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GENERALIZED COMMUNICATION CONDITIONS AND THE EIGENVALUE PROBLEM FOR A MONOTONE AND HOMOGENOUS FUNCTION

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This work is concerned with the eigenvalue problem for a monotone and homogenous self-mapping f of a finite dimensional positive cone. Paralleling the classical analysis of the (linear) Perron–Frobenius theorem, a verifiable communication condition is formulated in terms of the successive compositions of f, and under such a condition it is shown that the upper eigenspaces of f are bounded in the projective sense, a property that yields the existence of a nonlinear eigenvalue as well as the projective boundedness of the corresponding eigenspace. The relation of the communication property studied in this note with the idea of indecomposability is briefly discussed.

Keywords: projectively bounded and invariant sets, generalized Perron–Frobenius conditions, nonlinear eigenvalue, Collatz–Wielandt relations

Classification: 47J10, 47H09, 47H07

Dedicated to the memory of *Don Miguel Hidalgo y Costilla*

1. INTRODUCTION

This work is concerned with the class of monotone and homogeneous self-mappings of a finite-dimensional positive cone $\mathcal{P}$. Given one of those mappings, say f, the corresponding eigenvalue problem consists in finding verifiable conditions ensuring the existence of a pair $(\lambda, \mathbf{x}) \in (0, \infty) \times \mathcal{P}$ such that λ is an eigenvalue of f and $\mathbf{x}$ is an eigenvector corresponding to λ, that is, the equation $f(\mathbf{x}) = \lambda \mathbf{x}$ is satisfied, a problem that was recently studied in Gaubert and Gunawardena [5] via the idea of indecomposable function stated in Section 6 below. On the other hand, a classical solution of this problem, namely, the Perron–Frobenius theorem, establishes that if f is a linear function associated with a nonnegative and *communicating* matrix, then f has a unique positive eigenvalue $\lambda(f)$ and, moreover, two different eigenvectors associated with $\lambda(f)$ are linearly dependent (Seneta [12], Minc [9]). *The main objective* of the paper is to formulate a generalized communication condition, based on the idea of communicating matrix and applicable to a general monotone and homogeneous function f, under which the existence of a (necessarily unique) nonlinear eigenvalue $\lambda(f)$ can be ensured. On the other hand, in the nonlinear case it is known that two different eigenvectors may be linearly independent, and under the conditions of the paper it will be shown that the eigenspace associated with the eigenvalue $\lambda(f)$ is projectively bounded, a property that can be roughly described as follows: There exists a constant $b \geq 1$ such that, if $\mathbf{x}$ is an eigenvector of f, then the quotient of two components of $\mathbf{x}$ is always between $1/b$ and b.

The properties of monotone and homogeneous functions have been studied from diverse perspectives: Nussbaum [10, 11] analyzes the limit points of the successive compositions, connections with option pricing and max-plus algebra are considered in Kolokoltsov [7] and Gunawardena [6], respectively, whereas Lemmens and Scheutzow [8] gave an estimation of the period of periodic points; see also [2]. On the other hand, it is well-known that the class of monotone and homogeneous functions also arise in other fields, like mathematical economics, game theory and risk-sensitive optimal control, where finding an eigenvalue and an associated eigenvector corresponds to the problem of solving the dynamic programming equation; see, for instance, Zijm [13], Dellacherie [4], Akian and Gaubert [1], Cavazos–Cadena and Hernández–Hernández [3] and the references therein.

The approach used in this note relies heavily on the ideas of projectively bounded and invariant sets, which will be precisely stated in Section 2. These notions are involved in the following characterization result which plays a central role in the subsequent development: A monotone and homogeneous mapping f has an eigenvalue if and only if there exists a projectively bounded set which is invariant with respect to f. To apply this theorem, two canonical invariant sets associated with f are considered—namely, the upper and lower eigenspaces—and under appropriate communication conditions it is shown that those sets are projectively bounded, obtaining the existence of an eigenvalue form the above theorem.

The organization of the paper is as follows: In Section 2 the eigenvalue problem is formally described, the basic ideas are introduced, and the fundamental criterion on the existence of a nonlinear eigenvalue is stated; after this first step, the classical Perron–Frobenius theorem and the nonlinear extension by Gaubert and Gunawardena [5] are briefly discussed. Next, in Section 3 a communication assumption generalizing the one in the Perron–Frobenius theorem is formulated as Assumption 3.1, and after this point the main results of the paper are established. Thus, in Section 4 it is shown that if a monotone and homogeneous function f satisfies the generalized communication requirement, then the nonempty upper eigenspaces are projectively bounded, a property that yields the existence of a nonlinear eigenvalue, as well as the projective boundedness of the eigenspace associated with f. Next, a similar result involving lower eigenspaces is obtained via the idea of dual function in Section 5 and, finally, in Section 6 the relation between the the communication property in Assumption 3.1 and the notion of indecomposable function in Gaubert and Gunawardena [5] is analyzed. At this point, an example is given to show that the indecomposability requirement is weaker than Assumption 3.1.

Notation. Throughout the remainder $\vee$ and $\wedge$ are used as infix notations for the maximum and minimum operators, respectively, that is, for real numbers a and b,

$a \vee b := \max\{a, b\}$ and $a \wedge b := \min\{a, b\}$, whereas

$$\bigvee_{i=1}^{k} a_i = \max\{a_1, a_2, \dots, a_k\} \quad \text{and} \quad \bigwedge_{i=1}^{k} a_i = \min\{a_1, a_2, \dots, a_k\}$$

All vectors considered below are column vectors and, given a positive integer n, $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ stands for the canonical basis of $\mathbb{R}^n$, so that $\mathbb{1} := \mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_n \in \mathbb{R}^n$ is the vector with all its components equal to 1. Finally, for $\mathbf{x} = (x_1, x_2, \dots, x_n)' \in \mathbb{R}^n$, the following notation is used: $\max(\mathbf{x}) := \bigvee_{i=1}^{n} x_i$ and $\min(\mathbf{x}) := \bigwedge_{i=1}^{n} x_i$, and if $\emptyset \neq J \subset \{1, 2, \dots, n\}$ then

$$\mathbf{e}_J := \sum_{k \in J} \mathbf{e}_k. \tag{1.1}$$

2. THE EIGENVALUE PROBLEM

In the subsequent development $n \geq 2$ is a fixed integer and $\mathcal{MH}_n$ denotes the class of all monotone and homogeneous self-mappings of the n-dimensional positive cone $\mathcal{P}_n := (0, \infty)^n$, that is, $f \equiv (f_1, \dots, f_n)' : \mathcal{P}_n \to \mathcal{P}_n$ belongs to $\mathcal{MH}_n$ if and only if the following *homogeneity* and *monotonicity* properties hold:

