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SOME NEW RESULTS ABOUT BROOKS–JEWETT AND DIEUDONNÉ–TYPE THEOREMS IN (L)-GROUPS

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In this paper we present some new versions of Brooks-Jewett and Dieudonné-type theorems for (l)-group-valued measures.

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1. INTRODUCTION

Dieudonné-type theorems (see [13]) are subjects of deep studies of several mathematicians. There are many versions of theorems of this kind, for example, for maps taking values in topological groups and/or Banach spaces: we quote here Brooks and Jewett ([8, 9]), Candeloro and Letta ([10, 11]).

We now report the classical Brooks–Jewett theorem ([9, Theorem 2]).

Theorem 1.1. Let $\mathcal{X}$ be a Banach space, $\mathcal{A}$ be a σ-ring of subsets of an abstract set G, $m_j : \mathcal{A} \to \mathcal{X}$ be finitely additive and (s)-bounded measures, $j \in \mathbb{N}$. Suppose that $m(E) := \lim_j m_j(E)$ exists in $\mathcal{X}$ for every $E \in \mathcal{A}$.

Then the m_j's are uniformly additive.

In this paper we deal with some Brooks–Jewett (see [9]) and Dieudonné-type theorems in the context of (l)-groups. We observe that there are Riesz spaces, in which order convergence is not generated by *any* topology: for example, $L^0(X, \mathcal{B}, \mu)$, where μ is a σ-additive and σ-finite non-atomic positive $\widetilde{\mathbb{R}}$-valued measure. Indeed, in these spaces order convergence means almost everywhere convergence and it is not compatible with any group topology.

We also use the concept of (RO)*-convergence* for set functions, which is inspired by similar concepts of "equal" convergence ([12]) and convergence "with respect to the same regulator" ([5, 6]).

In [2] similar results were proved with respect to order convergence for *positive* finitely additive measures, taking values in spaces of the type $L^0(X, \mathcal{B}, \mu)$. In [5, 6] some limit theorems and Dieudonné-type theorems were proved in the context of (l)-groups, using another kind of convergence ((D)-convergence), which at least for sequences coincides with order convergence if the underlying (l)-group is Dedekind complete and weakly σ-distributive.

We remark that in those papers all types of convergence are related to the notion of "common regulator", while here at least the concepts of (s)-boundedness, σ-additivity and regularity are formulated in a more intuitive way, and not directly related to (o)-sequences or similar objects.

In [7] some limit theorems were proved, in which σ-additivity is considered not necessarily "with respect to the same regulator". In this paper, avoiding those technicalities, we obtain some Brooks–Jewett and Dieudonné-type theorems, only assuming that pointwise convergence of the involved measures takes place with respect to the same (o)-sequence.

2. PRELIMINARIES

Definitions 2.1. An Abelian group $(R,+)$ is called (l)-*group* if it is endowed with a compatible ordering $\leq$, and is a lattice with respect to it.
An (l)-group R is said to be *Dedekind complete* if every nonempty subset of R, bounded from above, has supremum in R.

A sequence $(p_n)_n \downarrow 0$ in R is said to be an (o)-*sequence.* We say that a sequence $(r_n)_n$ in R is *order-convergent* (or (o)-*convergent*) to r if there exists an (o)-sequence $(p_n)_n$ with $|r_n - r| \leq p_n$ for all $n \in \mathbb{N}$ (see also [15, 18]), and we will write $(o)\lim_n r_n = r$.

A sequence $(r_n)_n$ is said to be (o)-*Cauchy* if there exists an (o)-sequence $(p_n)_n$ such that $|r_n - r_m| \leq p_n$ for all $n \in \mathbb{N}$ and $m \geq n$.

Given a topological space Ω and a set $N \subset \Omega$, we say that N is *nowhere dense* in Ω if its closure has empty interior. We say that $N \subset \Omega$ is *meager* if N can be expressed as a countable union of nowhere dense subsets of Ω.

From now on we assume that R is a Dedekind complete (l)-group.

We now recall the following version of the Maeda-Ogasawara-Vulikh Theorem (see [18], Theorems V.4.2, p. 138 and V.3.1, p. 131; [1], Theorem 3, p. 610).

Theorem 2.2. Every Dedekind complete (l)-group R is algebraically and lattice isomorphic to an order dense ideal of $\mathcal{C}_\infty(\Omega) = \{f \in \widetilde{\mathbb{R}}^\Omega : f$ is continuous, and $\{\omega \in \Omega : |f(\omega)| = +\infty\}$ is nowhere dense in $\Omega\}$, where Ω is a suitable compact extremely disconnected topological space.

Furthermore, if we denote by $\widehat{a}$ the element of $\mathcal{C}_\infty(\Omega)$ which corresponds to $a \in R$ under the above isomorphism, then for any family $(a_\lambda)_{\lambda \in \Lambda}$ of elements of R such that $a_0 := \bigvee_\lambda a_\lambda \in R$ we have $\widehat{a}_0(\omega) = \sup_\lambda[\widehat{a}_\lambda(\omega)]$ in the complement of a meager subset of Ω. The same is true for $\bigwedge_\lambda a_\lambda$.

