KYBERNETIKA — VOLUME *46* (2010), NUMBER 6, PAGES 948–952

EVERY UNIFORMLY ARCHIMEDEAN ATOMIC MV–EFFECT ALGEBRA IS SHARPLY DOMINATING

Vladimír Olejček

Following the study of *sharp domination* in effect algebras, in particular, in *atomic Archimedean MV-effect algebras* it is proved that if an atomic MV-effect algebra is *uniformly Archimedean* then it is sharply dominating.

Keywords: lattice effect algebra, MV-algebra, sharp element, sharp domination, atom, Euclidean algorithm

Classification: 03G12, 06D35, 06F25, 81P10

1. INTRODUCTION AND BASIC DEFINITIONS

Effect algebras were introduced by D. J. Foulis and M. K. Bennett in 1994 [2] for modeling unsharp measurements in a Hilbert Space. In a general form they are very natural structures to be carriers of states or probability measures when events are unsharp, fuzzy or imprecise and some of them may be mutually non-compatible. Simultaneously, F. Kôpka and F. Chovanec [7, 8] introduced in a sense equivalent structures called D-posets.

Definition 1.1. (Foulis and Bennett [2]) A partial algebra $(E;\, \oplus,\, 0,\, 1)$ is called an *effect algebra* if 0, 1 are two distinct elements of E and $\oplus$ is a partially defined binary operation on E which satisfies the following conditions for any $x, y, z \in E$:

(i) $x \oplus y = y \oplus x$ if $x \oplus y$ is defined,

(ii) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ if one side is defined,

(iii) for every $x \in E$ there exists a unique $y \in E$ such that $x \oplus y = 1$ we put $x' = y$,

(iv) if $1 \oplus x$ is defined then $x = 0$.

We often denote the effect algebra $(E;\, \oplus,\, 0,\, 1)$ briefly by E. On every effect algebra E a partial order $\leq$ and a partial binary operation $\ominus$ can be introduced as follows:

$$x \leq y \text{ and } y \ominus x = z \text{ iff } x \oplus z \text{ is defined and } x \oplus z = y$$

If E, with the partial order $\leq$ defined above, is a lattice (a complete lattice) then $(E;\, \oplus,\, 0,\, 1)$ is called a *lattice effect algebra* (a *complete lattice algebra*).

Lattice effect algebras generalize orthomodular lattices and MV-algebras. A lattice effect algebra is called an *MV-effect algebra* iff every two elements $x, y \in E$ are compatible, i. e. $x \vee y = x \oplus (y \ominus (x \wedge y))$ [6].

Recall that a minimal non-zero element of an effect algebra E is called an *atom* and E is called *atomic* if under every non-zero element of E there is an atom.

In an effect algebra E elements x and *non* x, denoted by x', need not be disjoint. The notions of a *sharp element* and *sharply dominating effect algebra* are due to S. P. Gudder ([3, 4]). An element w of an effect algebra E is called *sharp*, if $w \wedge w' = 0$, and E is called *sharply dominating* if for every $x \in E$ there exists the smallest sharp element w among all the sharp elements v with the property $x \leq v$.

For an element x of an effect algebra E we write $\mathrm{ord}(x) = \infty$ if $nx = x \oplus x \oplus \cdots \oplus x$ (n-times) exists for every positive integer n and we write $\mathrm{ord}(x) = n_x$ if n_x is the greatest positive integer such that $n_x x$ exists in E (n_x is called *isotropic index* of x). An effect algebra is called *Archimedean* if $\mathrm{ord}(x) < \infty$ for all $x \in E$.

Definition 1.2. A *direct product* $\prod\{E_k \mid k \in H\}$ of effect algebras E_k is the Cartesian product with $\oplus$, 0, 1 defined "coordinate-wise", i. e. $(a_k)_{k\in H} \oplus (b_k)_{k\in H}$ exists iff $a_k \oplus b_k$ is defined for every $k \in H$ and then $(a_k)_{k\in H} \oplus (b_k)_{k\in H} = (a_k \oplus_k b_k)_{k\in H}$. Moreover, $0 = (0_k)_{k\in H}$, $1 = (1_k)_{k\in H}$.

A *sub-direct product* of a family $\{E_k\}_{k\in H}$ of lattice effect algebras is a sub-lattice sub-effect algebra Q (i. e. Q is simultaneously a sub-lattice and a sub-effect algebra) of the direct product $\prod\{E_k \mid k \in H\}$ such that each restriction of the natural projection pr_k to Q is onto E_k.

In [5] the following example of an atomic Archimedean MV-effect algebra that is not sharply dominating is given.