$$f(\alpha \mathbf{x}) = \alpha f(\mathbf{x}), \quad \mathbf{x} \in \mathcal{P}_n, \quad \alpha > 0, \tag{2.1}$$

and

$$f(\mathbf{x}) \leq f(\mathbf{y}) \quad \text{if } \mathbf{x}, \mathbf{y} \in \mathcal{P}_n \text{ are such that } \mathbf{x} \leq \mathbf{y}, \tag{2.2}$$

where the inequalities between vectors are interpreted componentwise. Notice that $\mathcal{MH}_n$ is itself a cone, that is, if $f, g \in \mathcal{MH}_n$ and $c > 0$, then $cf, f + g \in \mathcal{MH}_n$; moreover $\mathcal{MH}_n$ is closed under compositions and under the operations of taking the maximum and minimum, i.e., $f, g \in \mathcal{MH}_n$ implies that $f(g) \in \mathcal{MH}_n$ and $f \vee g, f \wedge g \in \mathcal{MH}_n$. In particular,

$$f \in \mathcal{MH}_n \implies \sum_{k=0}^{n-1} f^k, \quad \bigwedge_{k=0}^{n-1} f^k, \quad \bigvee_{k=0}^{n-1} f^k \in \mathcal{MH}_n, \tag{2.3}$$

where

$$f^k = (f_1^k, \dots, f_n^k)'$$

is the k-fold composition of f with itself, and f^0 is the identity function on $\mathcal{P}_n$; observe that

$$f_i^{k+1}(\mathbf{x}) = f_i^k(f(\mathbf{x})), \quad k = 0, 1, 2, \dots, \; i = 1, 2, \dots, n. \tag{2.4}$$

As already noted, the eigenvalue problem for a function $f \in \mathcal{MH}_n$ consists in providing (verifiable) conditions guaranteeing the existence of a pair $(\lambda, \mathbf{y}) \in (0, \infty) \times \mathcal{P}_n$ such that

$$f(\mathbf{y}) = \lambda \mathbf{y}; \tag{2.5}$$

when this equation holds, λ is an eigenvalue of f and $\mathbf{y}$ is an eigenvector. If (2.5) has a solution then the eigenvalue is uniquely determined by the following Collatz–Wielandt relations (Minc [9], Gaubert and Gunawardena [5]):

$$\inf_{\mathbf{z}\in\mathcal{P}_n} \max\{f_i(\mathbf{z})/z_i \;\; i=1,2,\ldots,n\} = \lambda = \sup_{\mathbf{z}\in\mathcal{P}_n} \min\{f_i(\mathbf{z})/z_i \;\; i=1,2,\ldots,n\}. \quad (2.6)$$

The existence of an eigenvalue of a function $f\in\mathcal{MH}_n$ can be characterized in terms of the following ideas of projectively bounded and invariant sets.

Definition 2.1. (*i*) A nonempty set $B\subset\mathcal{P}_n$ is projectively bounded if

$$\sup_{\mathbf{y}\in B}\frac{\max(\mathbf{y})}{\min(\mathbf{y})}<\infty.$$

(*ii*) A set $B\subset\mathcal{P}_n$ is invariant with respect to $f\in\mathcal{MH}_n$—for brevity, f-invariant—if

$$\mathbf{x}\in B \implies f(\mathbf{x})\in B.$$

The following result is an immediate consequence of Theorem 4.1 in Nussbaum [10] (where non-expansive mappings acting on cons in Banach spaces were studied), and was stated and proved as Theorem 3 in Gaubert and Gunawardena [5] in the finite-dimensional context.

Theorem 2.1. For each $f\in\mathcal{MH}_n$ the following conditions (*i*) and (*ii*) are equivalent:

(*i*) The function f has an eigenvalue, that is, there exists a pair $(\lambda,\mathbf{y})\in(0,\infty)\times\mathcal{P}_n$ such that $f(\mathbf{y})=\lambda\mathbf{y}$.

(*ii*) There exists a nonempty set $B\subset\mathcal{P}_n$ which is projectively bounded and invariant with respect to f.

This theorem shows that the existence of an eigenvalue of a function $f\in\mathcal{MH}_n$ can be ensured by providing conditions so that a nonempty f-invariant set is projectively bounded. The following two invariant sets can be always associated with a general $f\in\mathcal{MH}_n$ and will play a central role in the subsequent development.

Definition 2.2. Let $f\in\mathcal{MH}_n$ and $a>0$ be arbitrary. The upper and lower eigenspaces associated with the pair (f,a) are defined by

$$S^a(f):=\{\mathbf{x}\in\mathcal{P}_n \mid f(\mathbf{x})\le a\mathbf{x}\} \quad \text{and} \quad S_a(f):=\{\mathbf{x}\in\mathcal{P}_n \mid f(\mathbf{x})\ge a\mathbf{x}\},$$

respectively.

The following simple result will be useful.

Lemma 2.1. For each $f\in\mathcal{MH}_n$ and $a>0$, the assertions (*i*)–(*ii*) below hold:

(i) The upper and lower eigenspaces $S^a(f)$ and $S_a(f)$ are f-invariant;

(ii) If $a\ge\max(f(\mathbb{1}))$ then $S^a(f)$ is nonempty; similarly, $S_a(f)\ne\emptyset$ for each $a\le\min(f(\mathbb{1}))$.

P r o o f. Via (2.1) and (2.2), the first part follows from Definition 2.2. On the other hand, observing that $f(\mathbb{1}) \leq \max(f(\mathbb{1}))\mathbb{1}$, it follows that $\mathbb{1} \in S^a(f)$ when $a \geq \max(f(\mathbb{1}))$ and, similarly, $\mathbb{1} \in S_b(f)$ if $b \leq \min(f(\mathbb{1}))$, establishing the second assertion. □

In the remainder of the section the classical solution of the eigenvalue problem for a linear function, as well as a recent extension to the nonlinear case, are briefly discussed.

The Linear Case. Let $A = [A_{i\,j}]$ be a nonnegative matrix of order $n \times n$ with non-null rows, and define

$$f_A(\mathbf{x}) := A\mathbf{x}, \quad \mathbf{x} \in \mathcal{P}_n, \tag{2.7}$$

so that $f_A \in \mathcal{MH}_n$. The classical Perron–Frobenius theorem stated below, which provides an answer to the eigenvalue problem for such a function f_A, involves the following idea of communicating matrix.

