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FINITE-TIME BOUNDEDNESS AND STABILIZATION OF SWITCHED LINEAR SYSTEMS

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In this paper, finite-time boundedness and stabilization problems for a class of switched linear systems with time-varying exogenous disturbances are studied. Firstly, the concepts of finite-time stability and finite-time boundedness are extended to switched linear systems. Then, based on matrix inequalities, some sufficient conditions under which the switched linear systems are finite-time bounded and uniformly finite-time bounded are given. Moreover, to solve the finite-time stabilization problem, stabilizing controllers and a class of switching signals are designed. The main results are proven by using the multiple Lyapunov-like functions method, the single Lyapunov-like function method and the common Lyapunov-like function method, respectively. Finally, three examples are employed to verify the efficiency of the proposed methods.

Keywords: switched linear systems, finite-time boundedness, multiple Lyapunov-like functions, single Lyapunov-like function, common Lyapunov-like function

Classification: 93A14, 93C10,93D15, 93D21

1. INTRODUCTION

A switched system belongs to a special class of hybrid systems. It consists of a family of subsystems described by differential or difference equations and a switching law that orchestrates switching between these subsystems [18]. Switched systems arise as models for phenomena which can not be described by exclusively continuous or exclusively discrete processes. Examples include manufacturing control [20], traffic control [28], chemical processing [10] and communication networks [25], etc. Due to their success in practical applications and importance in theory development, switched systems have been attracting considerable attention during the last decades.

The basic research topics for switched systems are the issues of stability and stabilizability which have attracted most of the attention [14, 8, 16, 7, 12, 21, 23, 24, 33, 15, 32, 22]. Due to the hybrid nature of switched systems operation, the stability analysis of switched systems is more difficult to deal with than that of continuous systems or discrete systems, and so becomes a challenging issue. In this respect, Lyapunov stability theory and its variations or generalizations still play a dominant role. Analysis method can be roughly divided into the common Lyapunov function method, the single Lyapunov function method and the multiple Lyapunov functions method. For the former, a common Lyapunov function for all subsystems guarantees stability under an arbitrary switching signal. However, many switched systems, which may not possess a common Lyapunov function, yet still are stable under some properly chosen switching signals. The multiple Lyapunov functions and the single Lyapunov function methods have been proven to be powerful and effective tools for finding such a switching signal (e. g., time-dependent switching signals and state-dependent switching signals) [14, 7, 12, 21, 23, 24]. For more analysis and synthesis results of switched systems, the readers are referred to the literature [8, 16, 33, 15, 32, 22] and the references therein.

Up to now, most of the existing literature related to stability of switched systems focuses on Lyapunov asymptotic stability, which is defined over an infinite time interval. However, in many practical applications, the main concern is the behavior of the system over a fixed finite time interval. For instance, the problem of sending a rocket from a neighborhood of a point A to a neighborhood of a point B over a fixed time interval; the problem, in a chemical process, of keeping the temperature, pressure or some other parameters within specified bounds in a prescribed time interval. In these cases, finite-time stability could be used, which focuses its attention on the transient behavior over a finite time interval rather than on the asymptotic behavior of a system response. More specifically, a system is said to be finite-time stable if, given a bound on the initial condition, its state remains within prescribed bounds in the fixed time interval. It should be noted that finite-time stability and Lyapunov asymptotic stability are independent concepts: a system could be finite-time stable but not Lyapunov asymptotically stable, and vice versa.

Some early results on finite-time stability problems can be found in [13, 9, 30]. Recently, based on the linear matrix inequality theory, many valuable results have been obtained for this type of stability [3, 2, 34, 1, 35, 4, 5], to name just a few. In [3, 2], the authors have extended the concept finite-time stability to finite-time boundedness. In [34, 1], the definition of finite-time stability have been extended to the systems with impulsive effects. In [35], finite-time control problem has been discussed for the systems subject to time-varying exogenous disturbance. More analysis and synthesis results of finite control problem can be found in [4, 5] and the references therein. In addition, it should be pointed out that the authors of [6, 19, 17, 29] have presented some results of finite-time stability for different systems, but the finite-time stability which consists of Lyapunov stability and finite-time convergence is different from that in this paper and [13, 9, 30, 3, 2, 34, 1, 35, 4, 5].

So far, the stability analysis for switched systems and the finite-time stability for different systems have been extensively studied by many researchers. However, little work has been done for the finite-time stability of switched systems. It is well-known that a nonlinear time-varying system or a linear time-varying system can often be approximated by using several linear time-invariant systems around the equilibrium point. These linear systems being only valid around the equilibrium point, it is important to be able to avoid large excursions of the states. Considering such reason and the wide application of switched systems in practice, it motivates us to investigate the finite-time stability and stabilization problems for a class of switched linear systems. In [31], the authors have studied the finite-time stability and practical stability for a class of switched systems. However, the sufficient conditions are difficult to test and the problem of controller design is not considered. To the best of the authors' knowledge, the proposed work in this paper on finite-time boundedness and stabilization problems for switched linear systems has not been studied in the previous literature.

The contribution of this paper is twofold. First, the definition of finite-time boundedness is extended to switched linear systems and some sufficient conditions are given in terms of matrix inequalities which are easy to test. Second, the problem of control synthesis including both switching signals and feedback switching controllers is discussed. These switching signals contain time-dependent switching signals and state-dependent signals, which are suitable for different cases.

The rest of the paper is organized as follows. In Section 2, some notations and problem formulations are presented. Section 3 provides the main results of this paper. Some sufficient conditions which guarantee a class of switched linear systems finite-time bounded and uniformly finite-time bounded are given. In Section 4, the finite-time stabilization problem is investigated. Finally, three numerical examples are presented to illustrate the efficiency of the proposed methods in Section 5. Concluding remarks are given in Section 6.

2. PRELIMINARIES AND PROBLEM FORMULATION

In this paper, let $P > 0 (\geq, <, \leq 0)$ denote a symmetric positive definite (positive-semidefinite, negative definite, negative-semidefinite) matrix P. For any symmetric matrix P, $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ denote the maximum and minimum eigenvalues of matrix P, respectively. The identity matrix of order n is denoted as I_n (or, simply, I if no confusion arises).

Consider a class of switched linear control systems of the form

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) + G_{\sigma(t)}\omega(t), \quad x(0) = x_0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^p$ is the control input and $\omega(t) \in \mathbb{R}^r$ is the exogenous disturbance. $\sigma(t)$: $[0, \infty) \to M = \{1, 2, \dots, m\}$ is the switching signal which is a piecewise constant function depending on time t or state $x(t)$. A_i, B_i and G_i are constant real matrices for $i \in M$.

Assumption 2.1. The exogenous disturbance $\omega(t)$ is time-varying and satisfies the constraint $\int_0^\infty \omega^T(t)\omega(t)\,\mathrm{d}t \leq d, d \geq 0$.

It should be pointed out that the assumption about the disturbance $\omega(t)$ in this paper is different from that of [3, 34], where the external disturbance is constant.

Assumption 2.2. [14] The state of the switched linear system does not jump at the switching instants, i.e., the trajectory $x(t)$ is everywhere continuous. The switching signal $\sigma(t)$ has finite number of switching on any finite interval time.

Corresponding to the switching signal $\sigma(t)$, we have the following switching sequence

$$\{x_0; (i_0, t_0), \cdots, (i_k, t_k), \cdots, | i_k \in M, k = 0, 1, \cdots\},$$

which means that i_kth subsystem is activated when $t \in [t_k, t_{k+1})$.

The aim of this paper is to find a class of switching signals under which system (1) with zero input is finite-time bounded and deal with the finite-time stabilization problem for system (1).

3. FINITE-TIME BOUNDEDNESS ANALYSIS

In this section, some finite-time boundedness criteria for switched linear control system (1) with input $u(t) = 0$ are presented. Firstly, let us extend the definitions of finite-time stability and finite-time boundedness in [3] to switched linear systems.

