KYBERNETIKA — VOLUME *46* (2010), NUMBER 6, PAGES 1009–1024

FUZZIFICATION OF CRISP DOMAINS

Roman Frič and Martin Papčo

The present paper is devoted to the transition from crisp domains of probability to fuzzy domains of probability. First, we start with a simple transportation problem and present its solution. The solution has a probabilistic interpretation and it illustrates the transition from classical random variables to fuzzy random variables in the sense of Gudder and Bugajski. Second, we analyse the process of fuzzification of classical crisp domains of probability within the category ID of D-posets of fuzzy sets and put into perspective our earlier results concerning categorical aspects of fuzzification. For example, we show that (within ID) all nontrivial probability measures have genuine fuzzy quality and we extend the corresponding fuzzification functor to an epireflector. Third, we extend the results to simplex-valued probability domains. In particular, we describe the transition from crisp simplex-valued domains to fuzzy simplex-valued domains via a "simplex" modification of the fuzzification functor. Both, the fuzzy probability and the simplex-valued fuzzy probability is in a sense minimal extension of the corresponding crisp probability theory which covers some quantum phenomenon.

Keywords: domain of probability, fuzzy random variable, crisp random event, fuzzy observable, fuzzification, category of ID-poset, epireflection, simplex-valued domains

Classification: 60A86, 60A05

1. INTRODUCTION

Since the pioneering paper by L. A. Zadeh ([20]), who proposed to extend the domain of probability from classical random events to fuzzy random events, the fuzzy probability, underwent a considerable evolution. For example, fuzzy random variables and fuzzy observables (dual notion), as a generalization of classical random variables and classical observables, have been introduced in order to capture some quantum phenomena. Categorical methods are suitable when comparing different models of probability theory and help to understand the transition from classical probability theory to fuzzy probability theory.

The first part of the present paper is devoted to discrete probability spaces and a simple transportation problem. It illustrates some fundamental constructions of the fuzzy probability theory. The second part is devoted to the category ID of D-posets of fuzzy sets and the transition from classical to fuzzy probability. A crucial role is played by the so-called fuzzification functor. In the final part we study the fuzzification process of simplex-valued generalized probability.

Basic information on fuzzy probability theory and fundamental applications to quantum physics can be found in [1, 2, 13]. Information about quantum structures and generalized probability can be found in [4, 5, 14, 15, 19, 20] and information concerning a categorical approach to probability theory can be found in [3, 6, 7, 10, 11, 12, 16, 17, 18]. For the reader's convenience we recall here some basic notions.

Let $(\Omega, \mathbf{A}, p)$ be a probability space in the classical Kolmogorov sense (i.e. Ω is a set, $\mathbf{A}$ is a σ-field of subsets of Ω, we assume that singletons are measurable, and p is a probability measure on $\mathbf{A}$). A measurable map f of Ω into the real line R, called *random variable*, sends p into a probability measure p_f, called the *distribution* of f, defined on the real Borel sets $\mathbf{B}_R$ via $p_f(B) = p(f^{\leftarrow}(B))$, $B \in \mathbf{B}_R$. In fact, f induces a map sending probability measures $\mathcal{P}(\mathbf{A})$ on $\mathbf{A}$ into probability measures $\mathcal{P}(\mathbf{B}_R)$ on $\mathbf{B}_R$. The preimage map $f^{\leftarrow}$, called *observable*, maps $\mathbf{B}_R$ into $\mathbf{A}$. Points of Ω are called *elementary events*, sets in $\mathbf{A}$ are called *sample random events* and sets in $\mathbf{B}_R$ are called *real random events*. Each random variable f can be viewed as a channel through which the probability p of the original probability space is transported to the distribution p_f, a probability measure on the real Borel sets and hence, in fact, a channel through which the probability measures on the sample random events are transported to the probability measures on the real random events; observe that each *degenerated point probability measure* $\delta_\omega \in \mathcal{P}(\mathbf{A})$, $\omega \in \Omega$ (defined for $A \in \mathbf{A}$ by $\delta_\omega(A) = 1$ if $\omega \in A$ and $\delta_\omega(A) = 0$ otherwise), is transported to a degenerated point probability measure $\delta_{f(\omega)} \in \mathcal{P}(\mathbf{B}_R)$.

To compare the classical and the fuzzy probability theory we consider a more general situation. Let $(X, \mathbf{A})$, $(Y, \mathbf{B})$ be classical measurable spaces and let $f\colon X \to Y$ be a map. If f is measurable, then the (dual) preimage map $f^d\colon \mathbf{B} \to \mathbf{A}$ defined by $f^d(B) = f^{\leftarrow}(B) = \{x \in X;\, f(x) \in B\}$, $B \in \mathbf{B}$, is a sequentially continuous (with respect to the pointwise convergence of characteristic functions) Boolean homomorphism of $\mathbf{B}$ into $\mathbf{A}$. Indeed, the assertion is a corollary of the following straightforward observation. For each $B \subseteq Y$ we have $\chi_{f^{\leftarrow}(B)} = \chi_B \circ f$ and the measurability of f is equivalent to the following condition

$$(\forall B \in \mathbf{B})\,(\exists A \in \mathbf{A})\,[\chi_B \circ f = \chi_A]. \tag{M}$$

Now, if p is a probability measure on $\mathbf{A}$ and f is measurable, then the composition $p \circ f^d = p_f$ is a probability measure on $\mathbf{B}$. This sends probability measures $\mathcal{P}(\mathbf{A})$ on $\mathbf{A}$ to probability measures $\mathcal{P}(\mathbf{B})$ on $\mathbf{B}$; denote T_f the resulting *distribution map*.

In the fuzzy probability theory, we start with a map T of $\mathcal{P}(\mathbf{A})$ into $\mathcal{P}(\mathbf{B})$ satisfying a natural measurability condition which guarantees the existence of a dual map T^d of all measurable functions $\mathcal{M}(\mathbf{B})$ of Y into the closed unit interval $I = [0, 1]$ into all measurable functions $\mathcal{M}(\mathbf{A})$ of X into I so that T^d has some natural properties (it is sequentially continuous and preserves the D-poset structure, i.e., it is an ID-morphism; from a general duality theory, see [10, 16], it follows that for each ID-morphism h of $\mathcal{M}(\mathbf{B})$ into $\mathcal{M}(\mathbf{A})$ there exists a fuzzy random variable T sending $\mathcal{P}(\mathbf{A})$ into $\mathcal{P}(\mathbf{B})$ such that $h = T^d$). This way $\mathcal{M}(\mathbf{A})$ and $\mathcal{M}(\mathbf{B})$ become *fuzzy random events*, T becomes a *fuzzy random variable* and T^d becomes *fuzzy observable*. However, a degenerated point probability measure on $\mathbf{A}$ can be mapped to a nondegenerated probability measure on $\mathbf{B}$ and, consequently, fuzzy random variables and fuzzy observables do have genuine quantum and fuzzy properties. For example, a fuzzy observable, unlikely a classical observable, can map a crisp event (a set in $\mathbf{B}$) to a genuine fuzzy event (a function in $\mathcal{M}(\mathbf{A})$).

2. TRANSPORTATION PROBLEM

Let B be a bottle containing one litre of liquid, let $\Omega = \{\omega_1, \omega_2, \dots, \omega_n\}$ and $\Xi = \{\xi_1, \xi_2, \dots, \xi_m\}$ be two finite sets of empty glasses such that the content of each is one litre. Let q be a map of Ξ into [0,1] such that $\sum_{k=1}^{m} q(\xi_k) = 1$. Distribute the whole content (1 litre) of B into Ω so that each ω_l contains $p(\omega_l)$ of it, that is, $0 \le p(\omega_l) \le 1$ and $\sum_{l=1}^{n} p(\omega_l) = 1$.

2.1. Classical case

Question C. Is it possible to pour the whole content $p(\omega_l)$ of each glass ω_l into some (empty) glass ξ_k in such a way that the glass ξ_k, $k \in \{1, 2, \dots, m\}$, will contain exactly $q(\xi_k)$ of the liquid?