Definition 2.3. Let $A = [A_{i\,j}]$ be a nonnegative matrix of order $n \times n$. In this case, A is communicating if the following condition holds: For each $i, j \in \{1, 2, \ldots, n\}$ there exist $i_1, \ldots, i_k \in \{1, 2, \ldots, n\}$ satisfying

$$i_0 = i, \quad i_k = j \quad \text{and} \quad A_{i_{r-1}\,i_r} > 0, \quad r = 1, 2, \ldots k. \tag{2.8}$$

Theorem 2.2. [Perron–Frobenius] Given a nonnegative matrix A of order $n \times n$, let $f_A(\cdot)$ be as in (2.7). In this context, if

$$A \text{ is communicating,} \tag{2.9}$$

then assertion (i) and (ii) below hold.

(i) The function f_A has an eigenvalue, i. e., there exists a pair $(\lambda, \mathbf{y}) \in (0, \infty) \times \mathcal{P}_n$ such that $f_A(\mathbf{y}) = \lambda\mathbf{y}$.

Moreover,

(ii) If $(\lambda_1, \mathbf{y}_1) \in (0, \infty) \times \mathcal{P}_n$ satisfies $f_A(\mathbf{y}_1) = \lambda_1\mathbf{y}_1$, then $\lambda = \lambda_1$ and $\mathbf{y} = c\mathbf{y}_1$ for some $c > 0$, so that f_A has a unique eigenvalue and the the corresponding eigenvector is unique up to a multiplicative constant.

A proof of this result can be seen, for instance, in Seneta [12], or Minc [9].

A Nonlinear Perron–Frobenius Theorem. An extension of Theorem 2.2 for a general $f \in \mathcal{MH}_n$ was recently given in Gaubert and Gunawardena [5]. The conclusions in that paper involve a communication matrix $M(f)$ associated with f, whose specification is motivated by the following remarks concerning the linear case: If A is a nonnegative matrix, then

(a) The property of being communicating does not depend on the exact values of the entries of A, but just on which components of A are non-null, and

(b) The positivity of an entry of matrix A is characterized by

$$A_{i\,j} > 0 \iff \lim_{t\to\infty} f_{A,i}(t\mathbf{e}_j + \mathbb{1}) = \lim_{t\to\infty}\left[A_{i\,j}t + \sum_{k=1}^{n} A_{i\,k}\right] = \infty.$$

These properties naturally lead to formulate the following idea.

Definition 2.4. For a given function $f \in \mathcal{MH}_n$, the corresponding (communication) matrix $M(f) \equiv [M(f)_{i\,j}]$ of order $n \times n$ is defined as follows: For $i, j = 1, 2, \ldots, n$,

$$\begin{aligned} M(f)_{i\,j} &:= 1, \text{ if } \lim_{t\to\infty} f_i(t\mathbf{e}_j + \mathbb{1}) = \infty \\ &:= 0, \text{ otherwise.} \end{aligned} \tag{2.10}$$

Notice that for each $f \in \mathcal{MH}_n$ and $i, j \in \{1, 2, \ldots, n\}$, the mapping $t \mapsto f_i(t\mathbf{e}_j + \mathbb{1})$ is increasing in $t \in (-1, \infty)$, by (2.2), so that the limit in (2.10) certainly exists. The first part of the following result was established as Theorem 2 in Gaubert and Gunawardena [5], and then the part (ii) was obtained via Theorem 2.1 and (2.6).

Theorem 2.3. Let $f \in \mathcal{MH}_n$ be such that the matrix

$$M(f) \text{ is communicating.} \tag{2.11}$$

In this case the following assertions (i) and (ii) hold:

(i) If $a > 0$ is such that the upper eigenspace $S^a(f)$ is non-empty, then $S^a(f)$ is projectively bounded.

Consequently,

(ii) The function f has an eigenvalue, that is, there exists a pair $(\lambda, \mathbf{y}) \in (0, \infty) \times \mathcal{P}_n$ such that $f(\mathbf{y}) = \lambda\mathbf{y}$, where the eigenvalue $\lambda \equiv \lambda(f)$ is uniquely determined by (2.6). Moreover, the corresponding eigenspace $\mathcal{E}(f)$, specified by

$$\mathcal{E}(f) := \{\mathbf{x} \in \mathcal{P}_n \mid f(\mathbf{x}) = \lambda(f)\mathbf{x}\}, \tag{2.12}$$

is projectively bounded.

If $f_A \in \mathcal{MH}_n$ is as in (2.7), form Definitions 2.3 and 2.4 it follows that the matrix A is communicating if and only if so is $M(f_A)$, and then the existence of an eigenvalue of f_A can be obtained by an application of Theorem 2.3 to the linear function f_A. On the other hand, contrasting with the linear case, if a general $f \in \mathcal{MH}_n$ satisfies (2.11), then two different eigenvectors may be linearly independent, a fact that was illustrated in Gaubert and Gunawardena [5, p. 4932].

3. A GENERALIZED COMMUNICATION CONDITION

In this section a communication property of a function $f \in \mathcal{MH}_n$ is formally stated as Assumption 3.1 below. Such a requirement extends the idea of communicating matrix in the Perron–Frobenius theorem, and is weaker than the requirement (2.11) used in Theorem 2.3. The starting point is the following simple characterization result.

Lemma 3.1. Let A be a nonnegative matrix of order $n \times n$ whose rows are non-null, and let $f_A(\cdot)$ be as in (2.7). In this context, properties (i)–(iii) below are equivalent.
(i) A is communicating.
(ii) $\mathcal{B} := I + A + A^2 + \cdots + A^{n-1} > 0$.
(iii) For each $i, j \in \{1, 2, \ldots, n\}$, $\lim_{t\to\infty} \sum_{k=0}^{n-1} f^k_{A,i}(t\mathbf{e}_j + \mathbb{1}) = \infty$.

P r o o f. $(i) \implies (ii)$: Assume that A is communicating, let i and j be two different integers between 1 and n and let $k(i,j) \equiv k$ be the smallest integer such that (2.8) holds. From this specification of k it follows that the integers $i_0, i_1, \ldots, i_k$ in (2.8) are different elements of $\{1, 2, \ldots, n\}$, so that $k \le n-1$ and then $\mathcal{B} \ge A^k$, a relation that yields that $\mathcal{B}_{i\,j} \ge A^k_{i\,j} = A^k_{i_0\,i_k} \ge A_{i_0\,i_1} A_{i_1\,i_2} \cdots A_{i_{k-1},i_k} > 0$; since $\mathcal{B}_{i\,i} \ge I_{i\,i} = 1$ for every $i \in \{1, 2, \ldots, n\}$, it follows that $\mathcal{B} > 0$.
$(ii) \implies (i)$: Assume that $\mathcal{B} > 0$ and let i and j be different integers between 1 and n. In this case $\mathcal{B}_{i\,j} = \sum_{k=1}^{n-1} A^k_{i\,j} > 0$, so that there exists an integer $k(i,j)$ between 1 and $n-1$ such that

