})$ for any $a \in L$.

In the same way as it was done for a d-map, the following lemma can be proved by using (G3) from Definition 4.1 and mathematical induction.

**Lemma 4.3.** Let $\mathcal{L}$ be a quantum logic and let $a_1, a_2, \ldots, a_n, n \in N$ be mutually orthogonal elements of $L$. Then for any $c \in L$:

$$\sum_{i=1}^{n} G(a_i, c) = G\Big(\bigvee_{i=1}^{n} a_i, c\Big) + (n-1)G(0_L, c).$$

**Remark 4.4.** The invariance w.r.t. order is not true in general. It means, that there is a map $G \in \Gamma$ and $a, b \in L$ such that $G(a,b) \neq G(b,a)$. If $G(a,b) \neq G(b,a)$ we know, that $a, b$ are not compatible. This fact can be rewritten as follows: $a > (a \wedge b) \vee (a \wedge b^{\perp})$. On the other hand $G(a,b) = G(b,a)$ does not imply $a \leftrightarrow b$.

We can get eight families $\Gamma_i$ $(i = 1, \ldots, 8)$ of maps $G$ according to values in vertex points $(0_L, 0_L)$, $(0_L, 1_L)$, $(1_L, 0_L)$, $(1_L, 1_L)$:

| | $\Gamma_1$ | $\Gamma_2$ | $\Gamma_3$ | $\Gamma_4$ | $\Gamma_5$ | $\Gamma_6$ | $\Gamma_7$ | $\Gamma_8$ |
|---|---|---|---|---|---|---|---|---|
| $G(0_L, 0_L)$ | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| $G(0_L, 1_L)$ | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 1 |
| $G(1_L, 1_L)$ | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 1 |

**Remark 4.5.** It is clear that $\Gamma_1$ and $\Gamma_8$ are one element sets. It means that

$$\Gamma_1 = \{G_{0_L}\}, \text{ where } G_{0_L}(a,b) = 0 \text{ for all } a, b \in L;$$
$$\Gamma_8 = \{G_{1_L}\}, \text{ where } G_{1_L}(a,b) = 1 \text{ for all } a, b \in L.$$

Moreover, $G \in \Gamma_2$ is an s-map and $G \in \Gamma_4$ is a d-map. (see [12])

### 4.2. Characterization of a quantum logic via $G$

First we suppose a quantum logic $\mathcal{L}$ as a horizontal sum of $k$ blocks.

**Lemma 4.6.** Let $\mathcal{L}$ be a horizontal sum of blocks $\mathcal{B}_1, \mathcal{B}_2$ and let $A_i$ be the set of all atoms of the block $\mathcal{B}_i$, $i = 1, 2$. Let G be a special bivariable map on $L$. Then

$$\sum_{u \in A_1, v \in A_2} G(u,v) = G(1_L, 1_L) + (n_1 + n_2 - 2)G(0_L, 1_L) + (n_1 - 1)(n_2 - 1)G(0_L, 0_L).$$

P r o o f. Applying Lemma 4.3.

$$\begin{aligned} \sum_{u \in A_1, v \in A_2} G(u,v) &= G\Big(\bigvee_{u \in A_1} u, \bigvee_{v \in A_2} v\Big) + (n_1 - 1)G\Big(0_L, \bigvee_{v \in A_2} v\Big) \\ &\quad + (n_2 - 1)\Big(G\Big(\bigvee_{u \in A_1} u, 0_L\Big) + (n_1 - 1)G(0_L, 0_L)\Big). \end{aligned}$$

As $\bigvee_{u\in A_1} u = \bigvee_{v\in A_2} v = 1_L$, we get

$$\sum_{u\in A_1, v\in A_2} G(u,v) = G(1_L,1_L) + (n_1+n_2-2)G(0_L,1_L) + (n_1-1)(n_2-1)G(0_L,0_L)\,.$$

