KYBERNETIKA — VOLUME *46* (2010), NUMBER 6, PAGES 935–947

MAC NEILLE COMPLETION OF CENTERS AND CENTERS OF MAC NEILLE COMPLETIONS OF LATTICE EFFECT ALGEBRAS

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If element z of a lattice effect algebra $(E, \oplus, \mathbf{0}, \mathbf{1})$ is central, then the interval $[\mathbf{0}, z]$ is a lattice effect algebra with the new top element z and with inherited partial binary operation $\oplus$. It is a known fact that if the set $C(E)$ of central elements of E is an atomic Boolean algebra and the supremum of all atoms of $C(E)$ in E equals to the top element of E, then E is isomorphic to a subdirect product of irreducible effect algebras ([18]). This means that if there exists a MacNeille completion $\hat{E}$ of E which is its extension (i.e. E is densely embeddable into $\hat{E}$) then it is possible to embed E into a direct product of irreducible effect algebras. Thus E inherits some of the properties of $\hat{E}$. For example, the existence of a state in $\hat{E}$ implies the existence of a state in E. In this context, a natural question arises if the MacNeille completion of the center of E (denoted as $\mathcal{MC}(C(E))$) is necessarily the same as the center of $\hat{E}$, i.e., if $\mathcal{MC}(C(E)) = C(\hat{E})$ is necessarily true. We show that the equality is not necessarily fulfilled. We find a necessary condition under which the equality may hold. Moreover, we show also that even the completeness of $C(E)$ and its bifullness in E is not sufficient to guarantee the mentioned equality.

Keywords: lattice effect algebra, center, atom, MacNeille completion

Classification: 03G12, 03G27, 06B99

1. INTRODUCTION AND PRELIMINARIES

Effect algebras, introduced by D.J. Foulis and M.K. Bennett [3], have their importance in the investigation of uncertainty. Lattice ordered effect algebras generalize orthomodular lattices and MV-algebras. Thus they may include non-compatible pairs of elements as well as unsharp elements.

Definition 1.1. (Foulis and Bennett [3]) An *effect algebra* is a system $(E; \oplus, \mathbf{0}, \mathbf{1})$ consisting of a set E with two different elements $\mathbf{0}$ and $\mathbf{1}$, called *zero* and *unit*, respectively and $\oplus$ is a partially defined binary operation satisfying the following conditions for all $p, q, r \in E$:

(E1) If $p \oplus q$ is defined, then $q \oplus p$ is defined and $p \oplus q = q \oplus p$.

(E2) If $q \oplus r$ is defined and $p \oplus (q \oplus r)$ is defined, then $p \oplus q$ and $(p \oplus q) \oplus r$ are defined and $p \oplus (q \oplus r) = (p \oplus q) \oplus r$.

(E3) For every $p \in E$ there exists a unique $q \in E$ such that $p \oplus q$ is defined and $p \oplus q = \mathbf{1}$.

(E4) If $p \oplus \mathbf{1}$ is defined then $p = \mathbf{0}$.

The element q in (E3) will be called the *supplement* of p, and will be denoted as p'.

In the whole paper, for an effect algebra $(E, \oplus, \mathbf{0}, \mathbf{1})$, writing $a \oplus b$ for arbitrary $a, b \in E$ will mean that $a \oplus b$ exists. On an effect algebra E we may define another partial binary operation $\ominus$ by

$$a \ominus b = c \quad \Leftrightarrow \quad b \oplus c = a.$$

The operation $\ominus$ induces a partial order on E. Namely, for $a, b \in E$ $b \leq a$ if there exists a $c \in E$ such that $a \ominus b = c$. If E with respect to $\leq$ is lattice ordered, we say that E is a *lattice effect algebra*. For the sake of brevity we will write just LEA. Further, in this article we often briefly write 'an effect algebra E' skipping the operations.

If every pair x, y of elements of a LEA E is *compatible*, meaning that $x \vee y = x \oplus (y \ominus (x \wedge y))$ then E is called an *MV-effect algebra* [1, 9].

S. P. Gudder ([5, 6]) introduced the notion of sharp elements and sharply dominating lattice effect algebras. Recall that an element x of the LEA E is called *sharp* if $x \wedge x' = \mathbf{0}$. Jenča and Riečanová in [7] proved that in every lattice effect algebra E the set $S(E) = \{x \in E; x \wedge x' = \mathbf{0}\}$ of sharp elements is an orthomodular lattice which is a *sub-effect algebra* of E, meaning that if among $x, y, z \in E$ with $x \oplus y = z$ at least two elements are in $S(E)$ then $x, y, z \in S(E)$. Moreover $S(E)$ is a *full sublattice* of E, hence supremum of any set of sharp elements, which exists in E, is again a sharp element. Further, each maximal subset M of pairwise compatible elements of E, called *block* of E, is a sub-effect algebra and a full sublattice of E and $E = \bigcup\{M \subseteq E; M \text{ is a block of } E\}$ (see [15, 16]). *Central elements* and centers of effect algebras were defined in [4]. In [13, 14] it was proved that in every lattice effect algebra E the *center*