$$A^k_{i\,j} > 0, \quad k = k(i,j).$$

Setting $r = k(j,i)$ it follows that $A^r_{j\,i} > 0$ and then $A^{k(i,j)+k(j,i)}_{i\,i} \ge A^{k(i,j)}_{i\,j} A^{k(j,i)}_{j\,i} > 0$, that is

$$A^s_{i\,i} > 0, \quad s = k(i,j) + k(j,i).$$

From these two last displays it follows immediately that for every $i, j \in \{1, 2, \ldots, n\}$ there exist a positive integer r and $i_0, i_1, \ldots, i_r \in \{1, 2, \ldots, n\}$ such that (2.8) holds, i. e., A is communicating.
$(ii) \iff (iii)$: From (2.7) it follows that $f^k_A = f_{A^k}$, and then $f_{\mathcal{B}} = \sum_{k=0}^{n-1} f^k_A$. Consequently, for each $i, j \in \{1, 2, \ldots, n\}$

$$\sum_{k=0}^{n-1} f^k_{A,i}(t\mathbf{e}_j + \mathbb{1}) = f_{\mathcal{B},i}(t\mathbf{e}_j + \mathbb{1}) = t\mathcal{B}_{i\,j} + \sum_{s=1}^{n} \mathcal{B}_{i\,s}$$

and the equivalence of (ii) and (iii) follows immediately form this display. □

From the equivalence of properties (i) and (iii) in Lemma 3.1 it follows that the communication condition in Theorem 2.2, implying the existence of an eigenvalue for the linear function f_A in (2.7), can be naturally extended to the case of a general $f \in \mathcal{MH}_n$ as follows.

Assumption 3.1. For each $i, j \in 1, 2, \ldots, n$,

$$\lim_{t\to\infty} \sum_{k=0}^{n-1} f^k_i(t\mathbf{e}_j + \mathbb{1}) = \infty. \tag{3.1}$$

Remark 3.1. (i) Notice that

$$\bigvee_{k=0}^{n-1} f^k \le \sum_{k=0}^{n-1} f^k \le n \bigvee_{k=0}^{n-1} f^k, \tag{3.2}$$

so that Assumption 3.1 can be equivalently stated in the following way:
For each $i, j \in \{1, 2, \ldots, n\}$,

$$\lim_{t\to\infty} \bigvee_{k=0}^{n-1} f_i^k(t\mathbf{e}_j + \mathbb{1}) = \infty.$$

(ii) With the notation in Lemma 3.1 notice that $\sum_{k=0}^{n-1} f_{A,i}^k(\mathbf{e}_j + s\mathbb{1}) = f_{\mathcal{B},i}(\mathbf{e}_j + s\mathbb{1}) = \mathcal{B}_{i\,j} + s\sum_{r=1}^{n} \mathcal{B}_{i\,r}$, so that $\mathcal{B}_{i\,j} > 0$ is equivalent to $\lim_{s\to 0}\sum_{k=0}^{n-1} f_{A,i}^k(\mathbf{e}_j + s\mathbb{1}) > 0$. Thus, the communication property in Theorem 2.2 can be also extended to a general $f \in \mathcal{MH}_n$ as follows:

$$\text{For all } i, j \in \{1, 2, \ldots, n\}, \quad \lim_{s\to 0} \sum_{k=0}^{n-1} f_i^k(\mathbf{e}_j + s\mathbb{1}) > 0, \tag{3.3}$$

a requirement that is stronger than Assumption 3.1. To verify this assertion, let $f \in \mathcal{MH}_n$ be arbitrary, and notice that the homogeneity of $\sum_{k=0}^{n-1} f^k$ yields that, for all $t > 0$ and $i, j \in \{1, 2, \ldots, n\}$

$$\sum_{k=0}^{n-1} f_i^k(t\mathbf{e}_j + \mathbb{1}) = t\sum_{k=0}^{n-1} f_i^k\left(\mathbf{e}_j + \frac{1}{t}\mathbb{1}\right);$$

see (2.3). After taking the limit as t goes to ∞ in this relation, it follows that f satisfies Assumption 3.1 when (3.3) holds.

(iii) Let $f \in \mathcal{MH}_n$ be arbitrary. Since each $\mathbf{x} \in \mathcal{P}_n$ satisfies $\min(\mathbf{x})\mathbb{1} \le \mathbf{x} \le \max(\mathbf{x})\mathbb{1}$, the homogeneity and monotonicity properties of f^k yield that, for each $k \ge 0$ and $j \in \{1, 2, \ldots, n\}$

$$\begin{aligned}
\min(\mathbf{x}) f^k\left(\frac{t}{\min(\mathbf{x})}\mathbf{e}_j + \mathbb{1}\right) &= f^k(t\mathbf{e}_j + \min(\mathbf{x})\mathbb{1}) \\
&\le f^k(t\mathbf{e}_j + \mathbf{x}) \\
&\le f^k(t\mathbf{e}_j + \max(\mathbf{x})\mathbb{1}) = \max(\mathbf{x}) f^k\left(\frac{t}{\max(\mathbf{x})}\mathbf{e}_j + \mathbb{1}\right)
\end{aligned}$$

so that

$$\min(\mathbf{x})\sum_{k=0}^{n-1} f^k\left(\frac{t}{\min(\mathbf{x})}\mathbf{e}_j + \mathbb{1}\right) \le \sum_{k=0}^{n-1} f(t\mathbf{e}_j + \mathbf{x}) \le \max(\mathbf{x})\sum_{k=0}^{n-1} f^k\left(\frac{t}{\max(\mathbf{x})}\mathbf{e}_j + \mathbb{1}\right).$$

Therefore, Assumption 3.1 is equivalent to the following requirement: For some $\mathbf{x} \in \mathcal{P}_n$

$$\lim_{t\to\infty} \sum_{k=0}^{n-1} f_i(t\mathbf{e}_j + \mathbf{x}) = \infty, \quad i, j \in \{1, 2, \ldots, n\},$$

and in this case this property holds for every $\mathbf{x} \in \mathcal{P}_n$.

Next, it will be shown that Assumption 3.1 is weaker than the condition (2.11) used to establish the conclusions of Theorem 2.3.

Theorem 3.1. For each $f \in \mathcal{MH}_n$ the following assertion is valid:

If the matrix $M(f)$ is communicating then Assumption 3.1 is satisfied by f;

see Definitions 2.3 and 2.4.