Example 1.3. Let M be a direct product of countably many finite chains $C_n = 0, 1, \ldots, n$ (and consequently MV-effect algebras). Then $M = \prod_{n=1}^{\infty}\{0, 1, \ldots, n\}$ with coordinate-wise defined partial operation $\oplus$ is a complete (consequently Archimedean by [9, Theorem 3.3]) atomic MV-effect algebra. Consider the subset E of M as $E = F_0 \cup F_1$. F_0 is the set of all sequences of M with all but finitely many of even coordinates equal to 0 and all but finitely many of odd coordinates constant. F_1 is the set of all sequences of M with all but finitely many of even coordinates equal to n and all but finitely many of odd coordinates smaller than n by a constant.

The essential property of the MV-effect algebra E in the above example is that the set of isotropic indices of its elements is unbounded. It leads to an idea of a "bounded isotropic index" for all elements of E defined here as a "uniformly Archimedean" MV-effect algebra. Thereafter we prove that such an atomic MV-algebra is sharply dominating.

2. MAIN RESULT

Definition 2.1. An effect algebra E is called *uniformly Archimedean* if there is a positive number $m \in N$ such that for every non-zero element $x \in E$ the isotropic index n_x of x does not exceed m.

Other terms playing key role in the proof of the following Theorem are local versions of "atom" and "isotropic index" in an MV-effect algebra E. According to [1] and [10], E can be isomorfically embedded into a product M of intervals $\{0, 1, \ldots, n_p\}$, i.e.

$$E \cong Q \subseteq M = \prod\{\{0, 1, \ldots, n_p\} \mid p \in A\}$$

where A is the set of all atoms of E. Every element x of E is represented as a function $x : A \to N = \{0, 1, \ldots\}$, with the pth coordinate denoted by x_p.

Definition 2.2. Let x be an element of an MV-effect algebra E and let p be an atom in E with the isotropic index n_p, satisfying $p \leq x$. Denote q_p the greatest common divisor (GCD) of the numbers x_p and n_p. The element $q_p p$ is called *local atom* with respect to the atom p and to the element x. The number $r_p = n_p/q_p$ is called *local isotropic index* of the atom p with respect to the element x. Note that if $x_p = 0$ then $q_p = n_p$ and $r_p = 1$. Similarly, if $x_p > 0$ then $q_p p \leq x_p \leq n_p p$. In both cases $r_p \geq 1$.

Theorem 2.3. Every uniformly Archimedean atomic MV-effect algebra is sharply dominating.

Proof. Assume that E is a uniformly Archimedean atomic effect algebra represented as above. Consider an arbitrary element $x \in E$. Obviously, y is a sharp element of M iff for every $p \in A$, $y_p = 0$ or $y_p = n_p$. Hence, the element y with

$$y_p = \begin{cases} 0 & \text{if } x_p = 0 \\ n_p & \text{if } x_p > 0 \end{cases}$$

is the smallest element in M dominating x. It is enough to prove that y belongs to E.

In the following construction we will apply a simple version of the Euclidean algorithm for counting the greatest common divisor of two positive integers a, b. Define $c_0 = a$, $c_1 = b$ and

$$c_{n+2} = \max(c_{n+1}, c_n) - \min(c_{n+1}, c_n) \tag{1}$$

for $n = 0, 1, 2, \ldots$.

It is well known that after finitely many steps of the above construction zero output is obtained. The last non-zero output preceding the zero-output is the greatest common divisor d of the integers a, b. Note that if the algorithm continues, 0 is followed again by d and the pattern $d - 0 - d$ repeats ad infinitum. Apply the same algorithm for the inputs $x, 1 \in E$, i.e. denote $t^{(0)} = 1_E$, $t^{(1)} = x$ and

$$t^{(n+2)} = (t^{(n+1)} \vee t^{(n)}) \ominus (t^{(n+1)} \wedge t^{(n)}) \tag{2}$$

for $n = 0, 1, 2, \ldots$. For every $p \in A$, n_p and x_p are the two inputs and the last non-zero output q_p is GCD of the numbers n_p and x_p. Then element $q_p p$ is the local atom with respect to the atom p and to the element x.

The operations in (2) are lattice and MV-effect algebra operations. Thus, every output in each step is an element of E. Since the values of n_p are bounded, after finitely many, say L, steps of the algorithm, for every $p \in A$, the output $t_p^{(L)}$ is q_p or 0. Moreover, for every $p \in A$, at least one of the outputs $t^{(L)}$, $t^{(L+1)}$ does not equal 0. It follows that the join $t = t^{(L)} \vee t^{(L+1)}$ of the outputs after L and $L+1$ steps belongs to E and all its coordinates are equal to the local atoms coefficients q_p.