$$C(E) = \{x \in E; (\forall y \in E) y = (y \wedge x) \vee (y \wedge x')\} = S(E) \cap B(E), \tag{1}$$

where $B(E) = \bigcap\{M \subseteq E; M \text{ is a block of } E\}$. Since $S(E)$ is an orthomodular lattice and $B(E)$ is an MV-effect algebra, we obtain that $C(E)$ is a Boolean algebra. Note that E is an orthomodular lattice if and only if $E = S(E)$ and E is an MV-effect algebra if and only if $E = B(E)$. Thus E is a Boolean algebra if and only if $E = S(E) = B(E) = C(E)$.

Recall that an element p of an effect algebra E is called an *atom* if and only if p is a minimal non-zero element of E and E is *atomic* if for each $x \in E$, $x \neq \mathbf{0}$, there exists an atom $p \leq x$.

Definition 1.2. Let $(E, \oplus, 0)$ be an effect algebra. To each $a \in E$ we define its *isotropic index*, notation $ord(a)$, as the maximal positive integer n such that

$$na := \underbrace{a \oplus \cdots \oplus a}_{n\text{-times}}$$

exists. We set $ord(a) = \infty$ if na exists for each positive integer n. We say that E is *Archimedean*, if for each $a \in E$, $a \neq \mathbf{0}$, $ord(a)$ is finite.

An element $u \in E$ is called *finite*, if there exists a finite system of atoms $a_1, \ldots, a_n$ (which are not necessarily distinct) such that $u = a_1 \oplus \cdots \oplus a_n$. An element $v \in E$ is called *cofinite*, if there exists a finite element $u \in E$ such that $v = u'$.

We say that for a finite system $F = (x_j)_{j=1}^k$ of not necessarily different elements of an effect algebra $(E, \oplus, \mathbf{0}, \mathbf{1})$ is $\oplus$-*orthogonal* if $x_1 \oplus x_2 \oplus \cdots \oplus x_n = (x_1 \oplus x_2 \oplus \cdots \oplus x_{n-1}) \oplus x_n$ exists in E (briefly we will write $\bigoplus_{j=1}^n x_j$). We define also $\oplus\emptyset = \mathbf{0}$.

Definition 1.3. For a lattice $(L, \wedge, \vee)$ and a subset $D \subseteq L$ we say that D is a *bifull sublattice* of L, if and only if for any $X \subseteq D$, $\bigvee_L X$ exists if and only if $\bigvee_D X$ exists and $\bigwedge_L X$ exists if and only if $\bigwedge_D X$ exists, in which case $\bigvee_L X = \bigvee_D X$ and $\bigwedge_L X = \bigwedge_D X$.

Recall that an element $a \in L$, where $(L, \wedge, \vee)$ is a lattice, is called a *compact element* if for arbitrary $D \subset L$ with $\bigvee D \in L$, if $a \leq \bigvee D$ then $a \leq \bigvee F$ for some finite set $F \subseteq D$. The lattice L is called *compactly generated* if every element of L is a join of compact elements.

Lemma 1.4. Let $(E, \oplus, \vee, \wedge, \mathbf{0}, \mathbf{1})$ be an atomic Archimedean lattice effect algebra. Then

(i) (see [10]) a block M of E is atomic if there exists a maximal pairwise compatible set A of atoms of E such that $A \subseteq M$ and if M_1 is a block of E with $A \subseteq M_1$, then $M_1 = M$. Moreover for all $x \in E$ and all $a \in A$ the following holds

$$x \in M \quad \Leftrightarrow \quad x \leftrightarrow a,$$

(ii) (see [17]) to every nonzero element $x \in E$ there exist mutually distinct atoms $a_\alpha \in E$ and positive integers k_α for $\alpha \in \mathcal{I}$ such that

$$x = \bigoplus_{\alpha \in \mathcal{I}} (k_\alpha a_\alpha) = \bigvee_{\alpha \in \mathcal{I}} (k_\alpha a_\alpha).$$

It is known that if E is a distributive effect algebra (i. e., the effect algebra E is a distributive lattice – e. g., if E is an MV-effect algebra) then $C(E) = S(E)$. If moreover E is Archimedean and atomic then the set of atoms of $C(E) = S(E)$ is the set $\{n_a a; a \in E \text{ is an atom of } E\}$, where $n_a = ord(a)$ (see [19]). Since $S(E)$ is a bifull sublattice of E if E is an Archimedean atomic LEA (see [12]), we obtain that

$$\mathbf{1} = \bigvee_{C(E)} \{p \in C(E); p \text{ is an atom of } C(E)\} = \bigvee_{E} \{p \in C(E); p \text{ is an atom of } C(E)\}$$

for every Archimedean atomic distributive lattice effect algebra E. In [8] it was shown that there exists a LEA E for which this property fails to be true. Important properties of Archimedean atomic lattice effect algebras with atomic center were proven by Riečanová in [20].