□

**Proposition 4.7.** Let a quantum logic $\mathcal{L}$ be a horizontal sum of blocks $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k$, where $A_i$ is the set of all atoms of the block $\mathcal{B}_i$, $i = 1,2,\ldots,k$ and let $A$ be the set of all atoms of $\mathcal{L}$, $T = card(A)$. Let $G$ be a special bivariable map on $L$. Then

$$S_G = \sum_{u,v\in A} G(u,v) = k^2 G(1_L,1_L) + 2k(T-k)G(1_L,0_L) + (T-k)^2 G(0_L,0_L).$$

Proof.

$$\begin{aligned} S_G &= \sum_{u,v\in A} G(u,v) = \sum_{i=1}^{k}\sum_{j=1}^{k}\sum_{u\in A_i, v\in A_j} G(u,v) \\ &= \sum_{i=1}^{k}\sum_{j=1}^{k}(G(1_L,1_L) + (n_i+n_j-2)G(0_L,1_L) + (n_i-1)(n_j-1)G(0_L,0_L)) \\ &= \sum_{i=1}^{k}(kG(1_L,1_L) + (kn_i+T-2k)G(0_L,1_L) + (n_i-1)(T-k)G(0_L,0_L)) \\ &= k^2G(1_L,1_L) + 2k(T-k)G(1_L,0_L) + (T-k)^2G(0_L,0_L). \end{aligned}$$

□

**Corollary 4.8.** Let a quantum logic $\mathcal{L}$ be a horizontal sum of blocks $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k$, where $A_i$ is the set of all atoms of the block $\mathcal{B}_i$, $i = 1,2,\ldots,k$ and let $A$ be the set of all atoms of $\mathcal{L}$, $T = card(A)$. Let $G_i \in \Gamma_i$ $i = 1,\ldots,8$. Then $S_{G_1} = 0$, $S_{G_2} = k^2$, $S_{G_3} = k(2T-k)$, $S_{G_4} = 2k(T-k)$, $S_{G_5} = T^2 - k^2$, $S_{G_6} = (T-k)^2$, $S_{G_7} = (T-k)^2 - k^2$, $S_{G_8} = T^2$.

**Corollary 4.9.** If a quantum logic $\mathcal{L}$ is a horizontal sum of blocks,where $A$ is the set of all atoms of $\mathcal{L}$ then $S_G = \sum_{u,v\in A} G(u,v) \in N$ for each special bivariable function $G$.

From the corollary it results that if $S_G \notin N$, then $\mathcal{L}$ cannot be a horizontal sum of two Boolean algebras. The opposite is not true.

Now we concentrate to a quantum logic with a nontrivial center, which consists of two Boolean algebras. The proofs of the following lemma and propositions immediately result from the previous ones.

**Lemma 4.10.** Let a quantum logic $\mathcal{L}$ with a non-trivial center $C(\mathcal{L})$ consist of blocks $\mathcal{B}_i, i = 1, 2$. Let $A_i \cup C$ be the sets of all atoms of the blocks $\mathcal{B}_i, i = 1, 2$ and let $A$ be the set of all atoms of $L$. Let $G$ be a special bivariable map on $L$. Then

$$\begin{aligned}
(1) \quad & \sum_{u \in A_1, v \in A_2} G(u,v) = \sum_{u \in A_2, v \in A_1} G(u,v) \\
& = G(c^\perp, c^\perp) + (n_1 + n_2 - 2)G(0_L, c^\perp) + (n_1 - 1)(n_2 - 1)G(0_L, 0_L)\,, \\
(2) \quad & \sum_{u \in A_1, v \in C} G(u,v) = \sum_{u \in C, v \in A_1} G(u,v) \\
& = n_1 G(0_L, c) + rG(0_L, c^\perp) + ((n_1 - 1)(r - 1) - 1)G(0_L, 0_L)\,, \\
(3) \quad & \sum_{u,v \in C} G(u,v) = G(c,c) + 2(r-1)G(0_L, c) + (r-1)^2 G(0_L, 0_L)\,,
\end{aligned}$$

where $n_i = card(A_i), i = 1, 2, r = card(C), c = \bigvee_{u \in C} u$ and $c^\perp = \bigvee_{u \in A_1} u = \bigvee_{u \in A_2} u$.