From now on, when we regard R as a subset of $\mathcal{C}_\infty(\Omega)$, we shall denote by the symbols $\vee$ and $\wedge$ the supremum and infimum in R and by sup and inf the "pointwise" supremum and infimum, respectively.

Assumptions 2.3. From now on, we assume that G is any infinite set, and $\mathcal{A} \subset \mathcal{P}(G)$ is an algebra. We suppose that $\mathcal{F}$, $\mathcal{G} \subset \mathcal{A}$ are two fixed lattices, such that the complement (with respect to G) of every element of $\mathcal{F}$ belongs to $\mathcal{G}$ and $\mathcal{G}$ is closed with respect to countable disjoint unions.

If G is a normal topological space [resp. locally compact Hausdorff space], examples of lattices $\mathcal{A}$, $\mathcal{F}$ and $\mathcal{G}$, satisfying the above properties, are the following: $\mathcal{A} = \{\text{Borelian subsets of } G\}$, $\mathcal{F} = \{\text{closed sets}\}$ [resp.$\{\text{compact sets}\}$], $\mathcal{G} = \{\text{open sets}\}$.

Definitions 2.4. We say that a set function $m : \mathcal{A} \to R$ is *bounded* if there exists $w \in R$ such that $|m(A)| \leq w$ for all $A \in \mathcal{A}$. The maps m_j, $j \in \mathbb{N}$, are *equibounded* (or *uniformly bounded*) on $\mathcal{A}$ if there is $u \in R$, with $|m_j(A)| \leq u$ for all $j \in \mathbb{N}$ and $A \in \mathcal{A}$.

If $\mathcal{E}$ is any sublattice of $\mathcal{A}$, we say that a sequence of measures $(m_j : \mathcal{A} \to R)_j$ (RO)-*converges* to a map m_0 on $\mathcal{E}$ if there is an (o)-sequence $(p_l)_l$ such that to each $l \in \mathbb{N}$ and $A \in \mathcal{E}$ it is possible to associate $j_0 \in \mathbb{N}$ with $|m_j(A) - m_0(A)| \leq p_l$ whenever $j \geq j_0$.

Given a finitely additive bounded measure $m : \mathcal{A} \to R$, we define m^+, m^-, $\|m\| : \mathcal{A} \to R$, by setting

$$\begin{aligned} m^+(A) &= (m^+)_{\mathcal{A}}(A) := \vee_{B \in \mathcal{A}, B \subset A}\, m(B), \\ m^-(A) &= (m^-)_{\mathcal{A}}(A) := - \wedge_{B \in \mathcal{A}, B \subset A}\, m(B), \end{aligned} \tag{1}$$

$$\|m\|(A) = \|m\|_{\mathcal{A}}(A) := (m^+)_{\mathcal{A}}(A) + (m^-)_{\mathcal{A}}(A),\ \ A \in \mathcal{A}.$$

The set functions $m^+, m^-, \|m\|$ are called *positive part, negative part* and *total variation* of m (on $\mathcal{A}$), respectively. Moreover, define the *semivariation* of m on $\mathcal{A}$, $v_{\mathcal{A}}(m) : \mathcal{A} \to R$, by setting

$$v_{\mathcal{A}}(m)(A) = \vee_{B \in \mathcal{A}, B \subset A} |m(B)|, \quad A \in \mathcal{A}.$$

We have (see also [14]):

$$v_{\mathcal{A}}(m)(A) \leq \|m\|_{\mathcal{A}}(A) \leq 2 v_{\mathcal{A}}(m)(A), \quad \text{for all } A \in \mathcal{A}. \tag{2}$$

Moreover, for every $A \in \mathcal{A}$ set

$$\begin{aligned} (m^+)_{\mathcal{G}}(A) &:= \vee_{B \in \mathcal{G}, B \subset A}\, m(B),\ (m^-)_{\mathcal{G}}(A) := \vee_{B \in \mathcal{G}, B \subset A} [-m(B)], \\ v_{\mathcal{G}}(m)(A) &:= \vee_{B \in \mathcal{G}, B \subset A} |m(B)|; \end{aligned}$$

analogously it is possible to define $(m^{\pm})_{\mathcal{F}}$ and $v_{\mathcal{F}}$, the positive and negative parts with respect to $\mathcal{F}$ and the $\mathcal{F}$-semivariation respectively.

From now on, all involved finitely additive maps are assumed to be bounded. We now introduce the concept of (s)-boundedness, following an approach similar to the classical one.

A finitely additive set function $m : \mathcal{A} \to R$ is said to be (s)-*bounded on* $\mathcal{A}$ or $\mathcal{A}$-(s)-*bounded* if for every disjoint sequence $(H_n)_n$ in $\mathcal{A}$ we have $\limsup_n v_{\mathcal{A}}(m)(H_n) = 0$. We say that the maps $m_j : \mathcal{A} \to R$, $j \in \mathbb{N}$, are *uniformly* (s)-*bounded on* $\mathcal{A}$ or *uniformly* $\mathcal{A}$-(s)-*bounded* if $\limsup_n [\vee_j v_{\mathcal{A}}(m_j)(H_n)] = 0$ whenever $(H_n)_n$ is a sequence of pairwise disjoint elements of $\mathcal{A}$.