Theorem 1.5. (Riečanová [20]) Let E be an Archimedean atomic lattice effect algebras with atomic center $C(E)$. Denote by A_E the set of all atoms of E and by $A_{C(E)}$ the set of all atoms of $C(E)$. The following conditions are equivalent:

1. $\bigvee_E A_{C(E)} = \mathbf{1}$.
2. For every atom $a \in A_E$ there exists an atom $p_a \in A_{C(E)}$ such that $a \leq p_a$.
3. For every $z \in C(E)$ it holds
$$z = \bigvee_{C(E)} \{p \in A_{C(E)}; p \leq z\} = \bigvee_E \{p \in A_{C(E)}; p \leq z\}.$$
4. $C(E)$ is a bifull sub-lattice of E.

In this case E is isomorphic to a subdirect product of Archimedean atomic irreducible lattice effect algebras.

2. MACNEILLE COMPLETION OF A LEA E WHOSE CENTER IS NOT BIFULL IN E

This section is based on an example published by the author in [8]. For reader's comfort in Section 2.1 we repeat the substantial parts of this paper where the LEA E whose center is not bifull in E, is constructed. In Section 2.2 we make the completion of E.

2.1. Construction of a LEA E whose center is not bifull in E

Let us have the following sequences of elements (sets):

$$\begin{aligned}
a_0 &= \{(x,y) \in \mathbb{R}^2; 0 \leq x \leq 1,\ y \in \mathbb{R}\}, \\
a_l &= \{(x,y) \in \mathbb{R}^2; l < x \leq l+1,\ y \in \mathbb{R}\}, \quad \text{for } l = 1,2,\ldots, \\
b_0 &= \{(x,y) \in \mathbb{R}^2; -1 \leq x < 0,\ y \in \mathbb{R}\}, \\
b_l &= \{(x,y) \in \mathbb{R}^2; -l-1 \leq x < -l,\ y \in \mathbb{R}\}, \quad \text{for } l = 1,2,\ldots, \qquad (2) \\
c_j &= \{(x,y) \in \mathbb{R}^2; -j \leq x \leq j,\ y \leq j \cdot x\}, \quad \text{for } j = 1,2,\ldots, \\
d_j &= \{(x,y) \in \mathbb{R}^2; -j \leq x \leq j,\ y > j \cdot x\}, \quad \text{for } j = 1,2,\ldots, \\
p_j &= \{j\}, \quad \text{for } j = 1,2,\ldots.
\end{aligned}$$

For such a choice of elements, the elements $q_1 \neq q_2$ are compatible if and only if $q_1 \cap q_2 = \emptyset$.

Denote $\hat{B}_0$, $\hat{B}_j$ (for $j = 1,2,\ldots$) complete atomic Boolean algebras with the corresponding sets of atoms A_0, A_j ($j = 1,2,\ldots$), given by

$$A_0 = \bigcup_{i=0}^{\infty}\{a_i\} \cup \bigcup_{i=0}^{\infty}\{b_i\} \cup \bigcup_{j=1}^{\infty}\{p_j\}, \qquad (3)$$

$$A_j = \bigcup_{i=j}^{\infty}\{a_i\} \cup \bigcup_{i=j}^{\infty}\{b_i\} \cup \bigcup_{j=1}^{\infty}\{p_j\} \cup \{c_j, d_j\}. \qquad (4)$$

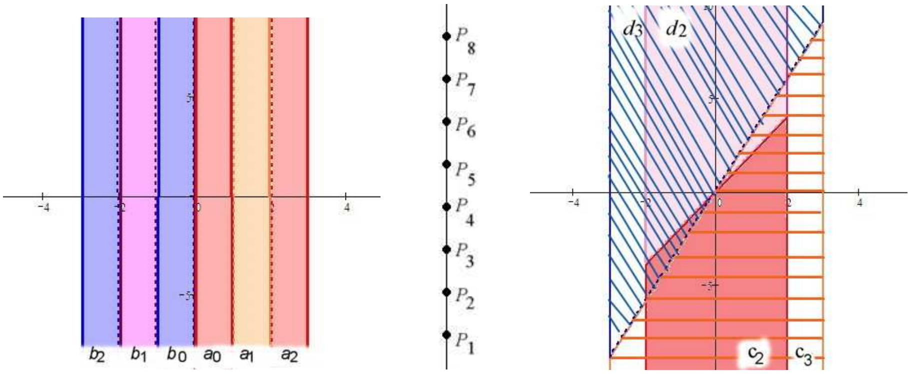

Fig. 1. Illustration of sequences of elements $(a_l)_l$, $(b_l)_l$, $(p_j)_j$, $(c_j)_j$, $(d_j)_j$.

Disjointness among some elements of the system (2) is equivalent with the fact that A_0 and A_j $(j = 1, 2, \dots)$ are unique maximal sets of pairwise compatible atoms.