**Proposition 4.11.** Let a quantum logic $\mathcal{L}$ with a non-trivial center $C(\mathcal{L})$ consist of blocks $\mathcal{B}_i, i = 1, 2$. Let $A_i \cup C$ be the sets of all atoms of the blocks $\mathcal{B}_i, i = 1, 2$ and let $A$ be the set of all atoms of $L$. Let $G$ be a special bivariable map on $L$. Then

$$\begin{aligned}
\sum_{u,v \in A} G(u,v) \quad = \quad & G(c,c) + 4G(c^\perp, c^\perp) + 2(n_1 + n_2 + r - 1)G(0_L, c) \\
& + 4(n_1 + n_2 + r - 2)G(0_L, c^\perp) \\
& + ((n_1 + n_2 + r - 3)^2 - 4)G(0_L, 0_L),
\end{aligned}$$

where $n_i = card(A_i), i = 1, 2, r = card(C), c = \bigvee_{u \in C} u$ and $c^\perp = \bigvee_{u \in A_1} u = \bigvee_{u \in A_2} u$.

And, finally, we assume that a quantum logic with a nontrivial center consists of $k$ blocks .

**Proposition 4.12.** Let a quantum logic $\mathcal{L}$ with a non-trivial center $C(\mathcal{L})$ consist of blocks $\mathcal{B}_i$ $i = 1, \ldots, k$. Let $A_i \cup C$ be the sets of all atoms of the block $\mathcal{B}_i$ $i = 1, \ldots, k$ and let $A$ be the set of all atoms of $L$. Let $G$ be a special bivariable map on $L$. Then

$$\begin{aligned}
\sum_{u,v \in A} G(u,v) \quad = \quad & G(c,c) + k^2 G(c^\perp, c^\perp) + 2(T-1)G(0_L, c) \\
& + 2k(T-k)G(0_L, c^\perp) + ((T-k-1)^2 - 2k)G(0_L, 0_L),
\end{aligned}$$

where $T = card(A), c = \bigvee_{u \in C} u$ and $c^\perp = \bigvee_{u \in A_i} u$ for each $i = 1, 2, \ldots, k$.

## 5. CONCLUSION

Let us realize the process of investigation of two events $A, B$, each of them expressed as $A = \{a_1, \ldots, a_n\}$, $B = \{b_1, \ldots, b_k\}$, according to its organization. How to face the situation, when simple events $a_i, b_j$ cannot be verified simultaneously, but, despite this fact, we are able to obtain some information about $a_i$ while one of $b_j$ does not come into being? Videlicet, how to deal with $f(a_i|b_j^{\perp})$ or $f(b_j|a_i^{\perp})$? For that reason we effort to find a basic structure created by these observations (e.g. whether some "property levels" of $A, B$ are the same). More precisely let $A, B$ be orthogonal partitions of unit $1_L$. Let us denote $B^{\perp} = \{b_1^{\perp}, \dots, b_k^{\perp}\}$ and $A^{\perp} = \{a_1^{\perp}, \dots, a_n^{\perp}\}$. Then

$$P(A, B^{\perp}) = \begin{pmatrix} p(a_1, b_1^{\perp}) & \cdots & p(a_1, b_k^{\perp}) \\ \vdots & \ddots & \vdots \\ p(a_n, b_1^{\perp}) & \cdots & p(a_n, b_k^{\perp}) \end{pmatrix},$$

where $p(a_i, b_j^{\perp}) = m(b_j)m(a_i|b_j^{\perp})$, $i = 1, \dots, n$ and $j = 1, \dots, k$. By analogy we get $P(B, A^{\perp})$. Let us denote $p_s = 0.5(p(a, b) + p(b, a))$. In [11], inter alia, it has been proved that $p_s$ is an s-map and $p_s(a, b) = p_s(b, a)$ for any $a, b \in L$ and, moreover, $p(a, a) = p_s(a, a)$ for each $a \in L$. As $d_p(a_i, b_j) = p(a_i, b_j^{\perp}) + p(a_i^{\perp}, b_j)$, the matrix

$$D_{p_s}(A, B) = \frac{1}{2}(P(A, B^{\perp}) + P(B, A^{\perp})^T)$$

is the matrix for the function $d_{p_s}$. The sum of $d_{p_s}$ throughout all levels gives us basic information about given structure. For example if $S_{d_{p_s}}$ is not integer number then $A, B$ do not create one Boolean algebra. Conversely if $S_{d_{p_s}} \in N$, it does not mean then $A$ and $B$ belong to one Boolean algebra.