A finitely additive set function $m : \mathcal{A} \to R$ is said to be *σ-additive* if for every disjoint sequence $(H_n)_n$ in $\mathcal{A}$, $\wedge_n [v_{\mathcal{A}}(m)(\bigcup_{l=n}^{\infty} H_l)] = 0$. We say that the measures $m_j : \mathcal{A} \to R$, $j \in \mathbb{N}$, are *uniformly σ-additive* if for each disjoint sequence $(H_n)_n$ in $\mathcal{A}$, $\wedge_n [\vee_j v_{\mathcal{A}}(m_j)(\bigcup_{l=n}^{\infty} H_l)] = 0$.

Analogously as above it is possible to formulate the concepts of (uniform) $\mathcal{G}$-(s)-boundedness and $\mathcal{G}$-σ-additivity, in which we replace the semivariation $v_{\mathcal{A}}$ with $v_{\mathcal{G}}$.

3. THE BROOKS–JEWETT THEOREM

We now state the following Brooks–Jewett type theorem.

Theorem 3.1. Let G, $\mathcal{A}$ and $\mathcal{G}$ be as in Assumptions 2.3, Ω be as in Theorem 2.2, and suppose that $(m_j : \mathcal{A} \to R)_j$ is a sequence of (not necessarily positive) finitely additive equibounded measures. Suppose that there is a map $m_0 : \mathcal{G} \to R$ such that the sequence $(m_j)_j$ (RO)-converges to m_0 on $\mathcal{G}$.

Then the real valued functions $m_j(\cdot)(\omega)$ are uniformly $\mathcal{G}$-(s)-bounded on $\mathcal{G}$ (with respect to j) for ω belonging to the complement of a meager subset of Ω. Moreover the m_j's are uniformly $\mathcal{G}$-(s)-bounded on $\mathcal{G}$.

Proof. Let Ω be as in Theorem 2.2. First of all we observe that, since the m_j's are equibounded, then there exists a nowhere dense set $N_0 \subset \Omega$ such that for all $\omega \notin N_0$ the maps $m_j(\cdot)(\omega)$, $j \in \mathbb{N}$, are real-valued, finitely additive and bounded on $\mathcal{G}$, and hence (s)-bounded on $\mathcal{G}$. Moreover, by (RO)-convergence, there is an (o)-sequence $(p_l)_l$ with the property that to every $l \in \mathbb{N}$ and $A \in \mathcal{G}$ there corresponds a positive integer j_0 with

$$|m_j(A) - m_0(A)| \le p_l \quad \text{for all}\, j \ge j_0. \tag{3}$$

Thanks to Theorem 2.2, a meager set $N \subset \Omega$ can be found, without loss of generality with $N \supset N_0$, such that the sequence $(p_l(\omega))_l$ is a real-valued (o)-sequence, whenever $\omega \notin N$. Thus for every $l \in \mathbb{N}$ and $A \in \mathcal{G}$ there is $j_0 \in \mathbb{N}$ such that for all $\omega \in \Omega \setminus N$ and $j \ge j_0$ we get:

$$|m_j(A)(\omega) - m_0(A)(\omega)| \le p_l(\omega). \tag{4}$$

This implies that $\lim_j m_j(A)(\omega) = m_0(A)(\omega)$ for each $A \in \mathcal{G}$ and $\omega \notin N$. Thus for such ω's the *real-valued* set functions $m_j(\cdot)(\omega)$ satisfy the hypotheses of the classical version of the Brooks–Jewett theorem (see [9, Theorem 2]), and so they are uniformly $\mathcal{G}$-(s)-bounded on $\mathcal{G}$. This concludes the first part of the assertion.

We now prove that the measures m_j, $j \in \mathbb{N}$, are uniformly $\mathcal{G}$-(s)-bounded on $\mathcal{G}$. Fix arbitrarily any disjoint sequence $(H_k)_k$ in $\mathcal{G}$ and let us check that

$$\wedge_s [\vee_{k \ge s}(\vee_j [\vee_{B \in \mathcal{G}, B \subset H_k} |m_j(B)|]) = 0. \tag{5}$$

Since the measures $m_j(\cdot)(\omega)$ are uniformly $\mathcal{G}$-(s)-bounded on $\mathcal{G}$ for all $\omega \in \Omega \setminus N$, where N is as in (4), then

$$\inf_s \big[\sup_{k \ge s} \big\{ \sup_j \big[v_{\mathcal{G}}(m_j(\cdot)(\omega))(H_k) \big] \big\} \big] = \lim_k \big\{ \sup_j \big[v_{\mathcal{G}}(m_j(\cdot)(\omega))(H_k) \big] \big\} = 0 \tag{6}$$

for every $\omega \notin N$. Since the union of countably many meager sets is still meager, then in the complement of a suitable meager set, without loss of generality containing N, for all $k \in \mathbb{N}$ we get:

$$\sup_j \Big[\sup_{B \in \mathcal{G}, B \subset H_k} |m_j(B)(\omega)| \Big] = \big\{ \vee_j \, [\vee_{B \in \mathcal{G}, B \subset H_k} |m_j(B)|] \big\}(\omega). \tag{7}$$

From (6) and (7) it follows that, again up to complements of meager sets,

$$\wedge_s \, [\vee_{k \geq s} (\vee_j [\vee_{B \in \mathcal{G}, B \subset H_k} |m_j(B)|])](\omega) = 0. \tag{8}$$

By a density argument we get (5).