For elements $u_1, u_2 \in \hat{B}_l$, $l = 0, 1, 2, \dots$, such that $u_1 \cap u_2 = \emptyset$ we introduce the partial operation $\oplus_l$ by

$$u_1 \oplus_l u_2 = u_1 \cup u_2. \tag{5}$$

Observe that if $u_1, u_2 \in \hat{B}_i \cap \hat{B}_j$, then

$$u_1 \oplus_i u_2 = u_1 \oplus_j u_2. \tag{6}$$

This is the reason why we will omit the index denoting operation $\oplus$ in the whole paper. Moreover we have the following equality

$$c_j \oplus d_j = \bigoplus_{i=0}^{j-1} (a_i \oplus b_i) = \{(x, y) \in \mathbb{R}^2; -j \le x \le j\}, \quad \text{for all } j = 1, 2, \dots. \tag{7}$$

The complete Boolean algebras $\hat{B}_0$, $\hat{B}_j$, $j = 1, 2, \dots$, have the following top elements:

$$\mathbb{R}^2 \cup \mathbb{N} = \mathbf{1} = 1_0 \quad = \quad a_0 \oplus b_0 \oplus \bigoplus_{i=1}^{\infty} (a_i \oplus b_i \oplus p_i) \tag{8}$$

$$\mathbb{R}^2 \cup \mathbb{N} = \mathbf{1} = 1_1 \quad = \quad (c_1 \oplus d_1) \oplus \bigoplus_{i=1}^{\infty} (a_i \oplus b_i \oplus p_i) \tag{9}$$

$$\mathbb{R}^2 \cup \mathbb{N} = \mathbf{1} = 1_j \quad = \quad (c_j \oplus d_j) \oplus \bigoplus_{i=j}^{\infty} (a_i \oplus b_i \oplus p_i) \oplus \bigoplus_{i=1}^{j-1} p_i, \tag{10}$$

$$\text{for all } j = 2, 3, \dots.$$

An element $u \in \hat{B}_l$ is finite if and only if $u = q_1 \oplus q_2 \oplus \cdots \oplus q_n$ for an $n \in \mathbb{N}$ and $q_1, q_2, \dots, q_n \in A_l$. Set $Q_l = \{u \in B_l; u \text{ is finite}\}$, $l = 0, 1, 2, \dots$. Then Q_l is a generalized Boolean algebra, since $B_l = Q_l \dot{\cup} Q_l^*$ is a Boolean algebra, where

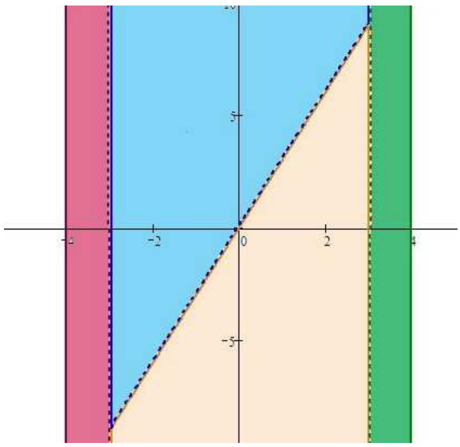

Fig. 2. Illustration of the element $a_3 \oplus b_3 \oplus c_3 \oplus d_3$.

$Q_l^* = \{u^*; u^* = 1_l \ominus u \text{ and } u \in Q_l\}$ (see [21], or [2, pp. 18-19]). This means that B_l is a Boolean subalgebra of finite and cofinite elements of $\hat{B}_l$ $(l = 0, 1, 2, \dots)$.

Theorem 2.1. (Kalina [8]) Denote $E = \bigcup_{l=0}^{\infty} B_l$. Then $(E, \oplus, \vee, \wedge, \mathbf{0}, \mathbf{1})$ is a compactly generated LEA with the family $(B_l)_{l=0}^{\infty}$ of atomic blocks of E. The center of E, $C(E)$, is not a bifull sublattice of E.

2.2. MacNeille completion of E

Let us denote

$$\hat{E} = \bigcup_{l=0}^{\infty} \hat{B}_l. \tag{11}$$

First we show the following lemma.

Lemma 2.2. $(\hat{E}, \oplus, \wedge, \vee, \mathbf{0}, \mathbf{1})$ is a lattice effect algebra.

P r o o f . Equation (6) shows that $\oplus$ is well defined. We show that this operation is commutative and associative. Let $q_1, q_2, q_3 \in \hat{E}$ are elements such that $q_1 \oplus q_2$ is defined and $(q_1 \oplus q_2) \oplus q_3$ is also defined. Then q_1, q_2 are disjoint sets and $(q_1 \oplus q_2)$ and q_3 are also disjoint sets. These imply that q_1, q_2, q_3 is a triple of pairwise disjoint sets and hence the commutativity and associativity follows immediately. Followingly $(\hat{E}, \oplus)$ is an effect algebra.