Proof. Given $f \in \mathcal{MH}_n$, for each integer $k \geq 0$ consider the following claim:

$$M(f)^k_{i\,j} > 0 \implies \lim_{t\to\infty} f_i^k(t\mathbf{e}_j + \mathbb{1}) = \infty, \quad i, j \in \{1, 2, \ldots, n\}. \tag{3.4}$$

It will be proved, by induction in k, that this assertion is always valid. First, notice that since $M(f)^0$ is the identity matrix and f^0 is the identity map in $\mathcal{P}_n$, (3.4) certainly holds for $k = 0$. Assume now that (3.4) is valid for some integer $k \geq 0$ and let $i, j \in \{1, 2, \ldots, n\}$ be such that

$$M(f)^{k+1}_{i\,j} > 0. \tag{3.5}$$

Since $M(f)^{k+1}_{i\,j} = \sum_{r=1}^n M(f)^k_{i\,r} M_{r\,,j}(f)$, there exists an integer r such that

$$M(f)^k_{i\,r} > 0 \quad \text{and} \quad M(f)_{r\,j} > 0.$$

In this case, the induction hypothesis yields that

$$\lim_{s\to\infty} f_i^k(s\mathbf{e}_r + \mathbb{1}) = \infty, \tag{3.6}$$

whereas the specification of the matrix $M(f)$ implies that

$$\lim_{t\to\infty} \varphi(t) = \infty, \quad \text{where } \varphi(t) := f_r(t\mathbf{e}_j + \mathbb{1}). \tag{3.7}$$

Notice now that for every $t \geq 0$ the monotonicity property of f yields that

$$f(t\mathbf{e}_j + \mathbb{1}) \geq f(\mathbb{1}) \geq a\mathbb{1},$$

where $a := \min(f(\mathbb{1}))$, and then

$$\begin{aligned} f(t\mathbf{e}_j + \mathbb{1}) &= \sum_{m=1}^n f_m(t\mathbf{e}_j + \mathbb{1})\mathbf{e}_m \\ &= \varphi(t)\mathbf{e}_r + \sum_{m\neq r} f_m(t\mathbf{e}_j + \mathbb{1})\mathbf{e}_m \\ &\geq \varphi(t)\mathbf{e}_r + \sum_{m\neq r} a\mathbf{e}_m = a\left[\frac{\varphi(t) - a}{a}\mathbf{e}_r + \mathbb{1}\right]. \end{aligned}$$

Since $f^k \in \mathcal{MH}_n$ and $f_i^{k+1}(t\mathbf{e}_j + \mathbb{1}) = f_i^k(f(t\mathbf{e}_j + \mathbb{1}))$ (see (2.4)), the above display yields that

$$f_i^{k+1}(t\mathbf{e}_j + \mathbb{1}) \geq f_i^k\left(a\left[\frac{\varphi(t) - a}{a}\mathbf{e}_r + \mathbb{1}\right]\right) = a f_i^k\left(\frac{\varphi(t) - a}{a}\mathbf{e}_r + \mathbb{1}\right)$$

and combining this relation with (3.6) and (3.7) it follows that

$$\lim_{t\to\infty} f_i^{k+1}(t\mathbf{e}_j + \mathbb{1}) = \infty.$$

In short, it has been proved that (3.5) implies that this last convergence holds, so that (3.4) is valid with $k+1$ instead of k, completing the induction argument. To establish the conclusion of the lemma, assume now that $M(f)$ is communicating and let $i, j \in \{1, 2, \ldots, n\}$ be arbitrary. In this context, applying Lemma 3.1 to the matrix $M(f)$ it follows that $\sum_{r=0}^{n-1} M(f)_{i\,j}^r > 0$, and then there exists an integer k between 0 and $n-1$ such that $M(f)_{i\,j}^k > 0$. From this point (3.4) yields that

$$\sum_{r=0}^{n-1} f_i^r(t\mathbf{e}_j + \mathbb{1}) \geq f_i^k(t\mathbf{e}_j + \mathbb{1}) \to \infty \quad \text{as } t \to \infty$$

so that (3.1) holds, completing the proof. □

The following example shows that Assumption 3.1 may hold even if the matrix $M(f)$ is not communicating.

Example 3.1. Consider the function $f \in \mathcal{MH}_3$ specified as follows:

$$f(\mathbf{x}) = \begin{bmatrix} a_1x_1 \vee a_2x_2 \\ (b_1x_1 \vee b_2x_3) + (b_3x_2 \vee b_4x_3) \\ c_1x_1 \wedge c_2x_2 \end{bmatrix}, \tag{3.8}$$

where the a_i's, b_i's and c_i's are positive constants. For this function it will be shown that $M(f)$ is not a communicating matrix and that (3.1) holds. To achieve these goals, notice that, *if* $t > 0$ *is large enough*,

$$\begin{aligned} f([t-1]\mathbf{e}_1 + \mathbb{1}) &= (a_1t,\ b_1t + (b_3 \vee b_4),\ c_2)', \\ f([t-1]\mathbf{e}_2 + \mathbb{1}) &= (a_2t,\ b_3t + (b_1 \vee b_2),\ c_1)', \\ f([t-1]\mathbf{e}_3 + \mathbb{1}) &= (a_1 \vee a_2,\ (b_2 + b_4)t,\ c_1 \wedge c_2)'. \end{aligned} \tag{3.9}$$

Taking the limit as t goes to ∞ and using Definition 2.4 it follows that

$$M(f) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix};$$

thus, $M(f)$ is not communicating, since its third row is null. On the other hand, combining the specification of the function f in (3.8) with (3.9), it is not difficult to see that if $t > 0$ is sufficiently large then

$$\begin{aligned} &f^2([t-1]\mathbf{e}_1 + \mathbb{1}) \\ &\quad = (a_1^2 t \vee a_2[b_1 t + (b_3 \vee b_4)],\ b_1 a_1 t + b_3[b_1 t + (b_3 \vee b_4)], \\ &\qquad\qquad c_1 a_1 t \wedge c_2[b_1 t + (b_3 \vee b_4)])', \\ &f^2([t-1]\mathbf{e}_2 + \mathbb{1}) \\ &\quad = (a_1 a_2 t \vee a_2[b_3 t + (b_1 \vee b_2)],\ b_1 a_2 t + b_3[b_3 t + (b_1 \vee b_2)], \\ &\qquad\qquad c_1 a_2 t \wedge c_2[b_3 t + (b_1 \vee b_2)])', \\ &f^2([t-1]\mathbf{e}_3 + \mathbb{1}) \\ &\quad = (a_2(b_1 + b_2)t,\ [b_1(a_1 \vee a_2) \vee b_2(c_1 \wedge c_2)] + b_3(b_2 + b_4)t,\ c_1(a_1 \vee a_2))'. \end{aligned}$$

Recalling that f^0 is the identity map on $\mathcal{P}_n$ and using that these relations as well as (3.9) are valid for $t > 0$ sufficiently large, it follows immediately that f satisfies Assumption 3.1.