Definition 3.1. Given three positive constants c_1, c_2, T_f, with $c_1 < c_2$, a positive definite matrix R, and a switching signal $\sigma(t)$, the switched linear system

$$\dot{x}(t) = A_{\sigma(t)}x(t), \quad x(0) = x_0 \tag{2}$$

is said to be finite-time stable with respect to $(c_1, c_2, T_f, R, \sigma)$, if $x_0^T R x_0 \le c_1 \Rightarrow x(t)^T R x(t) < c_2, \forall\, t \in [0, T_f]$.

Next, consider the case when the state is subject to some external disturbance signals. This leads us to give the definition of finite-time boundedness, which recovers Definition 3.1 as a special case.

Definition 3.2. Given four positive constants c_1, c_2, d, T_f, with $c_1 < c_2$, a positive definite matrix R, and a switching signal $\sigma(t)$, the switched linear system

$$\dot{x}(t) = A_{\sigma(t)}x(t) + G_{\sigma(t)}\omega(t), \quad x(0) = x_0 \tag{3}$$

is said to be finite-time bounded with respect to $(c_1, c_2, d, T_f, R, \sigma)$, if $x_0^T R x_0 \le c_1 \Rightarrow x(t)^T R x(t) < c_2, \forall\, t \in [0, T_f], \forall\, \omega(t) : \int_0^{T_f} \omega^T(t)\omega(t)\,\mathrm{d}t \le d$.

Remark 3.3. It should be remarked that the concepts of finite-time stability and finite-time boundedness are different from the concept of reachable set. Reachable set is defined as the set of states that a system attains under given some bounded inputs and starting from some given initial conditions. In most analysis about reachable set, it is assumed that the system is asymptotically stable [11]. However, a system is finite-time stable (or bounded) if, given a bound on the initial state (and bounded constant disturbances), the state remains within the prescribed bound in the fixed time interval. In the analysis of finite-time stability (or boundedness), the assumption of system asymptotic stability is unnecessary. For more detailed discussions about the difference between two approaches can be found in Remark 4 of [3].

In the sequel, based on the multiple Lyapunov-like functions and the single Lyapunov-like function methods, some sufficient conditions which guarantee system (3) finite-time bounded are given, respectively.

3.1. Sufficient conditions for finite-time boundedness based on multiple Lyapunov-like functions

In this subsection, some sufficient conditions which guarantee system (3) finite-time bounded are presented. Before giving the main result, let us review the definition of average dwell-time which will be useful in the subsequent analysis.

Definition 3.4. [12]. For any $T \geq t \geq 0$, let $N_\sigma(t,T)$ denote the number of switching of $\sigma(t)$ over (t,T). If

$$N_\sigma(t,T) \leq N_0 + (T-t)/\tau_a$$

holds for $\tau_a > 0$, N_0 is a nonnegative integer, then τ_a is called average dwell-time.

Without loss of generality, in this paper we choose $N_0 = 0$, as [24].

Now, based on the multiple Lyapunov-like functions method and an average dwell-time technique, the following theorem provides sufficient conditions for finite-time boundedness of system (3).

Theorem 3.5. For any $i \in M$, let $\widetilde{P}_i = R^{-1/2} P_i R^{-1/2}$ and suppose that there exist matrices $P_i > 0$, $Q_i > 0$ and constants $\alpha_i \geq 0$, $\gamma_i > 0$ such that

$$\begin{pmatrix} A_i\widetilde{P}_i + \widetilde{P}_i A_i^T - \alpha_i \widetilde{P}_i & G_i Q_i \\ Q_i G_i^T & -\gamma_i Q_i \end{pmatrix} < 0, \tag{4a}$$

$$\frac{c_1}{\lambda_1} + \frac{\gamma d}{\lambda_3} < \frac{c_2}{\lambda_2} e^{-\alpha T_f}. \tag{4b}$$

If the average dwell-time of the switching signal σ satisfies

$$\tau_a \geq \tau_a^* = \frac{T_f \ln \mu}{\ln(c_2/\lambda_2) - \ln(c_1/\lambda_1 + \gamma d/\lambda_3) - \alpha T_f}, \tag{4c}$$

then system (3) is finite-time bounded with respect to $(c_1, c_2, d, T_f, R, \sigma)$, where $\lambda_1 = \min_{\forall\, i \in M}(\lambda_{\min}(P_i))$, $\lambda_2 = \max_{\forall\, i \in M}(\lambda_{\max}(P_i))$, $\lambda_3 = \min_{\forall\, i \in M}(\lambda_{\min}(Q_i))$, $\alpha = \max_{\forall\, i\in M}(\alpha_i)$, $\gamma = \max_{\forall\, i \in M}(\gamma_i), \mu = \lambda_2/\lambda_1$.

Proof. Choose a piecewise Lyapunov-like function as follows

$$V(t) = V_{\sigma(t)}(t) = x^T(t)\widetilde{P}_{\sigma(t)}^{-1} x(t).$$

Step 1: When $t \in [t_k, t_{k+1})$, the derivative of $V(t)$ with respect to t along the trajectory of system (3) yields

$$\begin{aligned} \dot{V}(t) &= x^T(t)(\widetilde{P}_{\sigma(t_k)}^{-1} A_{\sigma(t_k)} + A_{\sigma(t_k)}^T \widetilde{P}_{\sigma(t_k)}^{-1}) x(t) + x^T(t) \widetilde{P}_{\sigma(t_k)}^{-1} G_{\sigma(t_k)} \omega(t) \\ &\quad + \omega^T(t) G_{\sigma(t_k)}^T P_{\sigma(t_k)}^{-1} x(t) \\ &= \begin{pmatrix} x^T(t), & \omega^T(t) \end{pmatrix} \begin{pmatrix} \widetilde{P}_{\sigma(t_k)}^{-1} A_{\sigma(t_k)} + A_{\sigma(t_k)}^T \widetilde{P}_{\sigma(t_k)}^{-1} & \widetilde{P}_{\sigma(t_k)}^{-1} G_{\sigma(t_k)} \\ G_{\sigma(t_k)}^T \widetilde{P}_{\sigma(t_k)}^{-1} & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ \omega(t) \end{pmatrix}. \end{aligned} \tag{5}$$

Assuming condition (4a) is satisfied, then pre- and post-multiplying (4a) by the positive symmetric matrix $\begin{pmatrix} \widetilde{P}_i^{-1} & 0 \\ 0 & Q_i^{-1} \end{pmatrix}$, we obtain the equivalent condition

$$\begin{pmatrix} \widetilde{P}_i^{-1}A_i + A_i^T\widetilde{P}_i^{-1} - \alpha_i\widetilde{P}_i^{-1} & \widetilde{P}_i^{-1}G_i \\ G_i^T\widetilde{P}_i^{-1} & -\gamma_i Q_i^{-1} \end{pmatrix} < 0. \tag{6}$$

Combining (5) and (6) leads to

$$\begin{aligned} \dot{V}(t) &< \begin{pmatrix} x^T(t), & \omega^T(t) \end{pmatrix} \begin{pmatrix} \alpha_{\sigma(t_k)}\widetilde{P}_{\sigma(t_k)}^{-1} & 0 \\ 0 & \gamma_{\sigma(t_k)}Q_{\sigma(t_k)}^{-1} \end{pmatrix} \begin{pmatrix} x(t) \\ \omega(t) \end{pmatrix} \\ &= \alpha_{\sigma(t_k)}V_{\sigma(t_k)}(t) + \gamma_{\sigma(t_k)}\omega^T(t)Q_{\sigma(t_k)}^{-1}\omega(t). \end{aligned} \tag{7}$$

By calculation, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-\alpha_{\sigma(t_k)}t}V_{\sigma(t_k)}(t)\right) < e^{-\alpha_{\sigma(t_k)}t}\gamma_{\sigma(t_k)}\omega^T(t)Q_{\sigma(t_k)}^{-1}\omega(t). \tag{8}$$