We show now that $(\hat{E}, \wedge, \vee, \mathbf{0}, \mathbf{1})$ is a bounded lattice.

Let $h_1, h_2 \in \hat{E}$ be arbitrary elements. First assume that $h_1 \leftrightarrow h_2$. Then there is an $i \in \{0, 1, 2, \dots\}$ such that $h_1 \in \hat{B}_i$, $h_2 \in \hat{B}_i$. Since $\hat{B}_i$ is a complete Boolean algebra, $h_1 \vee h_2$ and $h_1 \wedge h_2$ are well defined.

Assume that $h_1 \not\leftrightarrow h_2$. Then there are some $0 \le i < s$ such that $h_1 \in \hat{B}_i$ and $h_2 \in \hat{B}_s$. This means that for h_1 and h_2 we have

$$h_1 = \begin{cases} \bigoplus_{l=0}^{\infty} (\alpha_l a_l \oplus \beta_l b_l) \oplus \bigoplus_{j=1}^{\infty} \pi_j p_j, & \text{if } i = 0, \\ \gamma_i c_i \oplus \delta_i d_i \oplus \bigoplus_{l=i}^{\infty} (\alpha_l a_l \oplus \beta_l b_l) \oplus \bigoplus_{j=1}^{\infty} \pi_j p_j, & \text{if } i \neq 0, \end{cases} \tag{12}$$

$$h_2 = \gamma'_s c_s \oplus \delta_s d_s \oplus \bigoplus_{l=s}^{\infty} (\alpha'_l a_l \oplus \beta'_l b_l) \oplus \bigoplus_{m=1}^{\infty} \pi'_m p_m, \tag{13}$$

where $\alpha_l, \beta_l, \gamma_i, \delta_i, \pi_j \in \{0,1\}$ for $l = 0,1,2,\ldots,$ $i = 1,2,\ldots$ and $j = 1,2,\ldots,$ $\alpha'_l, \beta'_l, \gamma'_s, \delta'_s, \pi'_j \in \{0,1\}$ for $l = 1,2,\ldots,$ $s = 1,2,\ldots$ and $j = 1,2,\ldots.$ Because of formula (7) and the non-compatibility of h_1 and h_2, if we denote by Γ_i all atoms of A_i which are non-compatible with c_s (or equivalently, which are non-compatible with d_s), for h_1 we get that there exists a $q \in \Gamma_i$ such that $q \le h_1$ and at the same time

$$\bigoplus_{l=0}^{s-1} (a_l \oplus b_l) \not\le h_1, \qquad \text{if } i = 0,$$

$$c_i \oplus d_i \oplus \bigoplus_{l=i}^{s-1} (a_l \oplus b_l) \not\le h_1, \qquad \text{if } i \neq 0.$$

For h_2 we get that either $c_s \le h_2$ or $d_s \le h_2$, and $c_s \oplus d_s \not\le h_2$.
In all other cases we would get the compatibility of h_1 and h_2. Hence we have

$$h_1 \wedge h_2 = \bigoplus_{l=s}^{\infty} \left(\tilde{\alpha}_l a_l \oplus \tilde{\beta}_l b_l\right) \oplus \bigoplus_{m=1}^{\infty} \tilde{\pi}_m p_m, \tag{14}$$

$$\begin{aligned} h_1 \vee h_2 &= c_s \oplus d_s \oplus \bigoplus_{l=s}^{\infty} \left(\hat{\alpha}_l a_l \oplus \hat{\beta}_l b_l\right) \oplus \bigoplus_{m=1}^{\infty} \hat{\pi}_m p_m \\ &= \bigoplus_{l=0}^{s-1} (a_l \oplus b_l) \oplus \bigoplus_{l=s}^{\infty} \left(\hat{\alpha}_l a_l \oplus \hat{\beta}_l b_l\right) \oplus \bigoplus_{m=1}^{\infty} \hat{\pi}_m p_m, \end{aligned} \tag{15}$$

where $\tilde{\alpha}_l = \min\{\alpha_l, \alpha'_l\}$, $\tilde{\beta}_l = \min\{\beta_l, \beta'_l\}$, $\hat{\alpha}_l = \max\{\alpha_l, \alpha'_l\}$, $\hat{\beta}_l = \max\{\beta_l, \beta'_l\}$ for $l \in \{s, 2s+1, \ldots\}$, and $\tilde{\pi}_m = \min\{\pi_m, \pi'_m\}$, $\hat{\pi}_m = \max\{\pi_m, \pi'_m\}$ for $m \in \{1,2,\ldots\}$. The fact that $(\hat{E}, \oplus, \wedge, \vee, \mathbf{0}, \mathbf{1})$ is a LEA is due to formulas (5) and (6). □

In what follows we will denote the LEA $(\hat{E}, \oplus, \wedge, \vee, \mathbf{0}, \mathbf{1})$ just briefly as $\hat{E}$.

Theorem 2.3. $\hat{E}$ is a complete lattice.