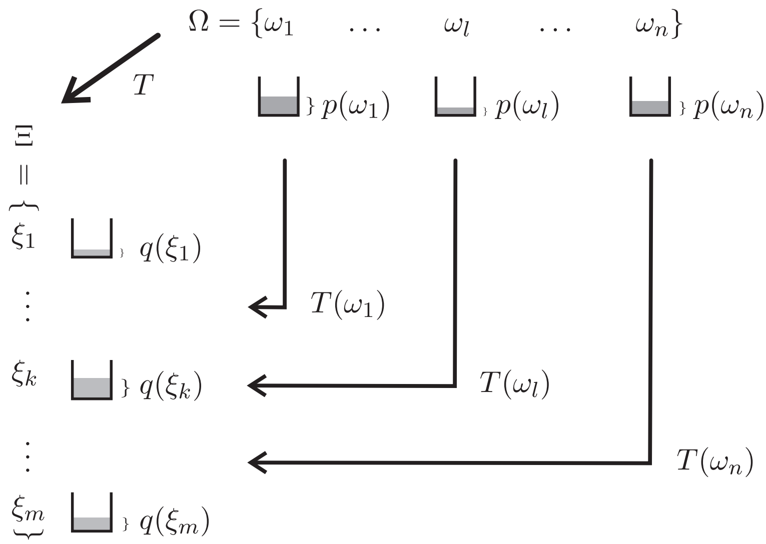

Fig. 1. Classical pipeline.

Answer C. It is easy to see that in general the answer is NO. Indeed, for instance, if $n = 2$, $m = 3$ and $q(\xi_1) = q(\xi_2) = q(\xi_3)$, then there is no way how to get the result.

Observe that our problem has the following purely probabilistic reformulation. Let (Ω, p) and (Ξ, q) be finite probability spaces, let T be a map of Ω into Ξ, and let $T^{\leftarrow}$ be the preimage map ($T^{\leftarrow}(\xi_k) = \{\omega_l; T(\omega_l) = \xi_k\}$). If $q = p \circ T^{\leftarrow}$, i.e., $q(\xi_k) = \sum_{\omega_l \in T^{\leftarrow}(\xi_k)} p(\omega_l)$, $k \in \{1, 2, \dots, m\}$, then T is said to be a *random map* and

(Ξ, q) is said to be a *random transform* of (Ω, p). Each random map T can be visualized as a system of n pipelines $\omega_l \mapsto T(\omega_l)$ through which $p(\omega_l)$ flows to $\xi_k = T(\omega_l)$. If ξ_k is the target of several pipelines, then $q(\xi_k)$ is the sum $\sum_{\omega_l \in T^{\leftarrow}(\xi_k)} p(\omega_l)$, i.e., the total influx through the pipelines in question. (See Figure 1.) Now the question is whether for each pair of finite probability spaces (Ω, p) and (Ξ, q) there exist a random map T transforming (Ω, p) into (Ξ, q).

Note that, for discrete probability spaces, random variables are special transformations, where the underlying set of the target probability space is a set of real numbers.

2.2. Fuzzy case

Question F. Is there a more complex way how to transport the liquid from Ω into Ξ so that we end up with $q : \Xi \to [0, 1]$, $\sum_{k=1}^{m} q(\xi_k) = 1$?

Strategy F. Instead of sending each $p(\omega_l)$ to some ξ_k via a simple "pipeline" $\omega_l \mapsto \xi_k = T(\omega_l)$, we can try to distribute $p(\omega_l)$, simultaneously sending to each ξ_k, $k \in \{1, 2, \dots, m\}$, via a complex "distribution pipeline" some fraction $w_{kl}p(\omega_l)$ of $p(\omega_l)$. Of course, not arbitrarily, but in such a way that the fractions sum up "properly", i.e., $\sum_{l=1}^{n} w_{kl}p(\omega_l) = q(\xi_k)$ and $\sum_{k=1}^{m} \sum_{l=1}^{n} w_{kl}p(\omega_l) = \sum_{l=1}^{n} p(\omega_l) \sum_{k=1}^{m} w_{kl} = \sum_{k=1}^{m} q(\xi_k) = 1$. (See Figure 2.) To comply with the second condition it suffices to guarantee that $\sum_{k=1}^{m} w_{kl} = 1$. In fact, this means that to each ω_l, $l \in \{1, 2, \dots, n\}$, we assign a suitable probability function $q_l = (w_{1l}, w_{2l}, \dots, w_{ml})$ on Ξ.

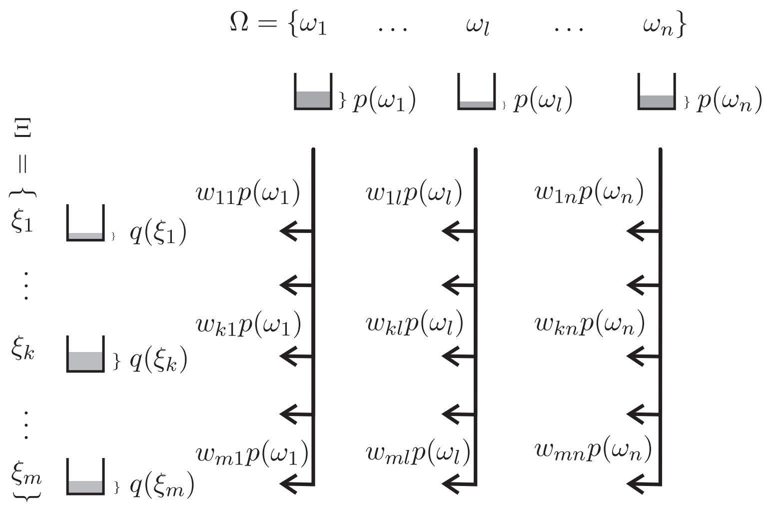

Fig. 2. Distribution pipeline.

Algorithm F. The construction of a "distribution pipeline" is based on a simple probabilistic idea: equip the product set $\Omega \times \Xi$ with a suitable probability r such that $p = \big(p(\omega_1), p(\omega_2), \dots, p(\omega_n)\big)$ and $q = \big(q(\xi_1), q(\xi_2), \dots, q(\xi_m)\big)$ are marginal probabilities (always possible, for example, put $r = p \times q$) and w_{kl} become conditional probabilities.

Let $\big\{r_{kl};\ l \in \{1, 2, \dots, n\}, k \in \{1, 2, \dots, m\}\big\}$ be non-negative numbers such that $\sum_{k=1}^{m} r_{kl} = p(\omega_l)$, $l \in \{1, 2, \dots, n\}$ and $\sum_{l=1}^{n} r_{kl} = q(\xi_k)$, $k \in \{1, 2, \dots, m\}$. For $l \in \{1, 2, \dots, n\}$ and $k \in \{1, 2, \dots, m\}$ define $w_{kl} = 1/m$ if $p(\omega_l) = 0$ (any choice such that $\sum_{k=1}^{m} w_{kl} = 1$ does the same trick) and

$$w_{kl} = \frac{r_{kl}}{p(\omega_l)} = \frac{P_r\big(\{\xi_k\} \cap \{\omega_l\}\big)}{P_r\big(\{\omega_l\}\big)} = P_r\big(\{\xi_k\} | \{\omega_l\}\big)$$

otherwise. Clearly, $\sum_{l=1}^{n} w_{kl} p(\omega_l) = q(\xi_k)$ for all $k \in \{1, 2, \dots, m\}$ and $\sum_{k=1}^{m} w_{kl} = 1$. Clearly, this defines a "distribution pipeline".

Answer F. YES, there is a "distribution pipeline" which transforms p to q.

Every "distribution pipeline" yields a generalized transformation of (Ω, p) to (Ξ, q); p flows trough the pipeline and it is transformed to q. The generalized transformation has a surprising background: fuzzy probability.