Integrating (8) from t_k to t gives

$$V(t) < e^{\alpha_{\sigma(t_k)}(t-t_k)}V_{\sigma(t_k)}(t_k) + \gamma_{\sigma(t_k)}\int_{t_k}^{t} e^{\alpha_{\sigma(t_k)}(t-s)}\omega^T(s)Q_{\sigma(t_k)}^{-1}\omega(s)\,\mathrm{d}s. \tag{9}$$

Then, due to the definitions of λ_1 and λ_2, for $\forall\, i,j \in M$ and $\forall\, x \in \mathbb{R}^n$, we have

$$x^TP_i^{-1}x \le \lambda_{\max}(P_i^{-1})x^Tx = (1/\lambda_{\min}(P_i))x^Tx \le (1/\lambda_1)x^Tx,$$

$$x^TP_j^{-1}x \ge \lambda_{\min}(P_j^{-1})x^Tx = (1/\lambda_{\max}(P_j))x^Tx \ge (1/\lambda_2)x^Tx.$$

Since $\lambda_2/\lambda_1 = \mu$, then $x^TP_i^{-1}x \le (1/\lambda_1)x^Tx \le (\lambda_2/\lambda_1)x^TP_j^{-1}x = \mu x^TP_j^{-1}x$. It follows that $x^TR^{1/2}P_i^{-1}R^{1/2}x \le \mu x^TR^{1/2}P_j^{-1}R^{1/2}x$, i.e., $x^T\widetilde{P}_i^{-1}x \le \mu x^T\widetilde{P}_j^{-1}x$. Without loss of generality, at the switching instant t_k, assume $\sigma(t_k) = i, \sigma(t_k^-) = j$, where $\sigma(t_k^-) = \lim_{v\to 0-}\sigma(t_k+v)$. Noticing that $x(t_k) = x(t_k^-)$ from Assumption (2.2), we obtain

$$V_{\sigma(t_k)}(t_k) \le \mu V_{\sigma(t_k^-)}(t_k^-), \tag{10}$$

where $x(t_k^-) = \lim_{v\to 0-}x(t_k+v)$.

By (9) and (10), and noticing that $\alpha = \max_{\forall\, i\in M}(\alpha_i), \gamma = \max_{\forall\, i\in M}(\gamma_i)$, we have

$$V(t) < e^{\alpha(t-t_k)}\mu V_{\sigma(t_k^-)}(t_k^-) + \gamma\int_{t_k}^{t} e^{\alpha(t-s)}\omega^T(s)Q_{\sigma(t_k)}^{-1}\omega(s)\,\mathrm{d}s. \tag{11}$$

Step 2: For any $t \in (0,T_f)$, let N be the number of switching of $\sigma(t)$ over $(0,T_f)$, which implies that $N_\sigma(0,t) \le N$. Noticing that $\mu \ge 1$ since $\lambda_2 \ge \lambda_1$, and using the iterative method in *step 1*, we have

$$\begin{aligned}
V(t) < {} & e^{\alpha t}\mu^N V_{\sigma(0)}(0) + \gamma\mu^N \int_0^{t_1} e^{\alpha(t-s)}\omega^T(s)Q_{\sigma(0)}^{-1}\omega(s)\,\mathrm{d}s \\
& + \gamma\mu^{N-1}\int_{t_1}^{t_2} e^{\alpha(t-s)}\omega^T(s)Q_{\sigma(t_1)}^{-1}\omega(s)\,\mathrm{d}s + \cdots + \gamma\int_{t_k}^{t} e^{\alpha(t-s)}\omega^T(s)Q_{\sigma(t_k)}^{-1}\omega(s)\,\mathrm{d}s \\
= {} & e^{\alpha t}\mu^N V_{\sigma(0)}(0) + \gamma\int_0^t e^{\alpha(t-s)}\mu^{N_\sigma(s,t)}\omega^T(s)Q_{\sigma(s)}^{-1}\omega(s)\,\mathrm{d}s \\
\leq {} & e^{\alpha T_f}\mu^N V_{\sigma(0)}(0) + \gamma e^{\alpha T_f}\mu^N\lambda_{\max}(Q_{\sigma(s)}^{-1})\int_0^{T_f}\omega^T(s)\omega(s)\,\mathrm{d}s \\
\leq {} & e^{\alpha T_f}\mu^N V_{\sigma(0)}(0) + \gamma e^{\alpha T_f}\mu^N d/\lambda_3,
\end{aligned} \tag{12}$$

where $\lambda_{\max}(Q_{\sigma(s)}^{-1}) = 1/\lambda_{\min}(Q_{\sigma(s)}) = 1/\lambda_3$. The relation $N \leq T_f/\tau_a$ leads to

$$V(t) < e^{\alpha T_f}\mu^{\frac{T_f}{\tau_a}}\left(V_{\sigma(0)}(0) + \gamma d/\lambda_3\right). \tag{13}$$

On the other hand,

$$V(t) \geq \lambda_{\min}(P_{\sigma(t)}^{-1})x^T(t)Rx(t) = \frac{1}{\lambda_{\max}(P_{\sigma(t)})}x^T(t)Rx(t) \geq \frac{1}{\lambda_2}x^T(t)Rx(t), \tag{14}$$

$$V_{\sigma(0)}(0) \leq \lambda_{\max}(P_{\sigma(0)}^{-1})x^T(0)Rx(0) = [1/\lambda_{\min}(P_{\sigma(0)})]x^T(0)Rx(0) \leq c_1/\lambda_1. \tag{15}$$

Putting together (13), (14) and (15), we get

$$x^T(t)Rx(t) \leq \lambda_2 V(t) < \lambda_2 e^{\alpha T_f}\mu^{\frac{T_f}{\tau_a}}(c_1/\lambda_1 + \gamma d/\lambda_3). \tag{16}$$

Assuming condition (4b) is satisfied, then the following proof can be divided into two cases.

Case 1: $\mu = 1$, which is a trivial case, from (4b), we have

$$\lambda_2 e^{\alpha T_f}(c_1/\lambda_1 + \gamma d/\lambda_3) < c_2. \tag{17}$$

Substituting (17) into (16) leads to $x^T(t)Rx(t) < c_2$.

Case 2: $\mu > 1$, from (4b), we obtain $\ln(c_2/\lambda_2) - \ln(c_1/\lambda_1 + \gamma d/\lambda_3) - \alpha T_f > 0$. Then, assuming condition (4c) is satisfied, we have

$$\frac{T_f}{\tau_a} \leq \frac{\ln\dfrac{c_2}{\lambda_2} - \ln(\dfrac{c_1}{\lambda_1} + \dfrac{\gamma d}{\lambda_3}) - \alpha T_f}{\ln\mu} = \frac{\ln(\dfrac{c_2}{\lambda_2(c_1/\lambda_1 + \gamma d/\lambda_3)}e^{-\alpha T_f})}{\ln\mu}.$$

This leads to

$$\mu^{\frac{T_f}{\tau_a}} \leq \frac{c_2}{\lambda_2(c_1/\lambda_1 + \gamma d/\lambda_3)}e^{-\alpha T_f}. \tag{18}$$

Therefore, it follows from (16) and (18) that

$$x^T(t)Rx(t) < c_2. \tag{19}$$

Noticing that the trajectory of system (3) remains continuous at instant T_f, we conclude that (19) holds for all $t \in [0, T_f]$. □

Remark 3.6. Let $N = 0$, which implies that there is no switching over $[0, T_f]$, then the system (3) degenerates into an ordinary linear system and Theorem 3.5 contains Lemma 6 of [3] as a special case.