Proof. Since $\hat{E}$ is the union of countably many blocks $\hat{B}_i$ and each block $\hat{B}_i$ is a complete Boolean algebra, it is enough to show that $\hat{E}$ is a σ-complete lattice. Each element $q \in \hat{E}$ has its supplement, hence we show just the σ-completeness with respect to $\vee$. Assume that $(h_{k_i})_{i=1}^{\infty}$ be a sequence of pairwise non-compatible elements of $\hat{E}$, where $h_{k_i} \in \hat{B}_{k_i}$ and $(k_i)_{i=1}^{\infty}$ is an increasing sequence of non-negative integers. Then the element h_{k_1} can be expressed by formula 12 replacing i by k_1, and h_{k_i} (for $i > 1$) can be expressed by formula (13) replacing s by k_i. Then by formula (15) we have that

$$\bigvee_{i=1}^{t} h_{k_i} = c_{k_t} \oplus d_{k_t} \oplus \bigoplus_{j=k_t}^{\infty} \left(\hat{\alpha}_j a_j \oplus \hat{\beta}_j b_j\right) \oplus \bigoplus_{m=1}^{\infty} \hat{\pi}_m p_m,$$

where

$$\begin{aligned} \hat{\alpha}_j &= \begin{cases} 1, & \text{if } a_j \le h_{k_i} \text{ for an } 1 \le i \le t, \\ 0, & \text{otherwise}, \end{cases} \\ \hat{\beta}_j &= \begin{cases} 1, & \text{if } b_j \le h_{k_i} \text{ for an } 1 \le i \le t, \\ 0, & \text{otherwise}, \end{cases} \\ \hat{\pi}_j &= \begin{cases} 1, & \text{if } p_j \le h_{k_i} \text{ for an } 1 \le i \le t, \\ 0, & \text{otherwise}. \end{cases} \end{aligned}$$

Formulas (2) imply

$$\bigvee_{i=1}^{t} (c_{k_t} \oplus d_{k_t}) = \mathbb{R}^2$$

which gives

$$\bigvee_{i=1}^{\infty} h_{k_i} = \mathbb{R}^2 \oplus \bigoplus_{m=1}^{\infty} \hat{\pi}_m p_m, \quad \text{where} \quad \hat{\pi}_j = \begin{cases} 1, & \text{if } p_j \le h_{k_i} \text{ for an } 1 \le i \le t, \\ 0, & \text{otherwise}. \end{cases}$$

This completes the proof that $\hat{E}$ is a complete lattice. □

Theorem 2.4. The atomic Archimedean LEA $E = \bigcup_{l=0}^{\infty} B_l$ can be densely embedded into $\hat{E} = \bigcup_{l=0}^{\infty} \hat{B}_l$.

P r o o f. Since each of the atomic complete Boolean algebras $\hat{B}_l$, for $l = 0, 1, 2, \dots$, is generated by countably many atoms, the completeness of each particular $\hat{B}_l$ is equivalent with its σ-completeness. Further, the atomic Boolean algebras B_l contain all finite elements of $\hat{B}_l$. This implies that each B_l can be densely embedded into $\hat{B}_l$. Hence we have that $E = \bigcup_{l=0}^{\infty} B_l$ can be densely embedded into $\hat{E} = \bigcup_{l=0}^{\infty} \hat{B}_l$, and the proof is finished. □

Let us denote by $\tilde{B}_0$, $\tilde{B}_j$ (for $j = 1, 2, \dots$) the following complete Boolean algebras generated by corresponding sets of atoms $\tilde{A}_0$, $\tilde{A}_j$:

$$\begin{aligned} \tilde{A}_0 &= \bigcup_{l=0}^{\infty} \{a_l\} \cup \bigcup_{l=0}^{\infty} \{b_l\}, \\ \tilde{A}_j &= \bigcup_{i=j}^{\infty} \{a_i\} \cup \bigcup_{i=j}^{\infty} \{b_i\} \cup \{c_j, d_j\}. \end{aligned}$$

Further we denote

$$\hat{E}_1 = \bigcup_{i=0}^{\infty} \tilde{B}_i. \tag{16}$$

We can embed $\hat{E}_1$ into $\hat{E}$. In this sense $\hat{E}_1$ is equipped with the partial operation $\oplus$ inherited from $\hat{E}$.

Lemma 2.5. $\hat{E}_1$ is a complete atomic Archimedean LEA with its center equal to $C(\hat{E}_1) = \{\mathbf{0}_{\hat{E}_1}, \mathbf{1}_{\hat{E}_1}\}$ and $\mathbf{1}_{\hat{E}_1}$ is an infinite element.

Proof. To show that $\hat{E}_1$ is a complete atomic Archimedean LEA we could repeat the proofs of Lemma 2.2 and of Theorem 2.3, just skipping the atoms $\{p_1, p_2, \dots\}$ from all formulas.