2.3. Distribution pipeline

The "distribution pipeline" can be viewed as a matrix $\mathbb{W}$ having m rows, n columns and having some additional properties. First, the elements of $\mathbb{W}$ are numbers from $I = [0, 1]$. Second, each column $\mathbf{q}_k$, $k \in \{1, 2, \dots, m\}$, is a probability function on Ξ. Third, each row $\mathbf{w}_k$, $k \in \{1, 2, \dots, m\}$ is a fuzzy subset of Ω. To transport p to q, it suffices to guarantee that $\sum_{l=1}^{n} w_{kl} p(\omega_l) = q(\xi_k)$, $k \in \{1, 2, \dots, m\}$. If r is a probability on the product set $\Omega \times \Xi$ such that p and q are marginal probabilities, then the case when p and q are independent, i.e., $r(\omega_l, \xi_k) = p(\omega_l) q(\xi_k)$, in symbols $r = p \times q$, gives a "trivial" solution: $q_l = q$, $l \in \{1, 2, \dots, n\}$, meaning that all columns of $\mathbb{W}$ are the same. Now, let $\mathbb{W}$ be any matrix having m rows and n columns and the elements of which are numbers from $I = [0, 1]$ such that each column q_k, $k \in \{1, 2, \dots, m\}$, is a probability function on Ξ. Then $\mathbb{W}$ represents a map of the set $\mathcal{P}(\Omega)$ of all probability functions on Ω into the set $\mathcal{P}(\Xi)$ of all probability functions on Ξ: for each $p \in \mathcal{P}(\Omega)$, put $\big(\mathbb{W}(p)\big)(k) = \sum_{l=1}^{n} w_{kl} p(\omega_l) = s(k)$, $k \in \{1, 2, \dots, m\}$. Since $\sum_{k=1}^{m} s(k) = 1$, $\mathbb{W}(p)$ is a probability on Ξ. In fact, the resulting map is a discrete fuzzy random variable in the sense of S. Gudder and S. Bugajski (see [13], [1], [2], [6]): each elementary event $\omega \in \Omega$ is mapped to some probability measure on Ξ. Dually, $\mathbb{W}$ represents a fuzzy observable sending each (crisp) event $\{\xi_k\}$ in Ξ to the fuzzy event $\mathbf{w}_k$ (the k-th row of $\mathbb{W}$) in Ω, $k \in \{1, 2, \dots, m\}$.

3. FUZZIFICATION – CLASSICAL CASE

In this section we briefly analyse the process of fuzzification of classical crisp domains of probability within the category ID of D-posets of fuzzy sets and put into perspective our earlier results concerning categorical aspects of fuzzification. The category ID is a natural larger category (containing fields of sets) in which probability measures and states (generalizations of probability measures) are morphisms of the same type as observables (maps dual to generalized random variables), namely, both are exactly "the structure preserving maps". Consequently, the additivity of probability measures becomes "the preservation of a less restrictive structure on events (the ID-structure) than the Boolean one" (see [3, 12]).

3.1. D-posets

D-posets have been introduced by F. Kôpka and F. Chovanec in [14] (see also [4]) in order to model events in quantum probability. They generalize MV-algebras and other probability domains and provide a category in which observables and states become morphisms. Recall that a D-poset is a partially ordered set with the greatest element 1, the least element 0, and a partial binary operation called *difference*, such that $a \ominus b$ is defined iff $b \leq a$, and the following axioms are assumed:

(D1) $a \ominus 0_X = a$ for each $a \in X$;

(D2) If $c \leq b \leq a$, then $a \ominus b \leq a \ominus c$ and $(a \ominus c) \ominus (a \ominus b) = b \ominus c$.

Fundamental to applications ([6]) are *D-posets of fuzzy sets*, i. e. systems $\mathcal{X} \subseteq I^X$, $I = [0,1]$, carrying the coordinatewise partial order, coordinatewise convergence of sequences, containing the top and bottom elements of I^X, and closed with respect to the partial operation difference defined coordinatewise; we always assume that $\mathcal{X}$ is *reduced*, i. e., if $x \neq y$ then $u(x) \neq u(y)$ for some $u \in \mathcal{X}$. Denote ID the category having D-posets of fuzzy sets as objects and having sequentially continuous D-homomorphisms as morphisms. Objects of ID are subobjects of the powers I^X.

3.2. Domains in ID

As in [11], our approach to domains of probability can be summarized as follows.

- Start with a "system $\mathcal{A}$ of events";
- Choose a "cogenerator C" – usually a structured set suitable for "measuring" (e.g., the two-element Boolean algebra {0,1}, the interval $I = [0,1]$ carrying the Łukasiewicz MV-structure, D-poset structure, ...);
- Choose a set X of "properties" measured via C so that X separates $\mathcal{A}$;
- Represent each event $a \in \mathcal{A}$ via the "evaluation" of $\mathcal{A}$ into C^X sending $a \in \mathcal{A}$ to $a_X \in C^X$, $a_X \equiv \{x(a);\ x \in X\}$;
- Form the minimal "subalgebra" D of C^X containing $\{a_X;\ a \in \mathcal{A}\}$;
- The subalgebra forms a *probability domain* $D \subseteq C^X$ which has nice categorical properties. For $C = \{0,1\}$ and $C = [0,1]$ (considered as ID-posets), respectively, the classical probability domains (σ-fields of sets) and fuzzy probability domains (measurable functions into [0,1]) become special cases.

- An *observable* is a "structure preserving map" of one probability domain into another one. The image of the former is a subdomain of the latter.

- A *state* (generalized probability measure) is a "structure preserving map" of the probability domain D into C.

3.3. From crisp to fuzzy

L. A. Zadeh in [20] proposed to extend the domain of probability from σ-fields of sets to suitable systems of fuzzy sets. Namely, to fuzzy subsets A of the Euclidean n-space R^n such that the membership function $\mu_A\colon\ R^n \to [0,1]$ is Borel measurable. If P is a probability measure over Borel sets, then the probability of A is defined as the Lebesgue–Stieltjes integral $\int_{R^n} \mu_A(x)\,\mathrm{d}P$. The fuzzification of probability theory underwent a considerable evolution. The reader is referred to a survey by R. Mesiar [15], to seminal papers by S. Gudder [13], S. Bugajski [1, 2], B. Riečan and D. Mundici [19].

Let $\mathbf{A}$ be a σ-algebra of (crisp) subsets of a set X; we consider $\mathbf{A}$ as the ID-poset of characteristic functions. Let $\mathcal{M}(\mathbf{A})$ be the set of all measurable functions into the interval $I = [0,1]$. It is known that both ID-posets $\mathbf{A}$ and $\mathcal{M}(\mathbf{A})$ are sequentially closed in I^X, each probability measure m on $\mathbf{A}$ can be uniquely extended to a state m_t on $\mathcal{M}(\mathbf{A})$ defined by $m_t(u) = \int u\,\mathrm{d}m,\ u \in \mathcal{M}(\mathbf{A})$, and both m and m_t are ID-morphisms into I (cf. [8]). Denote $CFSD$ the (full) subcategory of ID the objects of which are σ-fields of sets and denote $CGBID$ the (full) subcategory of ID the objects of which are of the form $\mathcal{M}(\mathbf{A})$. The objects of $CFSD$ are the domains of classical probability theory and the objects of $CGBID$ are the domains of fuzzy probability theory. This leads to the following question.

Question T. What is the transition from classical probability to fuzzy probability (fuzzification) from the viewpoint of category theory?

The question has been answered in [10], the crucial being the construction and understanding of "fuzzification functor" $\mathbf{F}\colon\ CFSD \to CGBID$ in [8]. The functor sends a classical probability domain, a σ-field $\mathbf{A}$, into its fuzzification $\mathcal{M}(\mathbf{A}) = \mathbf{F}(\mathbf{A})$ and sends a classical observable h, a Boolean homomorphism of one classical domain into another classical domain, into its fuzzification $\mathbf{F}(h)$, a D-homomorphism from one fuzzy domain into another fuzzy domain. In this sense, the identity map of $\mathbf{A}$ is sent to the identity map of $\mathcal{M}(\mathbf{A})$, hence crisp (classical) events are embedded in fuzzy events and $\mathbf{F}(h)$ is an extension of h.

Next, we try to put the ideas and results from [10, 11, 12] and [3] into a perspective. In particular, we point out the role of cogenerators.

To understand the transition from the classical probability theory to the fuzzy probability theory it is natural to understand the transition from $\{0,1\}$ (the cogenerator of classical domains of probability) to $I = [0,1]$ (the cogenerator of fuzzy domains of probability).

First, we identify $\{0,1\}$ and the trivial σ-field $\mathbf{T} = \{\emptyset, \{\omega\}\}$ of all subsets of a singleton–a classical probability domain containing only one elementary event ω

and, similarly, we identify $I = [0,1]$ and the fuzzy domain $I^{\{\omega\}}$ of all (measurable) fuzzy events in this trivial σ-field $\mathbf{T}$; observe that $I^{\{\omega\}} = \mathcal{M}(\mathbf{T})$.