Remark 3.7. The function $V(t)$ in the proof of Theorem 3.5 belongs to multiple Lyapunov-like functions. Unlike the classical (quadratic) Lyapunov function for switched (linear) systems in the case of asymptotical stability, there is no requirement of negative definiteness or negative semidefiniteness on $\dot{V}(t)$. This is the specifical difference between these two kinds of Lyapunov functions. Actually, if the exogenous disturbance $\omega(t) = 0$ and we limit the constants $\alpha_i < 0$ ($\forall\, i \in M$), then $\dot{V}(t)$ will be a negative definite function. In this case, we can obtain that the system (3) is asymptotically stable on the infinite interval $[0, +\infty)$ if the average dwell-time $\tau_a \geq -(\ln \mu)/\alpha$. The detailed proof can be found in [14].

In the case of finite-time stability for system (2), it is easy to obtain the sufficient conditions from Theorem 3.5, i. e., the case of $d = 0$.

Corollary 3.8. For any $i \in M$, let $\widetilde{P}_i = R^{-1/2} P_i R^{-1/2}$ and suppose that there exist matrices $P_i > 0$ and constants $\alpha_i \geq 0$ such that

$$A_i \widetilde{P}_i + \widetilde{P}_i A_i^T - \alpha_i \widetilde{P}_i < 0, \tag{20a}$$

$$\mu < \frac{c_2}{c_1} e^{-\alpha T_f}. \tag{20b}$$

If the average dwell-time of the switching signal σ satisfies

$$\tau_a \geq \tau_a^* = \frac{T_f \ln \mu}{\ln(c_2/c_1) - \ln \mu - \alpha T_f}, \tag{20c}$$

then system (2) is finite-time stable with respect to $(c_1, c_2, T_f, R, \sigma)$, where $\lambda_1 = \min_{\forall\, i\in M}(\lambda_{\min}(P_i)), \lambda_2 = \max_{\forall\, i\in M}(\lambda_{\max}(P_i)), \alpha = \max_{\forall\, i\in M}(\alpha_i), \mu = \lambda_2/\lambda_1$.

The advantage of multiple Lyapunov-like functions lies in their flexibility, since different Lyapunov-like functions are constructed for each subsystem. From Remark 3.6, we know that conditions (4a), (4b) imply that each subsystem is finite-time bounded with respect to (c_1, c_2, d, T_f, R). That is to say, if the switching is slow enough satisfying condition (4c), then the whole switched systems is finite-time bounded. However, in some practical cases, all subsystems are not finite-time bounded, then the condition (4a) can not be guaranteed. In this case, the whole switched systems may still be finite-time bounded by properly choosing the switching signal. In the next subsection, we will investigate this case.

3.2. Sufficient conditions for finite-time boundedness based on a single Lyapunov-like function

In this subsection, some sufficient conditions for finite-time boundedness of system (3) are derived by applying a single Lyapunov-like function method. Assume each subsystem of system (3) is not finite-time bounded. This assumption is due to that if at least one of the individual subsystems is finite-time bounded, this problem is trivial (just keep $\sigma(t) = p$, where p is the index of this finite-time bounded subsystem). In the sequel, a class of state-dependent switching signals are designed such that system (3) is finite-time bounded.

Theorem 3.9. For any $i \in M$, let $\widetilde{P} = R^{-1/2}PR^{-1/2}$ and suppose that there exist matrices $P > 0$, $Q > 0$ and constants $\alpha \geq 0$, $\beta_i \geq 0$, $\gamma > 0$, $\sum_{i=1}^{m} \beta_i = 1$, such that

$$\begin{pmatrix} \sum\limits_{i=1}^{m} \beta_i (A_i \widetilde{P} + \widetilde{P} A_i^T) - \alpha \widetilde{P} & \sum\limits_{i=1}^{m} \beta_i G_i Q \\ \sum\limits_{i=1}^{m} \beta_i Q G_i^T & -\gamma Q \end{pmatrix} < 0, \tag{21a}$$

$$\frac{c_1}{\lambda_{\min}(P)} + \frac{\gamma d}{\lambda_{\min}(Q)} < \frac{c_2}{\lambda_{\max}(P)} e^{-\alpha T_f}. \tag{21b}$$

If the switching signal $\sigma(t)$ is designed as

$$\sigma(t) = \begin{cases} i, & \text{if } y^T(t)\Omega_i y(t) < 0 \text{ and } \sigma(t^-) = i; \\ \operatorname{argmin}\{y^T(t)\Omega_j y(t), j \in M\}, & \text{if } y^T(t)\Omega_i y(t) \geq 0 \text{ and } \sigma(t^-) = i, \end{cases} \tag{21c}$$

then system (3) is finite-time bounded with respect to $(c_1, c_2, d, T_f, R, \sigma)$, where argmin stands for the index which attains the minimum among M,
$\sigma(0) = \operatorname{argmin}\{y^T(0)\Omega_j y(0), j \in M\}$,

$$y(t) = \begin{pmatrix} x(t) \\ \omega(t) \end{pmatrix}, \Omega_i = \begin{pmatrix} \widetilde{P}^{-1} A_i + A_i^T \widetilde{P}^{-1} - \alpha \widetilde{P}^{-1} & \widetilde{P}^{-1} G_i \\ G_i^T \widetilde{P}^{-1} & -\gamma Q^{-1} \end{pmatrix}.$$

Proof. Choose a single Lyapunov-like function as follows

$$V(t) = x^T(t) \widetilde{P}^{-1} x(t).$$

According to (21c), we assume that the ith subsystem is active when $t \in [t_k, t_{k+1})$ without loss of generality, which means that $\sigma(t) = \sigma(t_k) = i$, when $t \in [t_k, t_{k+1})$. Then the derivative of $V(t)$ with respect to t along the trajectory of system (3) yields

$$\begin{aligned} \dot{V}(t) &= x^T(t)(\widetilde{P}^{-1} A_{\sigma(t_k)} + A_{\sigma(t_k)}^T \widetilde{P}^{-1}) x(t) + x^T(t) \widetilde{P}^{-1} G_{\sigma(t_k)} \omega(t) \\ &\quad + \omega^T(t) G_{\sigma(t_k)}^T P^{-1} x(t) \\ &= \begin{pmatrix} x^T(t), & \omega^T(t) \end{pmatrix} \begin{pmatrix} \widetilde{P}^{-1} A_{\sigma(t_k)} + A_{\sigma(t_k)}^T \widetilde{P}^{-1} & \widetilde{P}^{-1} G_{\sigma(t_k)} \\ G_{\sigma(t_k)}^T \widetilde{P}^{-1} & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ \omega(t) \end{pmatrix}. \end{aligned} \tag{22}$$

In what follows, we will show that

$$\begin{aligned} &\begin{pmatrix} x^T(t), & \omega^T(t) \end{pmatrix} \begin{pmatrix} \widetilde{P}^{-1} A_{\sigma(t_k)} + A_{\sigma(t_k)}^T \widetilde{P}^{-1} & \widetilde{P}^{-1} G_{\sigma(t_k)} \\ G_{\sigma(t_k)}^T \widetilde{P}^{-1} & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ \omega(t) \end{pmatrix} \\ &\qquad < \begin{pmatrix} x^T(t), & \omega^T(t) \end{pmatrix} \begin{pmatrix} \alpha \widetilde{P}^{-1} & 0 \\ 0 & \gamma Q^{-1} \end{pmatrix} \begin{pmatrix} x(t) \\ \omega(t) \end{pmatrix}, \end{aligned} \tag{23}$$

holds for any $t \in [t_k, t_{k+1})$.