We show now that $C(\hat{E}_1) = \{\mathbf{0}_{\hat{E}_1}, \mathbf{1}_{\hat{E}_1}\}$. Formulas (2) imply that $\mathbf{1}_{\hat{E}_1} = \mathbb{R}^2$. Assume that there is yet another element of $C(\hat{E}_1)$. Let us denote this element by z. Assume that no atom from the set of atoms $\{c_1, d_1, c_2, d_2, \dots, c_j, d_j, \dots\}$ is below z. Since $z \neq \mathbf{0}_{\hat{E}_1}$, there exists an atom $a_i \leq z$ (or $b_i \leq z$). Then we get that $c_{i+1} \cap z \neq \emptyset$ and $c_{i+1} \not\leq z$ and hence $c_{i+1} \not\leftrightarrow z$. We may conclude that z is not a central element in this case. Assume that $c_j \leq z$ (or $d_j \leq z$) for some $j = 1, 2, \dots$ and there is a k such that $(c_k \oplus d_k) \not\leq z$. Then formulas (2) imply that either c_k or d_k in non-compatible with z and followingly z is not a central element. This consideration gives that if z is a central element then all atoms from the set of atoms $\{c_1, d_1, c_2, d_2, \dots, c_j, d_j, \dots\}$ are below z. Since

$$\bigvee_{j=1}^{\infty} (c_j \oplus d_j) = \mathbb{R}^2,$$

we get that $C(\hat{E}_1) = \{\mathbf{0}_{\hat{E}_1}, \mathbf{1}_{\hat{E}_1}\}$.

To conclude the proof we have to show that $\mathbf{1}_{\hat{E}_1}$ is an infinite element of $\hat{E}_1$. This is due to the fact that $\mathbf{1}_{\hat{E}_1}$ is an infinite element of each of the blocks $\tilde{B}_l$. □

Lemma 2.6. Let us denote by $\hat{\mathbf{B}}$ the complete Boolean algebra generated by the set of atoms $\{p_1, p_2, \dots, p_j, \dots\}$. Then $\hat{E}$ is isomorphic to the direct product $\hat{\mathbf{B}} \times \hat{E}_1$.

Proof. The isomorphism between $\hat{E} = \bigcup_{l=0}^{\infty} \hat{B}_l$ and the direct product $\hat{\mathbf{B}} \times \hat{E}_1$ follows from the fact that each of the blocks $\hat{B}_l$ is isomorphic to the direct product $\hat{\mathbf{B}} \times \tilde{B}_l$. □

Theorem 2.7. Let $E = \bigcup_{l=0}^{\infty} B_l$ and $\hat{E} = \bigcup_{l=0}^{\infty} \hat{B}_l$. Denote $\mathcal{MC}(C(E))$ the MacNeille completion of $C(E)$. Then the following holds

$$\mathcal{MC}(C(E)) \subsetneq C(\hat{E}).$$

Proof. Set $\mathbf{1}_{\hat{E}_1}$ the top element of $\hat{E}_1$. Then $\mathbf{1}_{\hat{E}_1} \in C(\hat{E})$. Since there is no non-zero central element of $\hat{E}$ below $\mathbf{1}_{\hat{E}_1}$, we may conclude that $\mathbf{1}_{\hat{E}_1}$ is an atom of $C(\hat{E})$.

On the other hand $\mathbf{1}_{\hat{E}_1}$ is neither a finite nor a cofinite element of $\hat{E}$ and hence $\mathbf{1}_{\hat{E}_1} \notin C(E)$. Since $\mathbf{1}_{\hat{E}_1}$ is an atom of $C(\hat{E})$, we get immediately $\mathbf{1}_{\hat{E}_1} \notin \mathcal{MC}(C(E))$ and the proof of the theorem is finished. □

Theorem 2.7 can be generalized into the following

Theorem 2.8. Let $\mathcal{E}$ be an atomic Archimedean LEA with atomic center $C(\mathcal{E})$ that is not a bifull sublattice of $\mathcal{E}$. Let $\mathcal{MC}\left(C(\mathcal{E})\right)$ be the MacNeille completion of $C(\mathcal{E})$ and $\hat{\mathcal{E}}$ the MacNeille completion of $\mathcal{E}$. Then the following holds

$$\mathcal{MC}\left(C(\mathcal{E})\right) \subsetneq C\left(\hat{\mathcal{E}}\right).$$

Proof. Because $C(\mathcal{E})$ is not a bifull sublattice of $\mathcal{E}$, due to Theorem 1.5 we have that

$$\bigvee_{\mathcal{E}} \{q \in C(\mathcal{E}); q \text{ is an atom of } C(\mathcal{E})\}$$

does not exist in $\mathcal{E}$ but

$$\bigvee_{C(\mathcal{E})} \{q \in C(\mathcal{E}); q \text{ is an atom of } C(\mathcal{E})\} = \mathbf{1}$$