Hence $\limsup_k (\vee_j [\vee_{B \in \mathcal{G}, B \subset H_k} |m_j(B)|]) = 0$, namely $\limsup_k (\vee_j v_{\mathcal{G}}(m_j)(H_k)) = 0$. Thanks to arbitrariness of the chosen sequence $(H_k)_k$, we get uniform (s)-boundedness of the m_j's on $\mathcal{G}$. □

We now prove a technical lemma, which will be useful in the sequel.

Lemma 3.2. Under the same hypotheses and notations as above, suppose that there exists a meager set $N \subset \Omega$ such that the real-valued measures $m_j(\cdot)(\omega)$, $j \in \mathbb{N}$, are uniformly (s)-bounded on $\mathcal{G}$ for all $\omega \notin N$. Fix $W \in \mathcal{F}$, and assume that the sequences $(G_n)_n$ and $(F_n)_n$, from $\mathcal{G}$ and $\mathcal{F}$ respectively, satisfy

$$W \subset F_{n+1} \subset G_n \subset F_n \quad \textit{for all } n \in \mathbb{N}$$

and the following equality:

$$\lim_n \Big[\sup_{A \in \mathcal{G}, A \subset G_n \setminus W} |m_j(A)(\omega)| \Big] = 0 \quad \textit{for all } j \in \mathbb{N} \tag{9}$$

for ω belonging to the complement of a meager set $N_W \subset \Omega$. Then

$$\lim_n \Big(\sup_j \big[\sup_{A \in \mathcal{G}, A \subset G_n \setminus W} |m_j(A)(\omega)| \big] \Big) = 0 \tag{10}$$

whenever $\omega \in \Omega \setminus (N \cup N_W)$.

P r o o f. Fix arbitrarily $\omega \in \Omega \setminus (N \cup N_W)$, set $\mathcal{W} := \{A \in \mathcal{G} : A \cap W = \emptyset\}$ and let $A \in \mathcal{W}$. Since $A \cap F_q \subset G_{q-1} \setminus W$ for all $q \in \mathbb{N}$, from (9) for all $j \in \mathbb{N}$ we get

$$m_j(A)(\omega) = \lim_q \, m_j(A \cap F_q^c)(\omega) \tag{11}$$

uniformly with respect to $A \in \mathcal{W}$.

If we deny the thesis of the lemma, then there exists $\varepsilon > 0$ with the property that to every $p \in \mathbb{N}$ there correspond $n \in \mathbb{N}$, $n > p$, $j \in \mathbb{N}$ and $A \in \mathcal{G}$ such that $A \subset G_n \setminus W$, $|m_j(A)(\omega)| > \varepsilon$, and hence, thanks to (11),

$$|m_j(A \cap F_q^c)(\omega)| > \varepsilon \tag{12}$$

for q large enough.

At the first step, in correspondence with $p = 1$, there exist: $A_1 \in \mathcal{G}$; three integers $n_1 \in \mathbb{N} \setminus \{1\}$, $j_1 \in \mathbb{N}$ and $q_1 > \max\{n_1, j_1\}$, with $A_1 \subset G_{n_1} \setminus W$ and

$$|m_{j_1}(A_1)(\omega)| > \varepsilon; \quad |m_{j_1}(A_1 \cap F^c_{q_1})(\omega)| > \varepsilon.$$

From (9), in correspondence with $j = 1, 2, \ldots, j_1$ we get the existence of an integer $h_1 > q_1$ such that

$$|m_j(A)(\omega)| \leq \varepsilon \tag{13}$$

whenever $n \geq h_1$ and $A \subset G_n \setminus W$.

At the second step, there exist: $A_2 \in \mathcal{G}$; three integers $n_2 > h_1$, $j_2 \in \mathbb{N}$ and $q_2 > \max\{n_2, j_2\}$, with $A_2 \subset G_{n_2} \setminus W$ and

$$|m_{j_2}(A_2)(\omega)| > \varepsilon; \quad |m_{j_2}(A_2 \cap F^c_{q_2})(\omega)| > \varepsilon. \tag{14}$$

From (13) and (14) it follows that $j_2 > j_1$.