Second, observe that $[0,1]$ is the minimal of all D-posets of fuzzy sets $\mathcal{X} \subseteq I^{\{\omega\}}$ containing $\mathbf{T}$ such that

(i) $\mathcal{X}$ is divisible (recall that a D-poset of fuzzy subsets $\mathcal{Y} \subseteq I^Y$ of Y is said to be divisible if for each $u \in \mathcal{Y} \subseteq I^Y$ and for each positive natural number n there exits $v \in \mathcal{Y} \subseteq I^Y$ such that for all $y \in Y$ we have $nv(y) = u(y)$;

(ii) $\mathcal{X}$ is sequentially closed in $I^{\{\omega\}}$.

While the second condition is a natural assumption in any "continuous" probability theory: domains are closed with respect to limits of sequences of events, the first condition is a necessary assumption guaranteeing positive "fuzzy solution" of the "Bottle problem".

Now, let h be an observable from a classical probability domain $\mathbf{A} \subseteq \{0,1\}^X$ into $\mathbf{T}$. Applying the fuzzification functor $\mathbf{F}$ we get a fuzzy observable $\mathbf{F}(h) : \mathcal{M}(\mathbf{A}) \to \mathcal{M}(\mathbf{T}) = I$. Observe that if $A = \chi_A$ is a crisp event, then $\big(\mathbf{F}(h)\big)(\chi_A) \in \{0,1\}$. Only a genuine fuzzy observable $g : \mathcal{M}(\mathbf{A}) \to \mathcal{M}(\mathbf{T}) = I$ can send χ_A to $g(\chi_A) \in (0,1) \subset I$. This of course means that each nontrivial probability measure p on $\mathbf{A}$ is the restriction of a genuine fuzzy observable g_p of $\mathcal{M}(\mathbf{A})$ to $\mathcal{M}(\mathbf{T})$. Surprising? Yes, *each genuine probability measure p is an intrinsic notion of the fuzzy probability theory within the category ID.*

There is another (not surprising) fuzzy feature of probability measures: each probability measure p on $\mathbf{A}$ is (as a map of $\mathbf{A}$ into I) a fuzzy subset of $\mathbf{A}$ and a sequentially continuous D-homomorphism, i. e., a morphism of ID.

Answer T. The transition from classical to fuzzy probability theory can be described via the fuzzification functor $\mathbf{F}$ sending $\mathbf{A}$ to $\mathcal{M}(\mathbf{A})$. The fuzzification is necessary to implement genuine fuzzy observables (sending some crisp event to a fuzzy event) and genuine fuzzy random variables (sending some degenerated point-probability measure to a non degenerated probability measure). Due to the one-to-one correspondence between σ-fields and measurable functions ranging in $I = [0,1]$, the former theory can be considered as a special case of the latter. Indeed, each $\mathbf{A}$ is embedded into $\mathbf{F}(\mathbf{A}) = \mathcal{M}(\mathbf{A})$ and for each classical observable g its image $\mathbf{F}(g)$ is its extension sending crisp events to crisp events. Within ID, the transition from classical probability domains to fuzzy domains is "the best possible": $\mathbf{F}$ "embeds" $\mathbf{A}$ into $\mathcal{M}(\mathbf{A})$, $\mathbf{A}$ and $\mathcal{M}(\mathbf{A})$ have "the same" probabilities and, finally, each probability measure is an intrinsic notion of the fuzzy probability theory.

4. EPIREFLECTION

Since the fuzzification functor $\mathbf{F}$ sends crisp domains to fuzzy domains and $CGBID$ is not a subcategory of $CFSD$ (the two categories have no object in common), to embed $\mathbf{A}$ into $\mathcal{M}(\mathbf{A})$ as an epireflector we need a larger category EID containing both $CFSD$ and $CGBID$ and a functor $\mathbf{E}$ such that $\mathbf{F}$ is the restriction of $\mathbf{E}$, i. e. $\mathbf{E}(\mathbf{A}) = \mathcal{M}(\mathbf{A})$ for all objects $\mathbf{A}$ of $CFSD$.

4.1. The category EID

Let $\mathbf{A}$ be a σ-field of subsets of X. Denote N^+ the set of all positive natural numbers. For $n \in N^+$, consider the set $\{0, 1/n, 2/n, \ldots, (n-1)/n\}$. Let C_n be the corresponding canonical D-poset and let $\mathcal{M}_n(\mathbf{A})$ be the D-poset of all measurable functions ranging in C_n; clearly, $\mathcal{M}_1(\mathbf{A}) = \mathbf{A}$. If $u \in \mathcal{M}_n(\mathbf{A})$, then u is a simple function of the form $\sum_{i=1}^{k} a_i \chi_{A_i}$, where $1 \le k \le n$, $a_i \in C_n$, and the sets A_i form a measurable partition of X, i. e., sets $A_i \in \mathbf{A}$ are mutually disjoint and $\bigcup_{i=1}^{k} A_i = X$. Denote $s(\mathbf{A})$ the set of all simple measurable functions, i. e., functions of the type $\sum_{i=1}^{k} a_i \chi_{A_i}$, where $k \in N^+$, $a_i \in I$, and the sets A_i form a measurable partition of X.

Denote EID the full subcategory of ID consisting of all objects of the form $\mathcal{M}_n(\mathbf{A})$ and $\mathcal{M}(\mathbf{A})$. We shall show that the assignment $\mathcal{M}_n(\mathbf{A}) \mapsto \mathcal{M}(\mathbf{A})$ yields the desired epireflector $\mathbf{E}$.

Lemma 4.1.1. Let $\mathbf{A}$ and $\mathbf{B}$ be σ-fields of subsets of X and Y, respectively. Let h, g be sequentially continuous D-homomorphisms of $\mathcal{M}(\mathbf{A})$ into $\mathcal{M}(\mathbf{B})$ such that $h(\chi_A) = g(\chi_A)$ for all $A \in \mathbf{A}$. Then

(i) $h(\chi_A/n) = g(\chi_A/n)$ for all $A \in \mathbf{A}, n \in N^+$;

(ii) $h(u) = g(u)$ for all $u = \sum_{i=1}^{k} a_i \chi_{A_i} \in \mathcal{M}_n(\mathbf{A}), n \in N^+$;

(iii) $h(u) = g(u)$ for all $u = \sum_{i=1}^{k} a_i \chi_{A_i} \in s(\mathbf{A})$;

(iv) $h(u) = g(u)$ for all $u \in \mathcal{M}(\mathbf{A})$.

P r o o f. (i) From the definition of a D-homomorphism it follows that $h(\chi_A/n) = h(\chi_A)/n = g(\chi_A/n) = g(\chi_A)/n$.

(ii) Let $u = \sum_{i=1}^{k} a_i \chi_{A_i} \in \mathcal{M}_n(\mathbf{A})$ for some $n \in N^+$. Clearly, for $a = k/n$, $1 < k < n$, $A \in \mathbf{A}$, we have $h(a\chi_A) = ah(\chi_A)$ and if $A, B \in \mathbf{A}$ are disjoint, then $h(\chi_A + \chi_B) = h(\chi_A) + h(\chi_B)$. Hence $h(u) = \sum_{i=1}^{k} a_i h(\chi_{A_i}) = \sum_{i=1}^{k} a_i g(\chi_{A_i}) = g(u)$.

(iii) Let $u = \sum_{i=1}^{k} a_i \chi_{A_i} \in s(\mathbf{A})$. Then there are functions $u_l = \sum_{i=1}^{k} a_{il} \chi_{A_i} \in \mathcal{M}_l(\mathbf{A})$, $l \in N^+$, such that $a_i = \lim_{l\to\infty} a_{il}$. Since $h(u_l) = \sum_{i=1}^{k} a_{il} h(\chi_{A_i}) = \sum_{i=1}^{k} a_{il} g(\chi_{A_i}) = g(u_l)$, $u = \lim_{l\to\infty} u_l$, and h, g are sequentially continuous, it follows that $h(u) = g(u)$.

(iv) Let $u \in \mathcal{M}(\mathbf{A})$. Then there is an increasing sequence of simple functions $u_l \in \mathcal{M}_l(\mathbf{A})$ such that $u = \lim_{l\to\infty} u_l$. Since $h(u_l) = g(u_l)$ and h, g are sequentially continuous, it follows that $h(u) = g(u)$. □

Corollary 4.1.2. Let $\mathbf{A}$ and $\mathbf{B}$ be σ-fields of subsets of X and Y, respectively. Let $\mathcal{O}(\mathbf{A})$ and $\mathcal{O}(\mathbf{B})$ be objects of EID and let h, g be sequentially continuous D-homomorphisms of $\mathcal{O}(\mathbf{A})$ into $\mathcal{O}(\mathbf{B})$. If $h(A) = g(A)$ for all $A \in \mathbf{A}$, then $h = g$.