If (23) is not true, then there exists a time $\bar{t} \in [t_k, t_{k+1})$ and a non-zero state $(x^T(\bar{t}), \omega^T(\bar{t}))^T$, such that

$$\begin{pmatrix} x^T(\bar{t}), & \omega^T(\bar{t}) \end{pmatrix} \begin{pmatrix} \widetilde{P}^{-1}A_{\sigma(t_k)} + A^T_{\sigma(t_k)}\widetilde{P}^{-1} - \alpha\widetilde{P}^{-1} & \widetilde{P}^{-1}G_{\sigma(t_k)} \\ G^T_{\sigma(t_k)}\widetilde{P}^{-1} & -\gamma Q^{-1} \end{pmatrix} \begin{pmatrix} x(\bar{t}) \\ \omega(\bar{t}) \end{pmatrix} \geq 0. \tag{24}$$

By the definitions of $y(t)$ and Ω_i, (24) implies that $y^T(\bar{t})\Omega_{\sigma(t_k)}y(\bar{t}) \geq 0$. Then according to the switching law (21c), we have $y^T(\bar{t})\Omega_j y(\bar{t}) \geq 0, \forall j \in M$. Since $\beta_i \geq 0$, $\sum_{i=1}^m \beta_i = 1$, we obtain $\sum_{i=1}^m \beta_i y^T(\bar{t})\Omega_i y(\bar{t}) \geq 0$, i. e.,

$$y^T(\bar{t}) \begin{pmatrix} \sum\limits_{i=1}^m \beta_i(\widetilde{P}^{-1}A_i + A_i^T\widetilde{P}^{-1}) - \alpha\widetilde{P}^{-1} & \sum\limits_{i=1}^m \beta_i\widetilde{P}^{-1}G_i \\ \sum\limits_{i=1}^m \beta_i G_i^T\widetilde{P}^{-1} & -\gamma Q^{-1} \end{pmatrix} y(\bar{t}) \geq 0. \tag{25}$$

Assuming condition (21a) is satisfied, then pre- and post-multiplying (21a) by the positive symmetric matrix $\begin{pmatrix} \widetilde{P}^{-1} & 0 \\ 0 & Q^{-1} \end{pmatrix}$, we obtain the equivalent condition

$$\begin{pmatrix} \sum\limits_{i=1}^m \beta_i(\widetilde{P}^{-1}A_i + A_i^T\widetilde{P}^{-1}) - \alpha\widetilde{P}^{-1} & \sum\limits_{i=1}^m \beta_i\widetilde{P}^{-1}G_i \\ \sum\limits_{i=1}^m \beta_i G_i^T\widetilde{P}^{-1} & -\gamma Q^{-1} \end{pmatrix} < 0. \tag{26}$$

Obviously, combining (25) and (26) leads a contradiction since $y(\bar{t}) \neq 0$. Thus, (23) holds for any $t \in [t_k, t_{k+1})$. By (22) and (23), we have for any $t \in [t_k, t_{k+1})$,

$$\dot{V}(t) < \begin{pmatrix} x^T(t), & \omega^T(t) \end{pmatrix} \begin{pmatrix} \alpha\widetilde{P}^{-1} & 0 \\ 0 & \gamma Q^{-1} \end{pmatrix} \begin{pmatrix} x(t) \\ \omega(t) \end{pmatrix} = \alpha V(t) + \gamma\omega^T(t)Q^{-1}\omega(t). \tag{27}$$

By calculation, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-\alpha t}V(t)\right) < e^{-\alpha t}\gamma\omega^T(t)Q^{-1}\omega(t). \tag{28}$$

Note that $V(t)$ remains continuous at any switching instant due to the definition of $V(t)$ and (28) holds for any subsystem if the subsystem is active. Integrating (28) from 0 to t gives

$$\begin{aligned} V(t) &< e^{\alpha t}V(0) + \gamma\int_0^t e^{\alpha(t-s)}\omega^T(s)Q^{-1}\omega(s)\,\mathrm{d}s \\ &\leq e^{\alpha T_f}V(0) + \gamma e^{\alpha T_f}\int_0^{T_f} \omega^T(s)Q^{-1}\omega(s)\,\mathrm{d}s \leq e^{\alpha T_f}\left(V(0) + \frac{\gamma d}{\lambda_{\min}(Q)}\right). \end{aligned} \tag{29}$$

On the other hand,

$$V(t) = x^T(t)\widetilde{P}^{-1}x(t) + \omega^T Q^{-1}\omega \geq \lambda_{\min}(P^{-1})x^T(t)Rx(t) = \frac{1}{\lambda_{\max}(P)}x^T(t)Rx(t), \tag{30}$$

$$V(0) = x^T(0)\widetilde{P}^{-1}x(0) \leq \lambda_{\max}(P^{-1})x^T(0)Rx(0) = [1/\lambda_{\min}(P)]x^T(0)Rx(0). \tag{31}$$

Putting together (29)-(31), we get

$$\begin{aligned} x^T(t)Rx(t) &\leq \lambda_{\max}(P)V(t) < \lambda_{\max}(P)e^{\alpha T_f}\left(V(0) + \gamma d/\lambda_{\min}(Q)\right) \\ &\leq \lambda_{\max}(P)e^{\alpha T_f}(c_1/\lambda_{\min}(P) + \gamma d/\lambda_{\min}(Q)). \end{aligned} \tag{32}$$

Assuming condition (21b) is satisfied, then we obtain $x^T(t)Rx(t) < c_2$. □

Remark 3.10. Note that even though each subsystem of system (3) is not finite-time bounded with respect to (c_1, c_2, d, T_f, R), the switched system (3) may still be finite-time bounded with respect to $(c_1, c_2, T_f, R, \sigma)$ by properly choosing the switching signal σ. In the case of asymptotical stability of switched systems, under the assumption that all the subsystems are not asymptotical stable, how to design the switching signals is an interesting topic. Here, in the case of finite-time boundedness of switched systems, we have discussed the similar topic in Theorem 3.9.

Similarly, in the case of finite-time stability for system (2), we have the following corollary.

Corollary 3.11. For any $i \in M$, let $\widetilde{P} = R^{-1/2}PR^{-1/2}$ and suppose that there exist a matrix $P > 0$ and constants $\alpha \geq 0$, $\beta_i \geq 0$, $\sum_{i=1}^m \beta_i = 1$, such that

$$\sum_{i=1}^{m} \beta_i(A_i\widetilde{P} + \widetilde{P}A_i^T) - \alpha\widetilde{P} < 0, \tag{33a}$$

$$\frac{c_1}{\lambda_{\min}(P)} + \frac{d}{\lambda_{\min}(Q)} < \frac{c_2}{\lambda_{\max}(P)}e^{-\alpha T_f}. \tag{33b}$$

If the switching signal $\sigma(t)$ is designed as

$$\sigma(t) = \begin{cases} i, & \text{if } x^T(t)\Omega_i x(t) < 0 \text{ and } \sigma(t^-) = i; \\ \operatorname{argmin}\{x^T(t)\Omega_j x(t), j \in M\}, & \text{if } x^T(t)\Omega_i x(t) \geq 0 \text{ and } \sigma(t^-) = i, \end{cases} \tag{33c}$$

then system (2) is finite-time stable with respect to $(c_1, c_2, T_f, R, \sigma)$, where $\sigma(0) = \operatorname{argmin}\{x^T(0)\Omega_j x(0), j \in M\}$, $\Omega_i = \widetilde{P}^{-1}A_i + A_i^T\widetilde{P}^{-1} - \alpha\widetilde{P}^{-1}$.

The multiple Lyapunov-like functions and the single Lyapunov-like function methods have been employed to design different switching signals in different cases. However, by the condition (4c), (21c), there are some constraints on the switching signals. Hence, Theorems 3.5 and 3.9 may not be suitable for the cases of fast switching or stochastic switching. In the following, we will give some conditions which guarantee system (3) finite-time bounded under an arbitrary switching.

3.3. Sufficient conditions for uniform finite-time boundedness

In the above two subsections, sufficient conditions for finite-time boundedness of switched linear system (3) are presented. To solve the case of arbitrary switching, the common Lyapunov-like function method will be used. In the sequel, some sufficient conditions are presented for finite-time boundedness of system (3) under un arbitrary switching, which is called uniform finite-time boundedness in this paper.