Thus, proceeding by induction, it is possible to construct a sequence $(A_k)_k$ in $\mathcal{G}$ and three strictly increasing sequences in $\mathbb{N}$, $(n_k)_k$, $(j_k)_k$, $(q_k)_k$, with $q_k > n_k > q_{k-1}$, $k \geq 2$; $q_k > j_k$ and

$$A_k \subset G_{n_k} \setminus W; \quad |m_{j_k}(A_k \cap F^c_{q_k})(\omega)| > \varepsilon$$

for all $k \in \mathbb{N}$. But this is impossible, since the sets $A_k \cap F^c_{q_k}$, $k \in \mathbb{N}$, are pairwise disjoint elements of $\mathcal{G}$, $\omega \in \Omega \setminus (N \cup N_W)$, and the maps $m_j(\cdot)(\omega)$, $j \in \mathbb{N}$ are uniformly (s)-bounded on $\mathcal{G}$ for each fixed $\omega \in \Omega \setminus N$. This concludes the proof. □

If $\mathcal{A}$ is a σ-algebra, then, analogously as in Lemma 3.2, by considering $\mathcal{G} = \mathcal{F} = \mathcal{A}$ and $W = \emptyset$ it is possible to prove the following:

Corollary 3.3. With the same assumptions as above, let $\mathcal{A}$ be a σ-algebra and suppose that there is a meager set $N \subset \Omega$ such that the real-valued measures $m_j(\cdot)(\omega)$, $j \in \mathbb{N}$, are uniformly (s)-bounded on $\mathcal{A}$ for all $\omega \notin N$. Assume that $(H_n)_n$ is a decreasing sequence in $\mathcal{A}$, $H_n \downarrow \emptyset$. If

$$\lim_n \Big[\sup_{A \in \mathcal{A}, A \subset H_n} |m_j(A)(\omega)| \Big] = 0 \quad \text{for all } j \in \mathbb{N} \tag{15}$$

for $\omega \in \Omega \setminus N_1$, where N_1 is a suitable meager set, then

$$\lim_n \Big(\sup_j \big[\sup_{A \in \mathcal{A}, A \subset H_n} |m_j(A)(\omega)| \big] \Big) = 0 \tag{16}$$

whenever $\omega \in \Omega \setminus (N \cup N_1)$.

4. REGULAR SET FUNCTIONS

In this section we investigate some fundamental properties of (l)-group-valued regular set functions. In [5] we formulated regularity of the involved measures "with respect to a same regulator". Here we do not assume any hypothesis of this kind.

From now on, assume that $\mathcal{A} \subset \mathcal{P}(G)$ is a σ-algebra.

Definitions 4.1. A finitely additive measure $m : \mathcal{A} \to R$ is said to be *regular* if for each $A \in \mathcal{A}$ and $W \in \mathcal{F}$ there exist four sequences $(F_n)_n$, $(F'_n)_n$ in $\mathcal{F}$, $(G_n)_n$, $(G'_n)_n$ in $\mathcal{G}$, such that:

$$F_n \subset F_{n+1} \subset A \subset G_{n+1} \subset G_n \quad \text{for all } n \in \mathbb{N}, \tag{17}$$

$$W \subset F'_{n+1} \subset G'_n \subset F'_n \quad \text{for any } n \in \mathbb{N}; \tag{18}$$

moreover, $\wedge_n[v_{\mathcal{A}}(m)(G_n \setminus F_n)] = \wedge_n[v_{\mathcal{A}}(m)(G'_n \setminus W)] = 0$.

The finitely additive measures $m_j : \mathcal{A} \to R$, $j \in \mathbb{N}$, are said to be *uniformly regular* if for all $A \in \mathcal{A}$ and $W \in \mathcal{F}$ there exist sequences $(F_n)_n, (G_n)_n, (F'_n)_n, (G'_n)_n$ satisfying (17) and (18), and such that

$$\wedge_n[\vee_j (v_{\mathcal{A}}(m_j)(G_n \setminus F_n))] = \wedge_n[\vee_j (v_{\mathcal{A}}(m_j)(G'_n \setminus W))] = 0.$$

We now prove that, if we deal with a regular measure m, for all $A \in \mathcal{A}$ the semivariations $v_{\mathcal{F}}(m)(A)$ and $v_{\mathcal{A}}(m)(A)$ coincide; moreover, when $A \in \mathcal{G}$, then $v_{\mathcal{A}}(m)(A)$ also coincides with $v_{\mathcal{G}}(m)(A)$.

Lemma 4.2. (see also [5], Lemma 3.1) *Let R, G, $\mathcal{A}$, $\mathcal{F}$, $\mathcal{G}$ be as above, and suppose that $m : \mathcal{A} \to R$ is any regular bounded finitely additive measure. Then for each $A \in \mathcal{A}$ we get:*

$$(m^{\pm})_{\mathcal{A}}(A) = (m^{\pm})_{\mathcal{F}}(A), \quad v_{\mathcal{A}}(m)(A) = v_{\mathcal{F}}(m)(A). \tag{19}$$

Moreover, for every $V \in \mathcal{G}$ one has:

$$(m^{\pm})_{\mathcal{A}}(V) = (m^{\pm})_{\mathcal{G}}(V), \quad v_{\mathcal{A}}(m)(V) = v_{\mathcal{G}}(m)(V). \tag{20}$$