P r o o f. 1. Let $\mathcal{O}(\mathbf{A}) = \mathcal{M}_n(\mathbf{A})$ for some $n \in N^+$. Then the assertion can be proved virtually in the same way as (i) and (ii) in the previous lemma.

2. Let $\mathcal{O}(\mathbf{A}) = \mathcal{M}(\mathbf{A})$. Then the assertion can be proved virtually in the same way as the previous lemma. □

Corollary 4.1.3. Let $\mathbf{A}$ be a σ-fields of subsets of X and let $\mathcal{O}(\mathbf{A})$ be an object of EID. Let let h be a sequentially continuous D-homomorphism of $\mathcal{O}(\mathbf{A})$ into I.

(i) Then there exists a unique probability measure m on $\mathbf{A}$ such that for each $u \in \mathcal{O}(\mathbf{A})$ we have $h(u) = \int u \,\mathrm{d}m$.

(ii) For $u \in \mathcal{M}(\mathbf{A})$ put $\overline{h}(u) = \int u \,\mathrm{d}m$. Then $\overline{h}$ is the unique sequentially continuous D-homomorphism of $\mathcal{M}(\mathbf{A})$ into I such that $\overline{h}(u) = h(u)$ for all $u \in \mathcal{O}(\mathbf{A})$.

Proof. Denote $h_{\mathbf{A}}$ the restriction of h to $\mathbf{A}$. It is known (Proposition 3.1. in [10]) that there exists a unique probability measure m on $\mathbf{A}$ such that $m(A) = h_{\mathbf{A}}(\chi_A)$ for all $A \in \mathbf{A}$. The Lebesgue integral $\int u \,\mathrm{d}m, u \in \mathcal{M}(\mathbf{A})$, is a sequentially continuous D-homomorphism of $\mathcal{M}(\mathbf{A})$ into I. Denote $\overline{h}(u) = \int u \,\mathrm{d}m, u \in \mathcal{M}(\mathbf{A})$. Then the restriction $\overline{h} \restriction \mathcal{O}(\mathbf{A})$ of $\overline{h}$ to $\mathcal{O}(\mathbf{A})$ is a sequentially continuous D-homomorphism of $\mathcal{O}(\mathbf{A})$ into $I = \mathcal{M}(\mathbf{T})$ and, according to the previous corollary, $\overline{h} \restriction \mathcal{O}(\mathbf{A}) = h$. Consequently, both (i) and (ii) are satisfied. □

Theorem 4.1.4. Let $\mathbf{A}$ and $\mathbf{B}$ be σ-fields of subsets of X and Y, respectively. Let $\mathcal{O}(\mathbf{A})$ and $\mathcal{O}(\mathbf{B})$ be objects of EID and let h be a sequentially continuous D-homomorphism of $\mathcal{O}(\mathbf{A})$ into $\mathcal{O}(\mathbf{B})$. Then there exists a unique sequentially continuous D-homomorphism $\overline{h}$ of $\mathcal{M}(\mathbf{A})$ into $\mathcal{M}(\mathbf{B})$ such that $\overline{h}(u) = h(u)$ for all $u \in \mathcal{O}(\mathbf{A})$

Proof. The case $\mathcal{O}(\mathbf{A}) = \mathcal{M}(\mathbf{A})$ is trivial. So, assume that $\mathcal{O}(\mathbf{A}) = \mathcal{M}_n(\mathbf{A})$ for some $n \in N^+$. To avoid technicalities, we consider $\mathbf{A}$, $\mathcal{M}_n(\mathbf{A})$, $s(\mathbf{A})$, and $\mathcal{M}_n(\mathbf{A})$ as canonical subobjects of I^X and, similarly, we consider $\mathbf{B}$, $\mathcal{M}_n(\mathbf{B})$, $s(\mathbf{B})$, and $\mathcal{M}(\mathbf{B})$ as canonical subobjects of I^Y. Further, we identify each point $x \in X$ and the degenerated point probability δ_x and, similarly, we identify each point $y \in Y$ and the degenerated point probability δ_y.

Denote $h_{\mathbf{A}}$ the restriction of h to $\mathbf{A}$. It is known that to $h_{\mathbf{A}}$ there corresponds a unique map T of $\mathcal{P}(\mathbf{A})$ into $\mathcal{P}(\mathbf{B})$ such that for each $A \in \mathbf{A}$ and each $y \in Y$ we have $\big(h_{\mathbf{A}}(\chi_A)\big)(y) = \big(T(\delta_y)\big)(A)$ (see Lemma 3.1 in [6]). Define a map h_T of $\mathcal{M}(\mathbf{A})$ into I^Y as follows: $\big(h_T(u)\big)(y) = \int u \,\mathrm{d}T(\delta_y)$. Then h_T is a sequentially continuous D-homomorphism (remember the Lebesgue Dominate Convergence Theorem). Since $h_T(\chi_A) = h(\chi_A)$ for each $A \in \mathbf{A}$, according to Corollary 4.1.2. we have $h_T(u) = h(u)$ for all $u \in \mathcal{O}(\mathbf{A})$.

Now, it suffices to prove that h_T maps $\mathcal{M}(\mathbf{A})$ into $\mathcal{M}(\mathbf{B})$. Indeed, then h_T determines the desired extension $\overline{h}$ of h, the uniqueness of which is guaranteed by Lemma 4.1.1.

If $l \in N^+$ and $u = \sum_{i=1}^{k} a_i \chi_{A_i} \in \mathcal{M}_l(\mathbf{A})$, then $h_T(u) = \sum_{i=1}^{k} a_i h_T(\chi_{A_i})$. But $h_T(\chi_{A_i}) = h(\chi_{A_i}) \in \mathcal{O}(\mathbf{B})$, hence $h_T(u) \in \mathcal{M}(\mathbf{B})$.

If $u \in s(\mathbf{A})$, then there are functions $u_l \in \mathcal{M}_l(\mathbf{A})$ such that $u = \lim_{l\to\infty} u_l$ and hence $h_T(u) \in \mathcal{M}(\mathbf{B})$.

Finally, if $u \in \mathcal{M}(\mathbf{A})$, then there are functions $u_l \in s(\mathbf{A})$ such that $u = \lim_{l\to\infty} u_l$ and hence $h_T(u) \in \mathcal{M}(\mathbf{B})$, as well. □

For an object $\mathcal{O}(\mathbf{A})$ in EID define $\mathbf{E}(\mathcal{O}(\mathbf{A})) = \mathcal{M}(\mathbf{A})$ and for a morphism h of $\mathcal{O}(\mathbf{A})$ into $\mathcal{O}(\mathbf{B})$ define $\mathbf{E}(h) = \overline{h}$, where $\overline{h}$ is the unique morphism of $\mathcal{M}(\mathbf{A})$ into $\mathcal{M}(\mathbf{B})$ determined as an extension of h.

Lemma 4.1.5. $\mathbf{E}$ is a functor of EID into $CGBID$.

Proof. We have to prove that $\mathbf{E}$ preserves the identity maps and compositions. Both assertions are straightforward consequences of Corollary 4.1.2. The identity map of $\mathcal{M}(\mathbf{A})$ onto $\mathcal{M}(\mathbf{A})$ is the unique extension of the identity map of $\mathcal{O}(\mathbf{A})$ onto $\mathcal{O}(\mathbf{A})$. Similarly, if h maps $\mathcal{O}(\mathbf{A})$ into $\mathcal{O}(\mathbf{B})$ and g maps $\mathcal{O}(\mathbf{B})$ into $\mathcal{O}(\mathbf{C})$, then the composition of extensions $\overline{g} \circ \overline{h}$ and the extension of the composition $\overline{g \circ h}$ coincide. Thus $\mathbf{E}(g \circ h) = \mathbf{E}(g) \circ \mathbf{E}(h)$. □

The next assertion follows directly from Corollary 4.1.2.

Theorem 4.1.6. $\mathbf{E}$ is an epireflection of EID into $CGBID$.