Definition 3.12. Given four positive constants c_1, c_2, d, T_f, with $c_1 < c_2$, and a positive definite matrix R, the switched linear system (3) is said to be uniformly finite-time bounded with respect to (c_1, c_2, d, T_f, R), if $x_0^T R x_0 \le c_1 \Rightarrow x(t)^T R x(t) < c_2, \forall\, t \in (0, T_f], \forall\, \omega(t) : \int_0^{T_f} \omega^T(t)\omega(t)\,\mathrm{d}t \le d$ holds for any switching signal $\sigma(t)$.

Remark 3.13. The meaning of 'uniformity' in Definition 3.12 and in [14] are identical, which is with respect to the switching signal, rather than time.

Theorem 3.14. For any $i \in M$, let $\widetilde{P} = R^{-1/2} P R^{-1/2}$ and suppose that there exist matrices $P > 0$, $Q > 0$ and constants $\alpha_i \ge 0$, $\gamma_i > 0$ such that

$$\begin{pmatrix} A_i\widetilde{P} + \widetilde{P}A_i^T - \alpha_i\widetilde{P} & G_iQ \\ QG_i^T & -\gamma_iQ \end{pmatrix} < 0, \tag{34a}$$

$$\frac{c_1}{\lambda_{\min}(P)} + \frac{\gamma d}{\lambda_{\min}(Q)} < \frac{c_2}{\lambda_{\max}(P)} e^{-\alpha T_f}, \tag{34b}$$

then system (3) is uniformly finite-time bounded with respect to (c_1, c_2, d, T_f, R), where $\alpha = \max_{\forall\, i\in M}(\alpha_i), \gamma = \max_{\forall\, i\in M}(\gamma_i)$.

Proof. Choose a common Lyapunov-like function as follows $V(x(t)) = x^T(t)\widetilde{P}^{-1}x(t)$. Substituting $\widetilde{P}_{\sigma(t)}, Q_{\sigma(t)}$ with $\widetilde{P}, Q$ into the proof of Theorem 3.5, it is easy to get the conclusion. □

4. FINITE-TIME STABILIZATION

Having given the finite-time boundedness analysis for the switched linear systems (3), in the following, let us investigate the finite-time stabilization issue. Here, in this paper, the following switching state feedback controller

$$u(t) = K_{\sigma(t)}x(t) \tag{35}$$

is designed to stabilize system (1). Substituting (35) into system (1), we get the closed-loop system (36) as follows

$$\dot{x}(t) = (A_{\sigma(t)} + B_{\sigma(t)}K_{\sigma(t)})x(t) + G_{\sigma(t)}\omega(t), \quad x(0) = x_0. \tag{36}$$

In the sequel, some sufficient conditions for finite-time stabilization and uniform finite-time stabilization of system (1) are presented, respectively.

Theorem 4.1. For any $i \in M$, let $\widetilde{P}_i = R^{-1/2} P_i R^{-1/2}$ and suppose that there exist matrices $P_i > 0$, $Q_i > 0$, M_i and constants $\alpha_i \geq 0$, $\gamma_i > 0$ such that

$$\begin{pmatrix} A_i \widetilde{P}_i + \widetilde{P}_i A_i^T + B_i M_i + M_i^T B_i^T - \alpha_i \widetilde{P}_i & G_i Q_i \\ Q_i G_i^T & -\gamma_i Q_i \end{pmatrix} < 0, \tag{37a}$$

$$\frac{c_1}{\lambda_1} + \frac{\gamma d}{\lambda_3} < \frac{c_2}{\lambda_2} e^{-\alpha T_f}. \tag{37b}$$

Then, under the feedback controller $u(t) = K_i x(t) = M_i \widetilde{P}_i^{-1} x(t)$ and any switching signal σ with average dwell-time satisfying

$$\tau_a \geq \tau_a^* = \frac{T_f \ln \mu}{\ln(c_2/\lambda_2) - \ln(c_1/\lambda_1 + \gamma d/\lambda_3) - \alpha T_f}, \tag{37c}$$

system (1) is finite-time bounded with respect to $(c_1, c_2, d, T_f, R, \sigma)$, where $\lambda_1 = \min_{\forall i \in M}(\lambda_{\min}(P_i))$, $\lambda_2 = \max_{\forall i \in M}(\lambda_{\max}(P_i))$, $\lambda_3 = \min_{\forall i \in M}(\lambda_{\min}(Q_i))$, $\alpha = \max_{\forall i \in M}(\alpha_i), \gamma = \max_{\forall i \in M}(\gamma_i), \mu = \lambda_2/\lambda_1$.

Proof. Applying Theorem 3.5 to the closed-loop system (36) and changing variables as $M_{\sigma(t)} \triangleq K_{\sigma(t)} \widetilde{P}_{\sigma(t)}$, it is easy to obtain the result. $\square$

Actually, in Theorem 4.1, it is easy to find that each subsystem of system (1) can be finite-time stabilized by linear state feedback controller (just keep $\sigma(t) = i, \forall i \in M$). However, in some cases, although each subsystem can not be finite-time stabilized by any linear state feedback, the whole switched system may still be finite-time stabilized under a suitable switching signal.

Theorem 4.2. For any $i \in M$, let $\widetilde{P} = R^{-1/2} P R^{-1/2}$ and suppose that there exist matrices $P > 0$, $Q > 0$, M_i, and constants $\alpha \geq 0$, $\beta_i \geq 0$, $\gamma > 0$, $\sum_{i=1}^m \beta_i = 1$, such that

$$\begin{pmatrix} \sum\limits_{i=1}^m \beta_i (A_i \widetilde{P} + \widetilde{P} A_i^T + B_i M_i + M_i^T B_i^T) - \alpha \widetilde{P} & \sum\limits_{i=1}^m \beta_i G_i Q \\ \sum\limits_{i=1}^m \beta_i Q G_i^T & -\gamma Q \end{pmatrix} < 0, \tag{38a}$$

$$\frac{c_1}{\lambda_{\min}(P)} + \frac{\gamma d}{\lambda_{\min}(Q)} < \frac{c_2}{\lambda_{\max}(P)} e^{-\alpha T_f}. \tag{38b}$$

Then, if the feedback controller is chosen as $u(t) = K_i x(t) = M_i \widetilde{P}^{-1} x(t)$ and the switching signal $\sigma(t)$ is designed as

$$\sigma(t) = \begin{cases} i, & \text{if } y^T(t)\Omega_i y(t) < 0 \text{ and } \sigma(t^-) = i; \\ \operatorname{argmin}\{y^T(t)\Omega_j y(t), j \in M\}, & \text{if } y^T(t)\Omega_i y(t) \geq 0 \text{ and } \sigma(t^-) = i, \end{cases} \tag{38c}$$

system (1) is finite-time bounded with respect to $(c_1, c_2, d, T_f, R, \sigma)$, where $\sigma(0) = \operatorname{argmin}\{y^T(0)\Omega_j y(0), j \in M\}$, $y^T(t) = (x^T(t), \omega^T(t))$,

$$\Omega_i = \begin{pmatrix} \widetilde{P}^{-1}(A_i + B_i M_i \widetilde{P}^{-1}) + (A_i + B_i M_i \widetilde{P}^{-1})^T \widetilde{P}^{-1} - \alpha \widetilde{P}^{-1} & \widetilde{P}^{-1} G_i \\ G_i^T \widetilde{P}^{-1} & -\gamma Q^{-1} \end{pmatrix}.$$

Proof. Applying Theorem 3.9 to the closed-loop system (36) and changing variables as $M_{\sigma(t)} \triangleq K_{\sigma(t)}\widetilde{P}$, it is easy to obtain the result. □

Consequently, the uniform finite-time stabilization problem for switched linear control system (1) can be easily solved by Theorem 3.14.