Define $z^{(1)} \in M$ as $z^{(1)} = x \wedge t$. Then $z^{(1)} \in E$ and

$$z_p^{(1)} = \begin{cases} 0 & \text{if } x_p = 0 \\ q_p & \text{if } x_p > 0. \end{cases}$$

Note that the zero element in E is a sharp element, thus, we can assume that $x \neq 0$, whence $z^{(1)}$ is not identically equal to 0. Denote $m_1 = \min\{r_p \mid p \in A, z_p^{(1)} > 0\}$ and put $z^{(2)} = z^{(1)} \wedge (m_1 z^{(1)})'$. Then $z^{(2)} \in E$ and

$$z_p^{(2)} = \begin{cases} 0 & \text{if } r_p \leq m_1 \\ q_p & \text{if } r_p > m_1. \end{cases}$$

Continue by induction. Suppose $z^{(i)}$, m_i were already constructed by the induction, i.e. $z^{(i)} \in E$, $z^{(i)}$ is not identically equal to 0 and $m_i = \min\{r_p \mid p \in A, z_p^{(i)} > 0\}$. Put $z^{(i+1)} = z^{(i)} \wedge (m_i z^{(i)})'$ and $m_{i+1} = \min\{r_p \mid p \in A, z_p^{(i)} > 0\}$. Then $z^{(i+1)} \in E$ and

$$z_p^{(i+1)} = \begin{cases} 0 & \text{if } r_p \leq m_i \\ q_p p & \text{if } r_p > m_i. \end{cases}$$

Note that $z_p^{(i)} > 0$, $i \geq 3$ for some p implies $z_p^{(i+1)} < z_p^{(i)} < \cdots < z_p^{(2)} < z_p^{(1)}$. Since E is uniformly Archimedean, the set $\{n_p \mid p \in A\}$ is bounded, hence finite. Consequently the set $\{r_p \mid p \in A\}$ is finite. Hence, there is an index $k+1$ such that $z^{(k+1)}$ is, and $z^{(k)}$ is not identically equal to 0. We will show that

$$y = \bigvee \{m_i z^{(i)} \mid i = 1, 2, \ldots, k\}.$$

Let $p \in A$. If $x_p = 0$ then $z_p^{(i)} = 0$ for all $i = 1, 2, \ldots, k$, whence $y_p = 0$. If $0 < x_p \leq m_1$ then $z_p^{(1)} = q_p$ and $z_p^{(i)} = 0$ for all $i = 2, \ldots, k$. Hence $y_p = m_1 z_p^{(1)} = r_p q_p = n_p$. Finally, for any $j = 2, \ldots, k$, if $m_{j-1} < x_p \leq m_j$ we have $z_p^{(i)} = q_p$ for all i, $i \leq j$ and $z_p^{(i)} = 0$ for all i, $i > j$. Hence $y_p = \max\{m_i z_p^{(i)} \mid i = 1, 2, \ldots, j\} = m_j z_p^{(j)} = r_p q_p = n_p$. □

(Received June 14, 2010)

REFERENCES

[1] R. Cignoli and D. Mundici: An elementary presentation of the equivalence between MV-algebras and ℓ-groups with strong unit. Studia Logica *61* (1998), 49–64.

[2] D. J. Foulis and M. K. Bennett: Effect algebras and unsharp quantum logics. Found. Phys. *24* (1994), 1325–1346.

[3] S. P. Gudder: Sharply dominating effect algebras. Tatra Mt. Math. Publ. *15* (1998), 23–30.

[4] S. P. Gudder: S-dominating effect algebras. Internat. J. Theor. Phys. *37* (1998), 915–923.

[5] M. Kalina, M. Olejček, J. Paseka, and Z. Riečanová: Sharply Dominating MV-Effect Algebras. Internat. J. Theor. Phys. (2010), doi: 10.1007/s10773-010-0338-x

[6] F. Kôpka: Boolean D-posets as factor spaces. Internat. J. Theor. Phys. *37* (1998), 93–101.

[7] F. Kôpka: D-posets of fuzzy sets. Tatra Mt. Math. Publ. *1* (1992), 83–87.

[8] F. Kôpka and F. Chovanec: D-posets. Math. Slovaca *44* (1994), 21–34.

[9] Z. Riečanová: Archimedean and block-finite lattice effect algebras. Demonstr. Math. *33* (2000), 443–452.

[10] Z. Riečanová: Subdirect Decompositions of Lattice Effect Algebras. Interernat. J. Theor. Phys. *42* (2003), 1415–1423.

Vladimír Olejček, Department of Mathematics, Faculty of Electrical Engineering and Information Technology, Slovak University of Technology, Ilkovičova 3, 812 19 Bratislava. Slovak Republic.

e-mail: vladimir.olejcek@stuba.sk