Finally for all $K \in \mathcal{F}$ we get:

$$\wedge_{H \in \mathcal{G}, K \subset H} \|m\|(H \setminus K) = 0. \tag{21}$$

P r o o f. We begin with the first part. To this aim, it is enough to show that

$$(m^{\pm})_{\mathcal{A}}(A) \le (m^{\pm})_{\mathcal{F}}(A), \quad v_{\mathcal{A}}(m)(A) \le v_{\mathcal{F}}(m)(A).$$

Fix arbitrarily $A \in \mathcal{A}$, and pick $B \subset A$, $B \in \mathcal{A}$: then there exists a sequence $(F_n)_n$ in $\mathcal{F}$, such that $F_n \subset F_{n+1} \subset B$ for all $n \in \mathbb{N}$ and $\wedge_n [v_{\mathcal{A}}(m)(B \setminus F_n)] = 0$. Then, by virtue of (2), $\wedge_n [\|m\|(B \setminus F_n)] = 0$: this clearly implies that $\wedge_n \big|\, |m(B)| - |m(F_n)| \,\big| = 0$, from which $|m(B)| \le \vee_n |m(F_n)| \le v_{\mathcal{F}}(m)(A)$.

So far, we have proved that, for every $A \in \mathcal{A}$:

$$m^+(A) = \vee_{F \subset A, F \in \mathcal{F}}\, m(F) \le \vee_{F \subset A, F \in \mathcal{F}}\, m^+(F) \le m^+(A), \tag{22}$$

and similarly

$$\begin{aligned} m^-(A) &= \vee_{F \subset A, F \in \mathcal{F}}\, (-m(F)) \le \vee_{F \subset A, F \in \mathcal{F}}\, m^-(F) \le m^-(A), \\ v_{\mathcal{A}}(m)(A) &= \vee_{F \subset A, F \in \mathcal{F}}\, |m(F)| \le \vee_{F \subset A, F \in \mathcal{F}}\, v_{\mathcal{A}}(m)(F) \le v_{\mathcal{A}}(m)(A). \end{aligned} \tag{23}$$

So, all inequalities in (22) and (23) are equalities. and, since $m^{\pm}$ are positive measures, then we deduce that

$$\wedge_{F\in\mathcal{F},F\subset A}\,\|m\|(A\setminus F)=0 \tag{24}$$

for all elements $A\in\mathcal{A}$.

Let us consider an arbitrary element $K\in\mathcal{F}$: since all elements F of $\mathcal{F}$ are complements of elements of $\mathcal{G}$, by (24) we get

$$0\le\wedge_{H\in\mathcal{G},K\subset H}\,\|m\|(H\setminus K)\le\wedge_{F\in\mathcal{F},F\subset G\setminus K}\,\|m\|((G\setminus K)\setminus F)=0. \tag{25}$$

Thus, all terms in (25) are equal to zero, and (21) is proved.

We now turn to (20): we just prove the last equality, the first ones are similar. To this aim, fix an arbitrary element $V\in\mathcal{G}$, and set $S:=v_{\mathcal{G}}(m)(V)$, $T:=v_{\mathcal{A}}(m)(V)$. Clearly $S\le T$, so we just prove the converse inequality. Thanks to the previous step, we have

$$T=\vee_{F\in\mathcal{F},F\subset V}|m(F)|,$$

hence all we must show is that $|m(F)|\le S$ for any element $F\subset V$, with $F\in\mathcal{F}$. So, let F be such a set; then, for every element $H\in\mathcal{G}$, with $F\subset H$, we have

$$|m(F)|=|m(H\cap V)|+|m(F)|-|m(H\cap V)|\le S+\big|\,|m(F)|-|m(H\cap V)|\,\big|,$$

i. e.

$$|m(F)|-S\le\big|\,|m(F)|-|m(H\cap V)|\,\big|.$$

Since H is arbitrary, taking into account of (25), we have

$$|m(F)|-S\le\wedge_{H\in\mathcal{G},F\subset H}(\big|\,|m(F)|-|m(H\cap V)|\,\big|)\le\wedge_{H\in\mathcal{G},F\subset H}\|m\|(H\setminus F)=0,$$

and we finally obtain $|m(F)|\le S$, as requested. Since F was arbitrary, this concludes the proof. □

The following proposition (see also [5, Proposition 2.6]) shows that, if $(m_j:\mathcal{A}\to R)_j$ is a sequence of equibounded regular means, even if they are not uniformly regular, the sequences $(F_n)_n$, $(G_n)_n$, $(F'_n)_n$, $(G'_n)_n$ above can be taken independently of j, satisfying the given definition of regularity.

Proposition 4.3. Let R, $\mathcal{A}$, $\mathcal{F}$, $\mathcal{G}$ be as in 2.3, $\mathcal{A}$ be a σ-algebra and $(m_j:\mathcal{A}\to R)_j$ be a sequence of regular means. Then for every $A\in\mathcal{A}$ and $W\in\mathcal{F}$ there exist four sequences $(F_n)_n$, $(F'_n)_n$ in $\mathcal{F}$, $(G_n)_n$, $(G'_n)_n$ in $\mathcal{G}$, satisfying (17) and (18), and such that

$$\wedge_n[v_{\mathcal{A}}(m_j)(G_n\setminus F_n)]=\wedge_n[v_{\mathcal{A}}(m_j)(G'_n\setminus W)]=0$$

for all $j\in\mathbb{N}$.