Theorem 4.3. For any $i \in M$, let $\widetilde{P} = R^{-1/2}PR^{-1/2}$ and suppose that there exist matrices $P > 0$, $Q > 0$, M_i and constants $\alpha_i \geq 0$, $\gamma_i > 0$ such that

$$\begin{pmatrix} A_i\widetilde{P} + \widetilde{P}A_i^T + B_iM_i + M_i^TB_i^T - \alpha_i\widetilde{P} & G_iQ \\ QG_i^T & -\gamma_iQ \end{pmatrix} < 0, \tag{39a}$$

$$\frac{c_1}{\lambda_{\min}(P)} + \frac{\gamma d}{\lambda_{\min}(Q)} < \frac{c_2}{\lambda_{\max}(P)}e^{-\alpha T_f}. \tag{39b}$$

Then, under the feedback controller $u(t) = K_ix(t) = M_i\widetilde{P}^{-1}x(t)$, system (1) is uniformly finite-time bounded with respect to (c_1, c_2, d, T_f, R), where $\alpha = \max_{\forall\, i\in M}(\alpha_i)$, $\gamma = \max_{\forall\, i\in M}(\gamma_i)$.

Proof. Applying Theorem 3.14 to the closed-loop system (36) and changing variables as $M_{\sigma(t)} \triangleq K_{\sigma(t)}\widetilde{P}$, we obtain the result. □

Remark 4.4. From a viewpoint of computation, two remarks are given. First, it should be noted that the conditions in Theorem 3.5 to Theorem 4.3, i. e., conditions (4a), (21a), (34a), (37a), (38a) and (39a), are not standard linear matrix inequalities (LMIs) conditions. Actually, these matrix inequalities are bilinear matrix inequalities (BMIs) [26, 27] due to the product of unknown scalars and matrices. As suggested in [15, 21], one possible way to compute the BMI problem is to grid up the unknown scalars, and then solve a set of LMIs for fixed values of these parameters. That is to say, once some values are fixed for α_i, β_i, γ_i, these conditions can be translated into LMIs conditions and thus solved involving Matlab's LMI control toolbox. Second, the conditions (4b), (21b), (34b), (37b), (38b) and (39b), are not linear matrix inequalities (LMIs) conditions. For convenience of computation, as in [3], these conditions can be guaranteed by the following LMI conditions, $\forall\, i \in M$,

$$\theta_1 I < P_i < I, \tag{40a}$$

$$\theta_2 I < Q_i, \tag{40b}$$

$$\begin{pmatrix} c_2e^{-\alpha T_f} & \sqrt{c_1} & \sqrt{\gamma d} \\ \sqrt{c_1} & \theta_1 & 0 \\ \sqrt{\gamma d} & 0 & \theta_2 \end{pmatrix} > 0, \tag{40c}$$

for some positive numbers θ_1 and θ_2. It should be pointed out that the condition (40) is just a sufficient condition for the conditions (4b), (21b), (34b), (37b), (38b) and (39b). To some extent, this sufficient condition is conservative.

Remark 4.5. Finite-time boundedness and stabilization problems have been discussed based on three kinds of Lyapunov-like functions. In practical applications, the choosing order is given as follows: First, we choose the common Lyapunov-like function method. If there exists a common Lyapunov-like function for all subsystems, then the finite-time boundedness can be guaranteed under an arbitrary switching. However, in many cases, it is difficult to find a common Lyapunov-like function. Second, we choose the multiple Lyapunov-like functions method. If there exist different Lyapunov-like functions for each subsystem, then the finite-time boundedness can be guaranteed if the switching is slow enough. However, if there are some subsystems which are not finite-time bounded, then the common Lyapunov-like function and the multiple Lyapunov-like functions methods can not be applied. Third, we need to employ the single Lyapunov-like function method. Even all the subsystems are not finite-time bounded, the whole system may still be finite-time bounded under a suitable switching signal.

5. NUMERICAL EXAMPLES AND SIMULATIONS

Example 5.1. Consider the switched linear system given by

$$\dot{x}(t) = A_{\sigma(t)}x(t) + G_{\sigma(t)}\omega \tag{41}$$

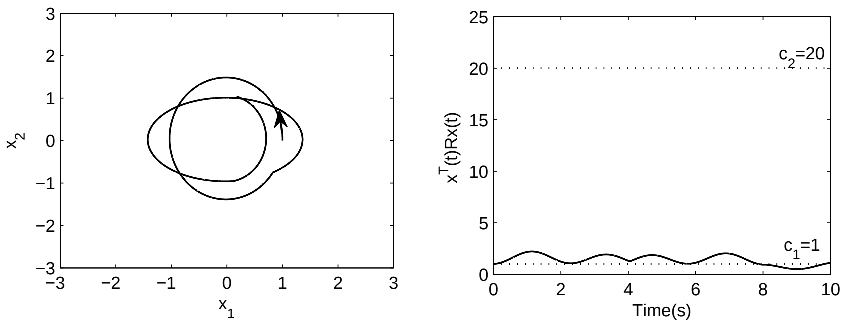
with $A_1 = \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}$, $G_1 = G_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $x(0) = (1 \;\; 0)^T$, $\omega(t) = (0.02\sin(2t+2) \;\; 0.01(\sin(3t) + \cos(0.5t)))^T$.

Note that each subsystem is not asymptotically stable because A_1 and A_2 are not Hurwitz. Then, we consider the finite-time boundedness. The parameters are given as $c_1 = 1$, $c_2 = 20$, $d = 0.0034$, $T_f = 10$ and matrix $R = I$. We first apply Theorem 3.14, i. e., the common Lyapunov-like function method. We can not find any feasible solution. Then, we apply Theorem 3.5 and solve corresponding matrix inequalities. Solving (4a) and (40) for $\alpha_i = \gamma_i = 0.01$ ($\forall\, i \in M$) leads to feasible solutions

$$P_1 = \begin{pmatrix} 0.4347 & 0.0001 \\ 0.0001 & 0.8693 \end{pmatrix}, P_2 = \begin{pmatrix} 0.8693 & 0.0001 \\ 0.0001 & 0.4347 \end{pmatrix},$$

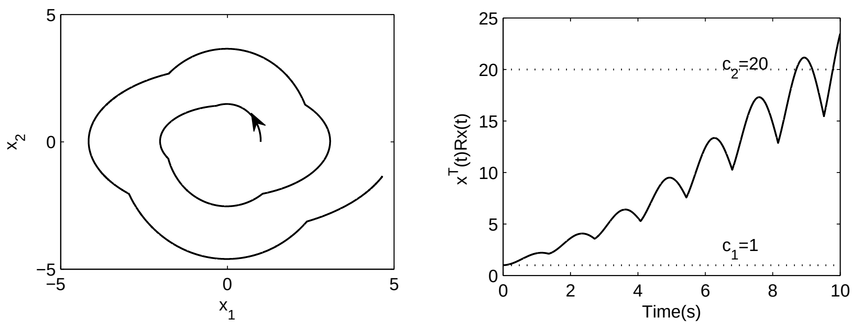
$$Q_1 = \begin{pmatrix} 0.0024 & 0.0000 \\ 0.0000 & 0.0029 \end{pmatrix}, Q_2 = \begin{pmatrix} 0.0029 & 0.0000 \\ 0.0000 & 0.0024 \end{pmatrix},$$

which satisfy the condition (4b). Therefore, according to (4c), for any switching signal $\sigma_1(t)$ with average dwell-time $\tau_a \geq \tau_a^* = 4.0245$, system (41) is finite-time bounded with respect to $(1, 20, 0.0034, 10, I, \sigma_1)$. Fig. 1(a) shows the state trajectory over $0 \sim 10s$ under a periodic switching signal σ_1 with interval time $\triangle T = 4.05s$ from the initial state $x(0)$. From Fig. 1(b), it is easy to see that system (41) is finite-time bounded with respect to $(1, 20, 0.0034, 10, I, \sigma_1)$. If the switching is too frequent, it is possible that the system is not finite-time bounded. Fig. 2(a) shows the state trajectory of system (41) over $0 \sim 10s$ under a periodic switching signal σ_2 with interval time $\triangle T = 1.35s$ from the initial state $x(0)$. From Fig. 1(b), we can find that system (41) is not finite-time bounded with respect to $(1, 20, 0.0034, 10, I, \sigma_2)$.