5. FUZZIFICATION – SIMPLEX CASE

5.1. Simplex-valued domains

In [11] we introduced the category S_nD cogenerated by a cogenerator $S_n = \{(x_1, x_2, \dots, x_n) \in I^n;\ \sum_{i=1}^n x_i \leq 1\}$ carrying the coordinatewise partial order, difference, and sequential convergence (essentially, the objects of S_nD are subobjects of the powers S_n^X) and we showed how basic probability notions can be defined within S_nD. In the resulting S_nD-probability we have n-component probability domains in which each event represents a body of competing components and the range of a state represents a simplex S_n of n-tuples of possible "rewards" — the sum of the rewards is a number from $[0, 1]$. For $n = 1$ we get fuzzy events and the corresponding fuzzy probability theory.

Let X be a nonempty set and let ${S_n}^X$ be the set of all maps of X into S_n; if X is a singleton $\{a\}$, then $S_n^{\{a\}}$ will be condensed to S_n. Let $\mathbf{f} \in {S_n}^X$. Then there are n maps $f_1, f_2, \dots, f_n$ of X into I such that for each $x \in X$ we have $\mathbf{f}(x) = \big(f_1(x), f_2(x), \dots, f_n(x)\big)$; we shall write $\mathbf{f} = (f_1, f_2, \dots, f_n)$. In what follows, ${S_n}^X$ carries the coordinatewise partial order ($\mathbf{g} \leq \mathbf{f}$ iff $g_i \leq f_i$ for all i, $1 \leq i \leq n$), the coordinatewise partial difference (for $\mathbf{g} \leq \mathbf{f}$ define $\mathbf{f} \ominus \mathbf{g} = (f_1 \ominus g_1, f_2 \ominus g_2, \dots, f_n \ominus g_n)$), and the coordinatewise sequential convergence inherited from S_n. Elements $(f_1, f_2, \dots, f_n) \in {S_n}^X$ such that $\sum_{i=1}^n f_i(x) = 1$, $x \in X$, are *maximal.* If for some index i, $1 \leq i \leq n$, we have $f_j(x) = 0$ for all $j \neq i$ and all $x \in X$, then $(f_1, f_2, \dots, f_n)$ is said to be *pure*; denote $\mathbf{p_i}$ the corresponding maximal pure element of ${S_n}^X$. Clearly, if for all i, $1 \leq i \leq n$, the functions f_i are constant zero functions, then $\big(f_1(x), f_2(x), \dots, f_n(x)\big)$ is the least element of ${S_n}^X$; it is called the *bottom* element and denoted by $\mathbf{b}$. To avoid complicated notation, if no confusion can arise, then the bottom element, resp. the i-th maximal pure elements, will be denoted by the same symbol $\mathbf{b}$, resp. $\mathbf{p_i}$, $1 \leq i \leq n$, independently of the ground set X. — For $n = 2$ see Figure 3.

Let X be a nonempty set. We are interested in subsets $\mathcal{X} \subseteq {S_n}^X$ closed with respect to the difference, containing the bottom element and all maximal pure elements of ${S_n}^X$. For $n = 1$ we get D-posets of fuzzy sets and for $n > 1$ we get a structure which generalizes fuzzy events to higher dimensions.

Let $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n \subseteq I^X$ be reduced ID-posets. Define $S(\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n)$ to be the set of all $(f_1, f_2, \ldots, f_n) \in {S_n}^X$ such that $f_i \in \mathcal{B}_i$, $1 \leq i \leq n$. If there exists an ID-poset $\mathcal{B} \subseteq I^{\ X}$ such that $\mathcal{B} = \mathcal{B}_i$, $1 \leq i \leq n$, then $S(\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n)$ is condensed to $S_n(\mathcal{B})$. In what follows we consider only the latter case.

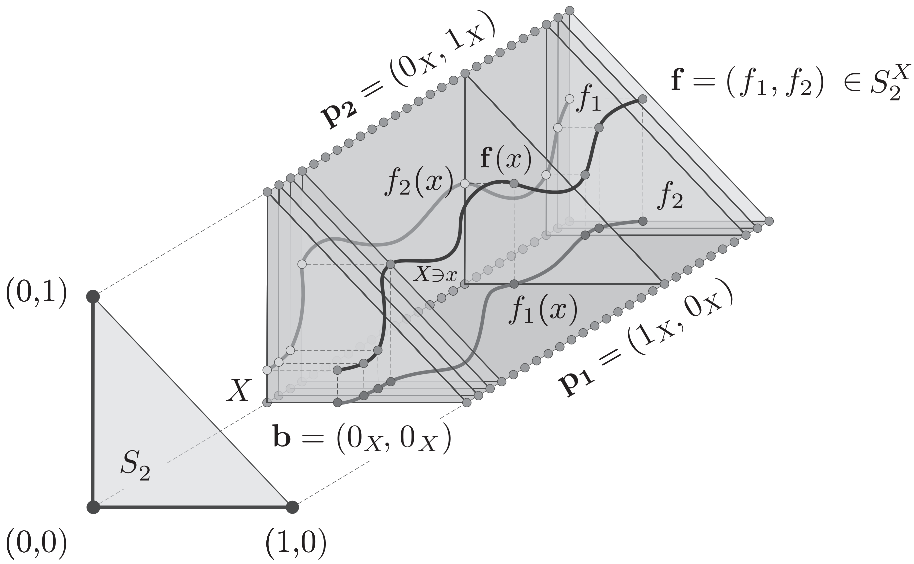

Fig. 3. Construction of S_n^X for $n = 2$, i. e. S_2^X.

Definition 5.1.1. Let X be a nonempty set. Let $\mathcal{X}$ be a subset of ${S_n}^X$, carrying the coordinatewise order, the coordinatewise convergence and closed with respect to the inherited difference. Assume that $\mathcal{X}$ contains the bottom element and all maximal pure elements. Then $(\mathcal{X}, \leq, \ominus, \mathbf{b}, \mathbf{p_1}, \ldots, \mathbf{p_n})$ is said to be an *S_nD-domain.* If there is a (reduced) ID-poset $\mathcal{B} \subseteq I^X$ such that $\mathcal{X} = S_n(\mathcal{B})$, then $(\mathcal{X}, \leq, \ominus, \mathbf{b}, \mathbf{p_1}, \ldots, \mathbf{p_n})$ is said to be a *simple S_nD-domain* and $\mathcal{B}$ is said to be the *base* of $\mathcal{X}$.

If no confusion can arise, then $(\mathcal{X}, \leq, \ominus, \mathbf{b}, \mathbf{p_1}, \ldots, \mathbf{p_n})$ will be reduced to $\mathcal{X}$. *In what follows, all S_nD-domains are assumed to be simple.*

Definition 5.1.2. Let $\mathbf{h}$ be a map of a simple S_nD-domain $\mathcal{X}$ into a simple S_nD-domain $\mathcal{Y}$ such that

(i) $\mathbf{h}(\mathbf{v}) \leq \mathbf{h}(\mathbf{u})$ whenever $\mathbf{u}, \mathbf{v} \in \mathcal{X}$ and $\mathbf{v} \leq \mathbf{u}$, and then $\mathbf{h}(\mathbf{u} \ominus \mathbf{v}) = \mathbf{h}(\mathbf{u}) \ominus \mathbf{h}(\mathbf{v})$;

(ii) $\mathbf{h}$ maps the bottom element of $\mathcal{X}$ to the bottom element of $\mathcal{Y}$ and the i-th maximal pure element of $\mathcal{X}$ to the i-th maximal pure element of $\mathcal{Y}$, for all $i, 1 \leq i \leq n$.

Then $\mathbf{h}$ is said to be an S_nD-*homomorphism.* A sequentially continuous S_nD-homomorphism of $\mathcal{X}$ into $\mathcal{Y}$ is said to be an S_nD-*observable.* A sequentially continuous S_nD-homomorphism of $\mathcal{X}$ into I is said to be an S_nD-*valued state* or, simply, a *state.*

Denote S_nD the category of simple S_nD-domains and sequentially continuous S_nD-homomorphisms. Clearly, the categories ID and S_1D coincide and each S_n^X is a simple S_nD-domain.

Lemma 5.1.3. (Theorem 3.1 in [9].) Let $\mathcal{X} = S_n(\mathcal{A}) \subseteq S_n^X$ and $\mathcal{Y} = S_n(\mathcal{B}) \subseteq S_n^Y$ be simple S_nD-domains.