In the last section it was shoved that this holds for every function $G$ and thus also for $G \in \Gamma_4$ (i. e. $G$ is d-map, see Remark 4.5). On a quantum logic a d-map need not create an s-map, in general [12], i. e. it is not a metrics.

In conclusion we call attention to the formula

$$S_d = 2(T-1)p(c, c) + 2k(T-k)p(c^{\perp}, c^{\perp})$$

- Assuming one Boolean algebra $p(c, c) = 1$ and $p(c^{\perp}, c^{\perp}) = 0$, we obtain $S_d = 2(T-1)$. For a horizontal sum of $k$ Boolean algebras we have $p(c, c) = 0$ and $p(c^{\perp}, c^{\perp}) = 1$ and we get $S_d = 2k(T-k)$.
  Then $Df = \dfrac{S_d}{2k}$ expresses degrees of freedom for a diagonal s-map. It is at least $T-k$, and $T-1$ at the most and it need not be an integer number.

- The value $p(c, c)$ can be denominated as a potency of a center $C(\mathcal{L})$. For $p(c, c) = 1$ we obtain $S_d = 2(T-1)$. This evokes a Boolean algebra with $T$ elements. If $p(c, c) = 0$ then $S_d = 2k(T-k)$ and this system seems to be a horizontal sum with $k$ blocks.


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## REFERENCES


[1] A. M. Al-Adilee and O. Nánásiová: Copula and s-map on a quantum logic. Inform. Sci. *179* (2009), 4199–4207.

[2] G. Dohnal: Markov property in quantum logic. A reflection. Inform. Sci. *179* (2009), 485–491.

[3] A. Dvurečenskij and S. Pulmannová: New Trends in Quantum Structure. Kluwer Acad. Publishers, Dortrecht/Boston/London, Ister Science, Bratislava 2000.

[4] R. J. Greechie: Orthogonal lattices admitting no states. J. Combin. Theory, Ser. *A10* (1971), 119–132.

[5] G. Kalmbach: Orthomodular Lattices. Academic Press, London 1983.

[6] O. Nánásiová: Map for simultaneous measurements for a quantum logic. Internat. J. Theoret. Phys. *42* (2003), 1889–1903.

[7] O. Nánásiová: Principle conditioning. Internat. J. Theoret. Phys. *43* (2004), 1757–1767.

[8] O. Nánásiová and A. Khrennikov: Representation theorem for observables on a quantum system. Internat. J. Theoret. Phys. *45* (2006), 469–482.

[9] O. Nánásiová and A. Khrennikov: Compatibility and marginality. Internat. J. Theoret. Phys. *46* (2007) 1083–1095.

[10] O. Nánásiová and S. Pulmannová: S-map and tracial states. Inform. Sci. *179* (2009) 515–520.

[11] O. Nánásiová and K. Trokanová, and I. Žembery: Commutative and non commutative s-maps. Forum Statist. Slovacum *2* (2007) 172–177.

[12] O. Nánásiová and Ľ. Valášková: Maps on a quantum logic. Soft Computing (2009). doi:10.1007/s00500-009-0483-4

[13] O. Nánásiová and Ľ. Valášková: Marginality and triangle inequality Internat. J. Theoret. Phys. (2010), accepted.

[14] M. Navara: An othomodular lattice admitting no group-valued measure. Proc. Amer. Math. Soc. *122* (1994), 7–12.

[15] J. von Neumann: Mathematische Grundlagen der Quantenmechanik. Springer-Verlag, Berlin 1932.

[16] P. Pták and S. Pulmannová: Quantum Logics. Kluwer Acad. Press, Bratislava 1991.

[17] V. Varadarajan: Geometry of Quantum Theory. D. Van Nostrand, Princeton, New Jersey 1968.



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