P r o o f. By hypothesis, for every $A\in\mathcal{A}$, $W\in\mathcal{F}$ and every $j\in\mathbb{N}$ there correspond four sequences $(G_n^{(j)})_n$, $(F_n^{(j)})_n$, $({G'}_n^{(j)})_n$, $({F'}_n^{(j)})_n$ such that: $F_n^{(j)}$, ${F'}_n^{(j)}\in\mathcal{F}$, $G_n^{(j)}$, ${G'}_n^{(j)}\in\mathcal{G}$ for all $j,n\in\mathbb{N}$;

$$F_n^{(j)}\subset F_{n+1}^{(j)}\subset A\subset G_{n+1}^{(j)}\subset G_n^{(j)}\quad j,n\in\mathbb{N}, \tag{26}$$

$$W \subset {F'}^{(j)}_{n+1} \subset {G'}^{(j)}_n \subset {F'}^{(j)}_n \quad j, n \in \mathbb{N}; \tag{27}$$

and with the property that

$$\wedge_n \, [v_{\mathcal{A}}(m_j)(G^{(j)}_n \setminus F^{(j)}_n)] = \wedge_n \, [v_{\mathcal{A}}(m_j)({G'}^{(j)}_n \setminus W)] = 0 \tag{28}$$

for all $j \in \mathbb{N}$.

For every $n \in \mathbb{N}$, set $G_n := \cap_{j \le n} G^{(j)}_n$, $F_n := \cup_{j \le n} F^{(j)}_n$, $F'_n := \cap_{j \le n} {F'}^{(j)}_n$, $G'_n := \cap_{j \le n} {G'}^{(j)}_n$: then $G_n, G'_n \in \mathcal{G}$, $F_n, F'_n \in \mathcal{F}$, and $F_n \subset F_{n+1} \subset A \subset G_{n+1} \subset G_n$ for all $n \in \mathbb{N}$. Moreover it is easy to see that the sequences $(G'_n)_n$, $(F'_n)_n$ satisfy (18).

Since $G_n \setminus F_n \subset G^{(j)}_n \setminus F^{(j)}_n$, $G_n \setminus W \subset G^{(j)}_n \setminus W$ for each $j, n \in \mathbb{N}$, then for all j we get:

$$\begin{aligned} 0 \;&\le\; \wedge_n \, [v_{\mathcal{A}}(m_j)(G_n \setminus F_n)] \le \wedge_n \, [v_{\mathcal{A}}(m_j)(G^{(j)}_n \setminus F^{(j)}_n)] = 0; \\ 0 \;&\le\; \wedge_n \, [v_{\mathcal{A}}(m_j)({G'}_n \setminus W)] \le \wedge_n \, [v_{\mathcal{A}}(m_j)({G'}^{(j)}_n \setminus W)] = 0. \end{aligned} \tag{29}$$

So all the terms in (29) are equal to 0. This concludes the proof. □

Before proving our versions of the Dieudonné theorem, we state the following

Theorem 4.4. Let G be any infinite set; $\mathcal{A} \subset \mathcal{P}(G)$ be any σ-algebra; $\mathcal{G}$, $\mathcal{F}$ be as in 2.3, where $\mathcal{G}$ and $\mathcal{F}$ are sublattices of $\mathcal{A}$ and $\mathcal{G}$ is closed with respect to countable disjoint unions. Assume that: $(m_j : \mathcal{A} \to R)_j$ is an equibounded sequence of regular set functions, (RO)-convergent to m_0 on $\mathcal{G}$; A, W, $(F_n)_n$, $(G_n)_n$, $(F'_n)_n$, $(G'_n)_n$ (independent of j) satisfy (17) and (18). Moreover, suppose that

$$\wedge_n \, [v_{\mathcal{A}}(m_j)(G_n \setminus F_n)] = \wedge_n \, [v_{\mathcal{A}}(m_j)(G'_n \setminus W)] = 0$$

for all $j \in \mathbb{N}$.

Then $\wedge_n [\vee_j \, v_{\mathcal{A}}(m_j)(G_n \setminus F_n)] = \wedge_n [\vee_j \, v_{\mathcal{A}}(m_j)(G'_n \setminus W)] = 0$.

P r o o f. First of all we observe that, by virtue of Lemma 4.2, $v_{\mathcal{A}}$ and $v_{\mathcal{G}}$ are equivalent, because, in the involved semivariations, we deal with elements of $\mathcal{G}$.

By Theorem 3.1 there exists a meager set $N \subset \Omega$ such that the real-valued measures $m_j(\cdot)(\omega)$ are uniformly (s)-bounded on $\mathcal{G}$ for all $\omega \notin N$.