4. PROJECTIVE BOUNDEDNESS OF UPPER EIGENSPACES

In this section the basic result of the paper will be proved, namely, if f satisfies Assumption 3.1 then f has an eigenvalue. Such a conclusion will be obtained from Theorem 2.1 after establishing that the nonempty upper eigenspaces corresponding to f are projectively bounded, providing an extension of Theorem 2.3.

Theorem 4.1. Let $f \in \mathcal{MH}_n$ and $a > 0$ be arbitrary and suppose that Assumption 3.1 holds. In this case

(i) If $S^a(f) \neq \emptyset$ then this set is projectively bounded.

Consequently,

(ii) The function f has a unique eigenvalue $\lambda \equiv \lambda(f)$, and the corresponding eigenspace $\mathcal{E}(f)$ is projectively bounded; see (2.12).

Proof. Set

$$F := \sum_{k=0}^{n-1} f^k \in \mathcal{MH}_n \tag{4.1}$$

and let $b > 0$ be arbitrary. It will be shown that

$$S^b(F) \neq \emptyset \implies S^b(F) \text{ is projectively bounded.} \tag{4.2}$$

Assuming that this property holds the desired conclusion can be obtained as follows: Let $a > 0$ be such that the f-invariant set $S^a(f)$ is non-empty. From Definition 2.2, (2.1) and (2.2) it follows that

$$\mathbf{x} \in S^a(f) \implies f(\mathbf{x}) \le a\mathbf{x} \implies f^k(\mathbf{x}) \le a^k \mathbf{x}, \quad k = 0, 1, 2, \ldots,$$

so that the above specification of F yields that

$$\mathbf{x} \in S^a(f) \implies \mathbf{x} \in S^b(F), \quad \text{where } b := \sum_{k=1}^{n-1} a^k,$$

and then

$$\emptyset \neq S^a(f) \subset S^b(F).$$

Combining this fact with (4.2) it follows that the non-empty f-invariant set $S^a(f)$ is projectively bounded, establishing the part (i). From this point, since the f-invariant $S^a(f)$ is non-empty for $a \geq \max(f(\mathbb{1}))$, by Lemma 2.1, the existence of an eigenvalue of f is a consequence of Theorem 2.1, whereas the projective boundedness of the (non-empty) eigenspace $\mathcal{E}(f)$ follows combining the inclusion $\mathcal{E}(f) \subset S^{\lambda(f)}(f)$ with the previous part; see (2.12). To conclude the argument (4.2) will be established. First, notice that (4.1) and Assumption 3.1 together yield that

$$\lim_{t\to\infty} F_i(t\mathbf{e}_j + \mathbb{1}) = \infty, \quad i, j = 1, 2, \ldots, n. \tag{4.3}$$

Now, let $b > 0$ be such that $S^b(F)$ is non-empty and, proceeding by contradiction, *assume* that $S^b(F)$ is not projectively bounded, so that there exists a sequence $\{\mathbf{x}^k\}_{k=1,2,\ldots}$ such that

$$\{\mathbf{x}^k\} \subset S^b(F) \quad \text{and} \quad \frac{\max(\mathbf{x}^k)}{\min(\mathbf{x}^k)} \to \infty \quad \text{as } k \to \infty.$$

Setting $\mathbf{y}_k := (\min(\mathbf{x}^k))^{-1}\mathbf{x}^k$ for each positive integer k, it follows that

$$\{\mathbf{y}^k\} \subset S^b(F), \quad \min(\mathbf{y}^k) = 1 \quad \text{and} \quad \max(\mathbf{y}^k) = \frac{\max(\mathbf{x}^k)}{\min(\mathbf{x}^k)} \to \infty \quad \text{as } k \to \infty.$$

Next, for each positive integer k let $i(k), j(k) \in \{1, 2, \ldots, n\}$ be such that $\min(\mathbf{y}^k) = y^k_{i(k)}$ and $\max(\mathbf{y}^k) = y^k_{j(k)}$; taking a subsequence, if necessary, it can be assumed that $\{i(k)\}$ and $\{j(k)\}$ are constant sequences, say $i(k) = i^*$ and $j(k) = j^*$ for every k, so that

$$1 = \min(\mathbf{y}^k) = y^k_{i^*}, \quad \max(\mathbf{y}^k) = y^k_{j^*}, \quad k = 1, 2, 3, \ldots \tag{4.4}$$

as well as

$$\lim_{k\to\infty} y^k_{j^*} = \infty. \tag{4.5}$$

Observing that (4.4) yields that $\mathbf{y}^k \geq [y^k_{j^*} - 1]\mathbf{e}_{j^*} + \mathbb{1}$, the monotonicity of F yields that

$$F([y^k_{j^*} - 1]\mathbf{e}_{j^*} + \mathbb{1}) \leq F(\mathbf{y}^k) \leq b\mathbf{y}^k$$

where the inclusion $\mathbf{y}^k \in S^b(F)$ was used to set the second inequality. Taking the i^*-th component in this display and using the first statement in (4.4) it follows that $F_{i^*}([y^k_{j^*} - 1]\mathbf{e}_{j^*} + \mathbb{1}) \leq b y^k_{i^*} = b$, and then

$$\lim_{k\to\infty} F_{i^*}([y^k_{j^*} - 1]\mathbf{e}_{j^*} + \mathbb{1}) \leq b,$$

a relation that, via (4.5), contradicts (4.3). Thus, (4.2) holds and, as already mentioned, this completes the proof. □

Example 3.1. [Continued.] The function f in (3.8) satisfies Assumption 3.1, so that f has an eigenvalue, by Theorem 4.1; such a conclusion cannot be derived from Theorem 2.3, since the matrix $M(f)$ is not communicating.

Given $f \in \mathcal{MH}_n$ and $\mathbf{r} \in \mathcal{P}_n$ let the function $f_\mathbf{r} \in \mathcal{MH}_n$ be defined by

$$f_\mathbf{r}(\mathbf{x}) = (r_1 f_1(\mathbf{x}), r_2 f_2(\mathbf{x}), \ldots, r_n f_n(\mathbf{x}))', \quad \mathbf{x} \in \mathcal{P}_n. \tag{4.6}$$

In the analysis of (controlled) Markov chains endowed with the risk-sensitive average criterion, the problem of determining the existence of an eigenvalue for the above function $f_\mathbf{r}$ arises in a natural way, and the vector $\mathbf{r}$ is associated with the one-step reward function; see, for instance, Cavazos–Cadena and Hernández-Hernández [3] were, for $f \in \mathcal{MH}_n$ satisfying a weak form of convexity, a characterization of the dependence structure of f was given so that each function $f_\mathbf{r}$ has an eigenvalue. For a general monotone and homogeneous function, the following consequence of Theorem 4.1 holds.