Set $z = \left(\bigvee_{\hat{\mathcal{E}}} \{q \in C(\mathcal{E}); q \text{ is an atom of } C(\mathcal{E})\}\right)'$. Then obviously

$$z \in \hat{\mathcal{E}}$$

holds and at the same time, since there is no non-zero element of $C(\mathcal{E})$ that is below z, $z \notin \mathcal{MC}\left(C(\mathcal{E})\right)$. □

3. SEARCHING FOR A SUFFICIENT CONDITION UNDER WHICH $\mathcal{MC}\left(C(\mathcal{E})\right) = C\left(\hat{\mathcal{E}}\right)$ HOLDS

Theorem 2.8 gives us a necessary condition under which, for an atomic Archimedean lattice effect algebra $\mathcal{E}$ the equality

$$\mathcal{MC}\left(C(\mathcal{E})\right) = C\left(\hat{\mathcal{E}}\right) \tag{17}$$

is valid. Once we have find a necessary condition, it is natural to look for a sufficient condition. We are going to present an example helping us to solve this problem.

Let us take the complete atomic Archimedean LEA $\hat{E}_1$ given by formula 16 and its isomorphic copy denoted by $\hat{E}_2$. Since all atoms of $\hat{E}_1$ are compact elements, the following assertion is straightforward

Lemma 3.1. The Archimedean atomic LEA $\hat{E}_1 \times \hat{E}_2$ is compactly generated. Further, its center $C(\hat{E}_1 \times \hat{E}_2)$ has the following elements

$$C\left(\hat{E}_1 \times \hat{E}_2\right) = \left\{\mathbf{0}, \mathbf{1}, \mathbf{1}_{\hat{E}_1}, \mathbf{1}_{\hat{E}_2}\right\},$$

where $\mathbf{1}_{\hat{E}_1}$ and $\mathbf{1}_{\hat{E}_2}$ are the top elements of $\hat{E}_1$ and $\hat{E}_2$, respectively.

Let us denote E_f the set of all finite and cofinite elements of $\hat{E}_1 \times \hat{E}_2$.

Theorem 3.2. E_f is an atomic Archimedean LEA which is densely embeddable into $\hat{E}_1 \times \hat{E}_2$. The center of E_f is the following

$$C(E_f) = \{\mathbf{0}, \mathbf{1}\}.$$

Proof. The fact that E_f is an atomic Archimedean LEA which is densely embeddable into $\hat{E}_1 \times \hat{E}_2$, follows from Lemma 3.1. Since $\mathbf{1}_{\hat{E}_1}$ and $\mathbf{1}_{\hat{E}_2}$ are neither finite nor cofinite elements of $\hat{E}_1 \times \hat{E}_2$, we have that $C(E_f) = \{\mathbf{0}, \mathbf{1}\}$. □

Let $\tilde{B}$ be an arbitrary atomic Boolean algebra and q_i, for i running throu an appropriate index set I, be atoms of $\tilde{B}$. Then, due to Theorem 1.5, $\tilde{B}$ is isomorphic with a subdirect product of $\{\mathbf{0}_{\tilde{B}}, z_i\}_{i \in I}$.

Theorem 3.3. There exists an atomic Archimedean LEA $E_{\tilde{B}}$ whose center is isomorphic with $\tilde{B}$ and for which equality (17) does not hold.

Proof. $\tilde{B}$ is a subdirect product of $\{\mathbf{0}_{\tilde{B}}, z_i\}$ for $i \in I$. Instead of $\{\mathbf{0}_{\tilde{B}}, z_1\}$ we take the atomic Archimedean LEA E_f. Then the center of the corresponding subdirect product of E_f and of the system $\{\mathbf{0}_{\tilde{B}}, z_i\}$ for $i \in I \setminus \{1\}$ is isomorphic to $\tilde{B}$, but due to Lemma 3.1 we have

$$\mathcal{MC}\left(C(E_{\tilde{B}})\right) = \mathcal{MC}(\tilde{B}) \subsetneq \mathcal{MC}\left(E_{\tilde{B}}\right).$$

□

4. CONCLUSIONS

In this paper we studied the equality

$$\mathcal{MC}(C(E)) = C\left(\hat{E}\right),$$

where E is an atomic Archimedean LEA and $\hat{E}$ its MacNeille completion. Particularly, we were interested in finding conditions expressible by means of properties of $C(E)$, under which the equality holds. We proved that there exists an atomic Archimedean LEA E for which equality is violated. Further, we proved that the bifullness of the center $C(E)$ in E is necessary for the equality to be true. Moreover we showed that even the completness of the center and the bifulness of $C(E)$ in E is not sufficient to guarantee the above equality and for an arbitrary atomic Boolean algebra B there exists an atomic Archimedean LEA whose center is equal to B and for which the above equality is not fulfilled.

ACKNOWLEDGEMENT

The support of Science and Technology Assistance Agency under the contract No. APVV-0375-06, and of the VEGA grant agency, grant number 1/0373/08, is kindly acknowledged. The author is grateful to the anonymous reviewers for their valuable comments helping to improve the manuscript.

(Received June 30, 2010)

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