(i) Let h be a D-homomorphism of $\mathcal{A}$ into $\mathcal{B}$. For $\mathbf{f} = (f_1, f_2, \ldots, f_n) \in \mathcal{X}$ put $\mathbf{h}(\mathbf{f}) = \big(h(f_1), h(f_2), \ldots, h(f_n)\big) \in \mathcal{Y}$ and denote $\mathbf{h}$ the resulting map of $\mathcal{X}$ into $\mathcal{Y}$. Then $\mathbf{h}$ is an S_nD-homomorphism.

(ii) Let $\mathbf{h}$ be an S_nD-homomorphism of $\mathcal{X}$ into $\mathcal{Y}$. Then there exists a unique D-homomorphism h of $\mathcal{A}$ into $\mathcal{B}$ such that for each $\mathbf{f} = (f_1, f_2, \ldots, f_n) \in \mathcal{X}$ we have $\mathbf{h}(\mathbf{f}) = \big(h(f_1), h(f_2), \ldots, h(f_n)\big)$.

P r o o f. The proof of (i) is straightforward and it is omitted.

(ii) Given $\mathbf{g} = (g_1, g_2, \ldots, g_n) \in S_n^Z$, for each k, $1 \le k \le n$, define $\mathrm{red}_k(\mathbf{g}) = (h_1, h_2, \ldots, h_n)$, where $h_k = g_k$ and $h_j = 0_Z$ otherwise.

Let $\mathbf{f} = (f_1, f_2, \ldots, f_n) \in S_n(\mathcal{A})$ and let $\mathbf{h}(\mathbf{f}) = (u_1, u_2, \ldots, u_n) \in S_n(\mathcal{B})$. Since $\mathbf{h}(\mathbf{f} \ominus \mathrm{red}_n(\mathbf{f})) = \mathbf{h}(\mathbf{f}) \ominus \mathbf{h}(\mathrm{red}_n(\mathbf{f})) = \mathbf{h}\big((f_1, f_2, \ldots, f_{n-1}, 0_X)\big) \in S_n(\mathcal{B})$ and $\mathbf{h}$ preserves order, necessarily there are elements $v_k \in S_n(\mathcal{B})$, $1 \le k \le n$, such that $\mathbf{h}\big((f_1, f_2, \ldots, f_{n-1}, 0_X)\big) = (v_1, v_2, \ldots, v_{n-1}, 0_Y) \in S_n(\mathcal{B})$ and $\mathbf{h}\big(\mathrm{red}_n(\mathbf{f})\big) = (0_Y, 0_Y, \ldots, 0_Y, v_n)$. Hence $\mathbf{h}\big(\mathrm{red}_n(\mathbf{f})\big) = (0_Y, 0_Y, \ldots, 0_Y, u_n) = \mathrm{red}_n(u_1, u_2, \ldots, u_n)$ and $\mathbf{h}\big((f_1, f_2, \ldots, f_{n-1}, 0_X)\big) = (u_1, u_2, \ldots, u_{n-1}, 0)$, i. e., $u_i = v_i$ for all i, $1 \le i \le n$. Inductively, $\mathbf{h}\big(\mathrm{red}_k(\mathbf{f})\big) = \mathrm{red}_k(u_1, u_2, \ldots, u_n)$, $1 \le k \le n$. For each k, $1 \le k \le n$, define $\mathcal{X}_k = \{(g_1, g_2, \ldots, g_n) \in S_n(\mathcal{A});\ g_l = 0_X \text{ for all } l \ne k,\ 1 \le l \le n\}$. Then $\mathbf{h}$ on $\mathcal{X}_k$ can be identified with an S_nD-homomorphism h_k on $\mathcal{A}$ into $\mathcal{B}$ and $\mathbf{h}(\mathbf{f}) = \big(h_1(f_1), h_2(f_2), \ldots, h_n(f_n)\big)$. Now, it suffices to prove that $h_i = h_j$ for all $i \ne j$, $1 \le i \le n$, $1 \le j \le n$. Contrariwise, suppose that there exists $f \in \mathcal{A}$ and $i < j$ such that $u = h_i(f) < h_j(f) = v$. Define $\mathbf{g} = (g_1, g_2, \ldots, g_n) \in S_n(\mathcal{A})$ as follows: $g_i = 1_X \ominus f$, $g_j = f$, and $g_k = 0_X$ otherwise. Then $\mathbf{h}(\mathbf{g}) = (w_1, w_2, \ldots, w_n) \in S_n(\mathcal{B})$, where $w_i = h_i(1_X \ominus f) = 1 - u$, $w_j = h_j(f) = v$, and $w_k = 0_Y$ otherwise. Then $\sum_{i=1}^n w_i = 1_Y - u + v > 1_Y$, a contradiction. □

5.2. Simplex-valued crisp and fuzzy

Denote CrS_nD the full subcategory of S_nD the objects of which are simple S_nD-domains of the form $S_n(\mathbb{A})$ (i. e. the base $\mathbb{A}$ is a σ-field of subsets considered as an ID-poset); such domains are said to be *crisp.*

Denote FuS_nD the full subcategory of S_nD the objects of which are simple S_nD-domains of the form $S_n\big(\mathcal{M}(\mathbb{A})\big)$ (i. e. the base $\mathcal{M}(\mathbb{A})$ is the set of all measurable functions into I considered as an ID-poset); such domains are said to be *fuzzy.*

We present a simple situation leading to n-dimensional crisp events: credit system — grading of university students.

Example 5.2.1. Consider a university and a student of a Bc program:

X	available courses
$x \in X$	a course
J	student JOHN
$J(x)$	the grade of JOHN at x, $J(x) \in S_5$
$(1,0,0,0,0)$	A
$(0,1,0,0,0)$	B
$(0,0,1,0,0)$	C
$(0,0,0,1,0)$	D
$(0,0,0,0,1)$	E
$(0,0,0,0,0)$	Fx — failed or NOT enrolled
$J \in S_5^X$	the performance of JOHN (crisp event)

Consider the fuzzification functor $\mathbf{F}$ sending each σ-field $\mathbf{A} \subseteq \{0,1\}^X$ to the set $\mathcal{M}(\mathbf{A}) \subseteq [0,1]^X$ of all measurable functions ranging in $I = [0,1]$, both considered as D-posets of fuzzy subsets of X. Recall that $\mathbf{F}$ sends objects of $CFSD$ (crisp events) into objects of $CGBID$ (fuzzy events) and each ID-morphism $h : \mathbf{A} \to \mathbf{B}$ to the unique ID-morphism $\mathbf{F}(h) : \mathcal{M}(\mathbf{A}) \to \mathcal{M}(\mathbf{B})$. Given a positive natural number n, define a map $\mathbf{F}_n$ sending each object $S_n(\mathbf{A}) \subseteq S_n^X$ of CrS_nD to the corresponding object $S_n\big(\mathcal{M}(\mathbf{A})\big) \subseteq S_n^X$ of FuS_nD. We show that $\mathbf{F}_n$ yields a functor sending each S_nD-morphism $\mathbf{h}$ of $S_n(\mathbf{A})$ into $S_n(\mathbf{B}) \subseteq S_n^Y$ to the unique S_nD-morphism $\mathbf{F}_n(\mathbf{h})$ of $S_n\big(\mathcal{M}(\mathbf{A})\big)$ into $S_n\big(\mathcal{M}(\mathbf{B})\big)$. Now, for $g = \mathbf{F}(h)$ put $\mathbf{F}_n(\mathbf{h}) = \mathbf{g}$.

The next assertion is a corollary of Lemma 5.1.3.

Theorem 5.2.2. For each positive natural number n, $\mathbf{F}_n$ is a functor from CrS_nD to FuS_nD.

We close with some remarks on simplex-valued probability. Using the relationship between the functors $\mathbf{F}$ and $\mathbf{F}_n$ it is possible to describe the transition from crisp to fuzzy simplex-valued probability.

Definition 5.2.3. (i) Let $\mathbf{A}$ be a σ-field of subsets of Ω, let $S_n(\mathbf{A})$ be the corresponding object of CrS_nD, let p be a probability measure on $\mathbf{A}$, let $\mathbf{p}$ be the corresponding state (S_nD-morphism ranging in S_n). Then $\big(\Omega, S_n(\mathbf{A}), \mathbf{p}\big)$ is said to be a *generalized crisp probability space*.