Corollary 4.1. Let $f \in \mathcal{MH}_n$ be such that Assumption 3.1 holds. In this case, for each $\mathbf{r} \in \mathcal{P}_n$ the function $f_\mathbf{r}$ in (4.6) has an eigenvalue and the corresponding eigenspace $\mathcal{E}(f_\mathbf{r})$ is projectively bounded.

Proof. Notice that (4.6) yields that $\min(\mathbf{r})f \leq f_\mathbf{r}$ and, using that $f_\mathbf{r}$ is monotone and homogenous, a simple induction argument yields that $\min(\mathbf{r})^k f^k \leq f_\mathbf{r}^k$ for each nonnegative integer k, so that

$$\min\{\min(\mathbf{r})^k \mid k = 0, 2, \ldots, n-1\} \sum_{k=0}^{n-1} f^k \leq \sum_{k=0}^{n-1} f_\mathbf{r}^k;$$

since Assumption 3.1 holds for the function f, this relation shows that such a condition is also satisfied by $f_\mathbf{r}$, and the conclusion follows from Theorem 4.1 applied to the function $f_\mathbf{r}$. □

5. THE DUAL FUNCTION

In this section an additional consequence of Theorem 4.1, concerning the projective boundedness of lower eigenspaces, will be obtained using the following idea of dual function.

Definition 5.1. Given $f \in \mathcal{MH}_n$, the corresponding dual function $\tilde{f} : \mathcal{P}_n \to \mathcal{P}_n$ is defined as follows:

$$\tilde{f}(\mathbf{y}) := f(\mathbf{y}^{-1})^{-1}, \quad \mathbf{y} \in \mathcal{P}_n$$

where, for each $\mathbf{y} = (y_1, \ldots, y_n)' \in \mathcal{P}_n$ the vector $\mathbf{y}^{-1}$ is defined by

$$\mathbf{y}^{-1} := (y_1^{-1}, \ldots, y_n^{-1})'.$$

From this specification it is not difficult to see that for arbitrary $f, g \in \mathcal{MH}_n$ the following properties hold:

$$\tilde{f} \in \mathcal{MH}_n, \quad \tilde{\tilde{f}} = f, \quad \widetilde{f(g)} = \tilde{f}(\tilde{g}), \quad \widetilde{f \vee g} = \tilde{f} \wedge \tilde{g} \quad \text{and} \quad \widetilde{f \wedge g} = \tilde{f} \vee \tilde{g}. \tag{5.1}$$

On the other hand, for each nonempty set $B \subset \mathcal{P}_n$ define

$$B^{-1} := \{\mathbf{x}^{-1} \,|\, \mathbf{x} \in B\}. \tag{5.2}$$

Observing that the equalities $\max(\mathbf{x}^{-1}) = \min(\mathbf{x})^{-1}$ and $\min(\mathbf{x}^{-1}) = \max(\mathbf{x})^{-1}$ hold for every $\mathbf{x} \in \mathcal{P}_n$, it follows that $\max(\mathbf{x}^{-1})/\min(\mathbf{x}^{-1}) = \max(\mathbf{x})/\min(\mathbf{x})$, and then the above display yields that

$$\sup_{\mathbf{y} \in B^{-1}} \frac{\max(\mathbf{y})}{\min(\mathbf{y})} = \sup_{\mathbf{x} \in B} \frac{\max(\mathbf{x})}{\min(\mathbf{x})},$$

so that

$$B \text{ is projectively bounded } \iff B^{-1} \text{ is projectively bounded}; \tag{5.3}$$

see Definition 2.1(i). The following result will be established combining Theorem 4.1 with the idea of dual function.

Theorem 5.1. Let $f \in \mathcal{MH}_n$ be arbitrary and suppose that Assumption 3.1 is satisfied by $\tilde{f}$, that is

$$\lim_{t \to \infty} \sum_{k=1}^{n-1} \tilde{f}_i^k(t\mathbf{e}_j + \mathbb{1}) = \infty, \quad i, j \in \{1, 2, \ldots, n\}. \tag{5.4}$$

In this case assertions (i) and (ii) below are valid:

(i) If $a > 0$ is such that the lower eigenspace $S_a(f)$ is nonempty, then $S_a(f)$ is projectively bounded.

(ii) The second assertion in the statement of Theorem 4.1 holds.

Before going any further, an alternative formulation of the condition (5.4) is stated below.

Remark 5.1. (i) Condition (5.4) can be expressed directly in terms of f. First, given $j \in \{1, 2, \ldots, n\}$ set

$$\mathbb{1}_{\{j\}^c} := \mathbb{1} - \mathbf{e}_j = \sum_{\substack{s \neq j \\ 1 \leq s \leq n}} \mathbf{e}_s \tag{5.5}$$

and notice that, for each $g \in \mathcal{MH}_n$, Definition 5.1 yields that $\tilde{g}(t\mathbf{e}_j + \mathbb{1}_{\{j\}^c}) = g(t^{-1}\mathbf{e}_j + \mathbb{1}_{\{j\}^c})^{-1}$; since $\lim_{t\to\infty} \tilde{g}_i(t\mathbf{e}_j + \mathbb{1}) = \lim_{t\to\infty} \tilde{g}_i(t\mathbf{e}_j + \mathbb{1}_{\{j\}^c})$ it follows that

$$\lim_{t \to \infty} \tilde{g}_i(t\mathbf{e}_j + \mathbb{1}) = \infty \iff \lim_{t \to 0} g_i(t\mathbf{e}_j + \mathbb{1}_{\{j\}^c}) = 0. \tag{5.6}$$

Next, for a given $f \in \mathcal{MH}_n$ observe that (5.4) is equivalent to

$$\lim_{t\to\infty} \bigvee_{k=0}^{n-1} \tilde{f}_i^k(t\mathbf{e}_j + \mathbb{1}) = \infty, \quad i, j \in \{1, 2, \ldots, n\};$$

see Remark 3.1(i). On the other hand, from (5.1) it follows that $\bigvee_{k=0}^{n-1} \tilde{f}^k$ is the dual function of $\bigwedge_{k=0}^{n-1} f^k$, so that (5.6) yields that the above display is equivalent to

$$\lim_{t\to 0} \bigwedge_{k=0}^{n-1} f_i^k(t\mathbf{e}_j + \mathbb{1}_{\{j\}^c}) = 0 \quad i, j \in \{1, 2, \ldots, n\},$$

expressing condition (5.4) in terms of f.

(ii) For $\mathbf{r} \in \mathcal{P}_n$ and $f \in \mathcal{MH}_n$ it is not difficult to see that $\widetilde{f_\mathbf{r}} = \mathbf{r}^{-1}\tilde{f}$, so that $\widetilde{f_\mathbf{r}} \geq \min(\mathbf{r}^{-1})\tilde{f}$. From this point it follows that when f satisfies (5.4) then so does $f_\mathbf{r}$, and then the conclusions of Theorem 5.1 also hold for the function $f_\mathbf{r}$.