Fix now arbitrarily $A \in \mathcal{A}$, $W \in \mathcal{F}$, and let $(F_n)_n$, $(G_n)_n$, $(F'_n)_n$, $(G'_n)_n$ be as in the hypotheses. By arguing analogously as in (5-8), we get the existence of a meager set $N^* \subset \Omega$ (depending on A and W), with

$$\begin{aligned} &\lim_n [v_{\mathcal{G}}(m_j(\cdot)(\omega))(G_n \setminus F_n)] = \inf_n [v_{\mathcal{G}}(m_j(\cdot)(\omega))(G_n \setminus F_n)] \\ = \; &\lim_n [v_{\mathcal{G}}(m_j(\cdot)(\omega))(G'_n \setminus W)] = \inf_n [v_{\mathcal{G}}(m_j(\cdot)(\omega))(G'_n \setminus W)] = 0 \end{aligned}$$

for all $j \in \mathbb{N}$ and $\omega \notin N^*$. By Lemma 3.2 and Corollary 3.3, we get

$$\inf_n \{\sup_j [v_{\mathcal{G}}(m_j(\cdot)(\omega))(G_n \setminus F_n)]\} = \inf_n \{\sup_j [v_{\mathcal{G}}(m_j(\cdot)(\omega))(G'_n \setminus W)]\} = 0 \tag{30}$$

for all $\omega \notin N \cup N^*$.

The assertion follows from (30), proceeding again analogously as in (5-8). □

5. THE DIEUDONNÉ THEOREM

In this section we prove that, if a sequence $(m_j)_j$ of equibounded regular finitely additive measures (RO)-converges in $\mathcal{G}$, then they are uniformly regular and have pointwise limit on the whole of $\mathcal{A}$.

Theorem 5.1. With the same notations as in the previous sections, fix $A \in \mathcal{A}$, and let $(G_n)_n$, $(F_n)_n$ satisfy the hypotheses of Theorem 4.4. Moreover, suppose that $(m_j)_j$ is (RO)-convergent to m_0 on $\mathcal{G}$.

Then the following assertions hold.

(j) The measures m_j, $j \in \mathbb{N}$, are uniformly regular.

(jj) The sequence $(m_j(A))_j$ is (o)-Cauchy in R for each $A \in \mathcal{A}$.

(jjj) Letting A run in $\mathcal{A}$, if we define

$$m_0(A) := (o)\lim_j m_j(A), \tag{31}$$

then m_0 is regular on $\mathcal{A}$.

Proof. (j) Uniform regularity of the m_j's follows easily from Theorem 4.4.

(jj) Fix arbitrarily $A \in \mathcal{A}$. By uniform regularity of m_j, $j \in \mathbb{N}$, there is a sequence $(G_n)_n$ in $\mathcal{G}$ with the property that $A \subset G_{n+1} \subset G_n$ for all $n \in \mathbb{N}$ and

$$\wedge_n[\vee_j (v_{\mathcal{A}}(m_j)(G_n \setminus A))] = (o)\lim_n[\vee_j (v_{\mathcal{A}}(m_j)(G_n \setminus A))] = 0.$$

Let $(v_n)_n$ be an (o)-sequence with $|m_j(G_n) - m_j(A)| \leq v_n$ for all $j, n \in \mathbb{N}$, and let $(p_l)_l$ be an (o)-sequence, related with (RO)-convergence of $(m_j)_j$ to m_0 on $\mathcal{G}$.

For all l, $n \in \mathbb{N}$ there exists $j^* \in \mathbb{N}$ with $|m_p(G_n) - m_q(G_n)| \leq 2\,p_l$ whenever $p, q \geq j^*$. In particular, to each $n \in \mathbb{N}$ we can associate a positive integer $j_n > n$ such that

$$\begin{aligned} |m_p(A) - m_q(A)| &\leq |m_p(A) - m_p(G_n)| + |m_p(G_n) - m_q(G_n)| + |m_q(G_n) - m_q(A)| \\ &\leq 2\,p_n + 2\,v_n \end{aligned}$$

for all $p, q \geq j_n$. Set $j_0 := 0$, $p_0 := p_1$, $v_0 := v_1$. Without loss of generality, we can suppose $j_{n-1} < j_n$ for all $n \in \mathbb{N}$. To every j there corresponds an integer $n = n(j) \in \mathbb{N} \cup \{0\}$ with $j_n \leq j < j_{n+1}$. Put $w_j := 2\,p_{n(j)} + 2\,v_{n(j)}$, $j \in \mathbb{N}$. It is easy to check that $(w_j)_j$ is an (o)-sequence and that

$$|m_j(A) - m_{j+r}(A)| \leq w_j$$

for all $j, r \in \mathbb{N}$. Therefore we obtain that the sequence $(m_j(A))_j$ is (o)-Cauchy.

(jjj) For each fixed $A \in \mathcal{A}$, define $m_0(A) := (o)\lim_j m_j(A)$. This limit exists in R, since by (jj) the sequence $(m_j(A))_j$ is (o)-Cauchy (see also [15]). Regularity of m_0 is an easy consequence of definition of m_0 and uniform regularity of the measures m_j, $j \in \mathbb{N}$. □

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