(a) Phase plot of state $x(t)$ (b) Time history of $x(t)^T Rx(t)$

Fig. 1. The response of system (41) under the switching signal σ_1.



(a) Phase plot of state $x(t)$ (b) Time history of $x(t)^T Rx(t)$

Fig. 2. The response of system (41) under the switching signal σ_2.

In the next example, we consider the case that each subsystem is not finite-time bounded. However, by properly choosing the switching signal, the switched system is still finite-time bounded.

Example 5.2. Consider the switched linear system given by

$$\dot{x}(t) = A_{\sigma(t)}x(t) + G_{\sigma(t)}\omega \tag{42}$$

with $A_1 = \begin{pmatrix} -0.7 & 0.4 & 0 \\ 0.67 & 0.7 & -0.7 \\ 0.7 & 1 & -1.4 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0.2 & 1.1 & 0 \\ 1.37 & -1.7 & -0.7 \\ 0.7 & 1 & 0.3 \end{pmatrix}$, $G_1 = G_2 = I$, $\omega(t) = \begin{pmatrix} 0.1\sin(t), & 0.1\cos(t), & -0.16(\sin(t) + 2\cos(t)) \end{pmatrix}^T$, $x(0) = (1, 0, 0)^T$.

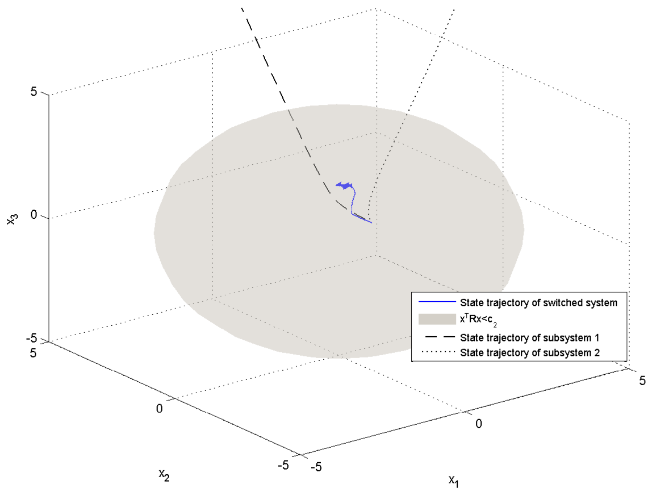


Fig. 3. The state trajectories for switched system (42) and each subsystem.

The parameters are given as $c_1 = 1$, $c_2 = 20$, $d = 0.37$, $T_f = 10$ and matrix $R = I$. Firstly, we apply Theorems 3.5 and 3.14, we can not get any feasible solution. Actually, it is shown that both the subsystems are not finite-time founded with respect to $(1, 20, 0.37, 10, I)$ by simulations. Fig. 3 shows the state trajectories of all subsystems over $0 \sim 10s$ from the initial state $x(0)$. Then, we apply Theorem 3.9 and solve corresponding matrix inequalities. Solving (21a) and (40) for $\alpha = 0.05$, $\beta_1 = 0.65, \beta_2 = 0.35$, $\gamma = 0.01$ leads to feasible solutions

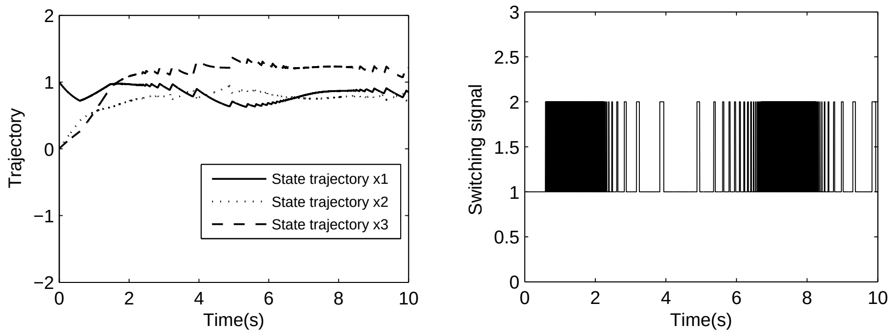
$$P = \begin{pmatrix} 0.5472 & -0.1919 & 0.2962 \\ -0.19190 & 0.5534 & 0.2545 \\ 0.2962 & 0.2545 & 0.7419 \end{pmatrix}, Q = \begin{pmatrix} 0.0057 & -0.0052 & 0.0003 \\ -0.0052 & 0.0071 & 0.0006 \\ 0.0003 & 0.0006 & 0.0040 \end{pmatrix}.$$

Therefore, according to Theorem 3.9, if the switching signal $\sigma_3(t)$ is designed as (21c), system (42) is finite-time bounded with respect to $(1, 20, 0.37, 10, I, \sigma_3)$. Fig. 3 shows the state trajectory of switched system (42) over $0 \sim 10s$ under the switching signal $\sigma_3(t)$ from the initial state $x(0)$. It is easy to see that system (42) is finite-time bounded with respect to $(1, 20, 0.37, 10, I, \sigma_3)$ from Fig. 4(a), which presents the state trajectory of system (42) under the switching signal $\sigma_3(t)$. Moreover, the switching signal $\sigma_3(t)$ is shown in Fig. 4(b).

Example 5.3. Consider the finite-time stabilization problem for switched system

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) + G_{\sigma(t)}\omega(t). \tag{43}$$

The corresponding parameters are specified as follows



(a) The state trajectory of switched system (42) (b) The switching signal σ_3

Fig. 4. The state trajectory of switched system (42) and the switching signal σ_3.

$A_1 = \begin{pmatrix} 1 & 3 \\ 0 & -0.25 \end{pmatrix}, B_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0.01 & 0 \\ -1 & -3 \end{pmatrix}, B_2 = \begin{pmatrix} 0 \\ 0.5 \end{pmatrix}, G_1 =$ $G_2 = I, \omega(t) = \begin{pmatrix} -0.4\cos(10t+3) \\ 0.2\sin(3t) \end{pmatrix}, c_1 = 1, c_2 = 10, d = 1, T_f = 1, R = I.$

By Theorem 4.3, solving (39a) and (40), we obtain the matrix solutions for $\alpha_i = \gamma_i = 0.6$ ($\forall\, i \in M$) as follows $P = \begin{pmatrix} 0.9115 & 0 \\ 0 & 0.8934 \end{pmatrix}, Q = \begin{pmatrix} 0.3054 & 0 \\ 0 & 0.4798 \end{pmatrix}$, $M_1 = \begin{pmatrix} -50.3443 & -2.6803 \end{pmatrix}, M_2 = \begin{pmatrix} 1.8230 & -93.8059 \end{pmatrix}$. Thus, according to Theorem 4.3, under the following state feedback controllers

$$u_1(t) = \begin{pmatrix} -55.2313 & -3.0000 \end{pmatrix} x(t), \quad u_2(t) = \begin{pmatrix} 2.0000 & -104.9933 \end{pmatrix} x(t),$$

system (43) is uniformly finite-time bounded with respect to $(1, 10, 1, 1, I)$.

6. CONCLUSION

In this paper, finite-time boundedness and stabilization problems have been investigated for a class of switched linear systems. Some sufficient conditions have been provided for finite-time boundedness of switched linear systems. In addition, the finite-time stabilization problem has also been studied. A challenging and interesting future research topic is how to extend the results in this paper to switched nonlinear systems.

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