Let $\big(\Omega, S_n(\mathbf{A}), \mathbf{p}\big)$ and $\big(\Xi, S_n(\mathbf{B}), \mathbf{q}\big)$ be generalized crisp probability spaces and let $\mathbf{h}$ be an S_nD-morphism of $S_n(\mathbf{A})$ into $S_n(\mathbf{B})$. Then $\mathbf{h}$ is said to be a *generalized crisp observable*. Moreover, if $\mathbf{p} = \mathbf{q} \circ \mathbf{h}$, then $\mathbf{h}$ is said to be a *generalized crisp random transformation*.

(ii) Let $\mathbf{A}$ be a σ-field of subsets of Ω, let $S_n\big(\mathcal{M}(\mathbf{A})\big)$ be the corresponding object of FuS_nD, let p be a probability measure on $\mathbf{A}$, let p_t be the state (ID-morphism ranging in I) on $\mathcal{M}(\mathbf{A})$ defined via integral and let $\mathbf{p}_t$ be the corresponding S_n-valued state on $S_n\big(\mathcal{M}(\mathbf{A})\big)$ defined by $\mathbf{p}_t(\mathbf{f}) = \big(p_t(f_1), p_t(f_2), \ldots, p_t(f_n)\big)$, $\mathbf{f} =$

$= (f_1, f_2, \dots, f_n) \in S_n\big(\mathcal{M}(\mathbf{A})\big)$. Then $\big(\Omega, S_n\big(\mathcal{M}(\mathbf{A})\big), \mathbf{p}_t\big)$ is said to be a *generalized fuzzy probability space.*

Let $\big(\Omega, S_n\big(\mathcal{M}(\mathbf{A})\big), \mathbf{p}_t\big)$ and $\big(\Xi, S_n\big(\mathcal{M}(\mathbf{B})\big), \mathbf{q}_t\big)$ be generalized fuzzy probability spaces and let $\mathbf{h}$ be an S_nD-morphism of $S_n\big(\mathcal{M}(\mathbf{A})\big)$ into $S_n\big(\mathcal{M}(\mathbf{B})\big)$. Then $\mathbf{h}$ is said to be a *generalized fuzzy observable.* Moreover, if $\mathbf{p} = \mathbf{q} \circ \mathbf{h}$, then $\mathbf{h}$ is said to be a *generalized fuzzy random transformation.*

Question GT. What is the transition from generalized crisp probability to generalized fuzzy probability (fuzzification) from the viewpoint of category theory?

Answer GT. Analogously as in the case of classical and fuzzy probability theories, we can describe the relationships between the two proposed generalized probability theories using the properties of the functor $\mathbf{F}_n\colon\ CrS_nD \to FuS_nD$.

First, observe that there is a one-to one correspondence between the objects of CrS_nD and the objects of FuS_nD: the correspondence between $S_n(\mathbf{A})$ and $\mathbf{F}_n\big(S_n(\mathbf{A})\big) = S_n\big(\mathcal{M}(\mathbf{A})\big)$ yields a bijection.

Second, there is a one-to-one correspondence between states (S_nD-morphisms ranging in S_n) on $S_n(\mathbf{A})$ and $S_n\big(\mathcal{M}(\mathbf{A})\big)$.

Third, each observable $\mathbf{h}$ (S_nD-morphisms) from $S_n(\mathbf{A})$ to $S_n(\mathbf{B})$ can be uniquely extended to an observable $\mathbf{F}_n(\mathbf{h}) = \mathbf{g}$ from $\mathbf{F}_n\big(S_n(\mathbf{A})\big) = S_n\big(\mathcal{M}(\mathbf{A})\big)$ into $\mathbf{F}_n\big(S_n(\mathbf{B})\big) = S_n\big(\mathcal{M}(\mathbf{B})\big)$.

Fourth, it follows from the properties of $\mathbf{F}$ and its relationships to $\mathbf{F}_n$ that there are observables $\mathbf{g}$ from $\mathbf{F}_n\big(S_n(\mathbf{A})\big) = S_n\big(\mathcal{M}(\mathbf{A})\big)$ into $\mathbf{F}_n\big(S_n(\mathbf{B})\big) = S_n\big(\mathcal{M}(\mathbf{B})\big)$ such that for no observable $\mathbf{h}$ from $S_n(\mathbf{A})$ to $S_n(\mathbf{B})$ we have $\mathbf{F}_n(\mathbf{h}) = \mathbf{g}$. Such observables have genuine generalized "quantum and fuzzy" qualities. In particular, if $\big(\Omega, S_n(\mathbf{A}), \mathbf{p}\big)$ is a generalized crisp probability space, then $\mathbf{p}$ is the restriction of a genuine generalized fuzzy observable.

Consequently, passing from the generalized crisp probability to the generalized fuzzy probability is a minimal extension within the category S_nD in which the objects are "divisible", generalized probability measures are morphisms, and some simple genuine generalized "quantum and fuzzy" situations can be modelled.

ACKNOWLEDGEMENT

We express our gratitude to the referee for his valuable suggestions leading to an improvement of the manuscript.

This work was partially supported by VEGA 1/0539/08 and VEGA 2/0032/09.

(Received July 12, 2010)

REFERENCES

[1] S. Bugajski: Statistical maps I. Basic properties. Math. Slovaca *51* (2001), 321–342.

[2] S. Bugajski: Statistical maps II. Basic properties. Math. Slovaca *51* (2001), 343–361.

[3] F. Chovanec and R. Frič: States as morphisms. Internat. J. Theoret. Phys. *49* (2010), 3050–3100.

[4] F. Chovanec and F. Kôpka: *D*-posets. In: Handbook of Quantum Logic and Quantum Structures: Quantum Structures. (K. Engesser, D. M. Gabbay and D. Lehmann, eds.), Elsevier, Amsterdam 2007, pp. 367–428.

[5] A. Dvurečenskij and S. Pulmannová: New Trends in Quantum Structures. Kluwer Academic Publ. and Ister Science, Dordrecht and Bratislava 2000.

[6] R. Frič: Remarks on statistical maps and fuzzy (operational) random variables. Tatra Mt. Math. Publ. *30* (2005), 21–34.

[7] R. Frič: Statistical maps: a categorical approach. Math. Slovaca *57* (2007), 41–57.

[8] R. Frič: Extension of domains of states. Soft Comput. *13* (2009), 63–70.

[9] R. Frič: Simplex-valued probability. Math. Slovaca *60* (2010), 607–614.

[10] R. Frič: States on bold algebras: Categorical aspects. J. Logic Comput. (To appear). DOI:10.1093/logcom/exp014

[11] R. Frič and M. Papčo: On probability domains. Internat. J. Theoret. Phys. *49* (2010), 3092–3063.

[12] R. Frič and M. Papčo: A categorical approach to probability theory. Studia Logica *94* (2010), 215–230.

[13] S. Gudder: Fuzzy probability theory. Demonstratio Math. *31* (1998), 235–254.

[14] F. Kôpka and F. Chovanec: D-posets. Math. Slovaca *44* (1994), 21–34.

[15] R. Mesiar: Fuzzy sets and probability theory. Tatra Mt. Math. Publ. *1* (1992), 105–123.

[16] M. Papčo: On measurable spaces and measurable maps. Tatra Mt. Math. Publ. *28* (2004), 125–140.

[17] M. Papčo: On fuzzy random variables: examples and generalizations. Tatra Mt. Math. Publ. *30* (2005), 175–185.

[18] M. Papčo: On effect algebras. Soft Comput. *12* (2007), 26–35.

[19] B. Riečan and D. Mundici: Probability on *MV*-algebras. In: Handbook of Measure Theory, Vol. II (E. Pap, ed.), North-Holland, Amsterdam 2002, pp. 869–910.

[20] L. A. Zadeh: Probability measures of fuzzy events. J. Math. Anal. Appl. *23* (1968), 421–427.

Roman Frič, Mathematical Institute, Slovak Academy of Sciences, Grešákova 6, 040 01 Košice, and Catholic University in Ružomberok, Hrabovská cesta 1, 034 01 Ružomberok. Slovak Republic.

e-mail: fric@saske.sk

Martin Papčo, Catholic University in Ružomberok, Hrabovská cesta 1, 034 01 Ružomberok, and Mathematical Institute, Slovak Academy of Sciences, Štefánikova 49, 814 73 Bratislava. Slovak Republic.

e-mail: martin.papco@murk.sk