Proof of **Theorem 5.1.** Assume that the condition (5.4) is satisfied by $f \in \mathcal{MH}_n$.

(i) Given $a > 0$ such that $S_a(f) \neq \emptyset$, notice that Definition 5.1 yields that $f(\mathbf{y}) \geq a\mathbf{y}$ if and only if $\tilde{f}(\mathbf{y}^{-1}) \leq a^{-1}\mathbf{y}^{-1}$, an inequality that is equivalent to $\mathbf{y}^{-1} \in S^{a^{-1}}(\tilde{f})$, and it follows that

$$\emptyset \neq S_a(f) = \left[S^{a^{-1}}(\tilde{f})\right]^{-1};$$

see (5.2). Next, an application of Theorem 4.1(i) to the dual function $\tilde{f}$ yields that $S^{a^{-1}}(\tilde{f})$ is projectively bounded, and the corresponding property for $S_a(f)$ follows combining the above display with (5.3).

(ii) Since the f-invariant set $S_a(f)$ is nonempty for $a > 0$ sufficiently small (see Lemma 2.1), the existence of an eigenvalue of f is obtained combining the previous part with Theorem 2.1. Finally, since $\emptyset \neq \mathcal{E}(f) \subset S_{\lambda(f)}(f)$, the projective boundedness of $\mathcal{E}(f)$ follows from the corresponding property of $S_{\lambda(f)}(f)$ established in the first part. □

Remark 5.2. Given $(\mu, \mathbf{y}) \in (0, \infty) \times \mathcal{P}_n$ notice that Definition 5.1 yields that $\tilde{f}(\mathbf{y}) = \mu\mathbf{y}$ is equivalent to $f(\mathbf{y}^{-1}) = \mu^{-1}\mathbf{y}^{-1}$, so that

$$f \text{ has an eigenvalue} \iff \tilde{f} \text{ has an eigenvalue.} \tag{5.7}$$

Under condition (5.4) the existence of an eigenvalue for $\tilde{f}$ is ensured by Theorem 4.1(ii), and then the existence of an eigenvalue of f can be also obtained from the above display.

Now, an explicit example will be given to show that Theorem 5.1 can be used to establish the existence of an eigenvalue, even when the precedent results can not be applied to obtain such a conclusion.

Example 5.1. Consider the following function $h \in \mathcal{MH}_3$:

$$h(\mathbf{x}) = (a_1x_1 \wedge a_2x_2,\ b_1x_1 \wedge b_2x_2 \wedge b_3x_3,\ c_1x_1 \vee c_2x_2)', \quad \mathbf{x} \in \mathcal{P}_3, \tag{5.8}$$

where the a_i's, b_i's and c_i's are positive constants. In the case, direct calculations yield that the corresponding dual function is given by

$$\tilde{h}(\mathbf{x}) = (a_1^{-1}x_1 \vee a_2^{-1}x_2,\ b_1^{-1}x_1 \vee b_2^{-1}x_2 \vee b_3^{-1}x_3,\ c_1^{-1}x_1 \wedge c_2^{-1}x_2)', \quad \mathbf{x} \in \mathcal{P}_3, \tag{5.9}$$

(a) Notice that for $t > 0$ large enough

$$\begin{aligned} h([t-1]\mathbf{e}_1 + \mathbb{1}) &= (a_2, b_2 \wedge b_3, c_1 t)' \\ h^2([t-1]\mathbf{e}_1 + \mathbb{1}) &= (a_1a_2 \wedge a_2(b_2 \wedge b_3),\ b_1a_2 \wedge b_2(b_2 \wedge b_3),\ c_1a_2 \vee c_2(b_2 \wedge b_3))'. \end{aligned}$$

These equalities yield that $\lim_{t\to\infty} \sum_{k=0}^{2} h_2^k([t-1]\mathbf{e}_1 + \mathbb{1}) < \infty$, so that Assumption 3.1 is not satisfied by h, a conclusion that yields that $M(h)$ is not a communicating matrix, by Theorem 3.1. Thus, the existence of an eigenvalue of h can not be obtained neither from Theorem 4.1 nor from Theorem 2.3.

(b) Using (5.9), form Definition 2.4 it is not difficult to see that

$$M(\tilde{h}) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

which is not a communicating matrix; see Definition 2.3. Thus, the existence of an eigenvalue of h can not be derived combining (5.7) with an application of Theorem 2.3 to the function $\tilde{h}$.

(c) It will be shown that Theorem 5.1 yields that the function h has an eigenvalue. To achieve this goal notice that, comparing the above expression for $\tilde{h}$ and the formula (3.8) for the function f in Example 3.1, it is not difficult to see that there exists a constant $C \geq 1$ such that $f \leq C\tilde{h}$ a relation that immediately leads to $f^2 \leq C^2\tilde{h}^2$, and it follows that

$$\sum_{k=0}^{2} f^k \leq \sum_{k=0}^{2} C^k\tilde{h}^k \leq C^2 \sum_{k=0}^{2} \tilde{h}^k.$$

Therefore, $\tilde{h}$ satisfies Assumption 3.1—since, as verified in Example 3.1, so does f—and then h has an eigenvalue, by Theorem 5.1.

6. INDECOMPOSABILITY PROPERTY

In Gaubert and Gunawardena [5], the projective boundedness of the upper-eigenspaces associated to a function $f \in \mathcal{MH}_n$ was characterized in terms of the idea of *indecomposability*, and in this section the relation between this property and Assumption 3.1 is analyzed.

Definition 6.1. Let the positive integer $n \geq 2$ be arbitrary. A mapping $f \in \mathcal{MH}_n$ is decomposable if there exists a partition I, J of the set $S = \{1, 2, \ldots, n\}$ such that

$$\lim_{t\to\infty} f_i(t\mathbf{e}_J + \mathbb{1}) < \infty, \quad i \in I; \tag{6.1}$$

see (1.1) for the specification of $\mathbf{e}_J$. The function f is indecomposable if f is not decomposable.

The following basic result, relating the above notion of indecomposability with the projective boundedness of upper-eigenspaces, was established as Theorem 5 in Gaubert and Gunawardena [5].

Theorem 6.1. Let the positive integer $n \geq 2$ and $f \in \mathcal{MH}_n$ be arbitrary. In this context, the following conditions (i) and (ii) are equivalent:

(i) Every non-empty upper eigenspace $S^a(f)$ is projectively bounded;

(ii) f is indecomposable.

Suppose now that $f \in \mathcal{MH}_n$ satisfies Assumption 3.1. In this case, Theorem 4.1 yields that each nonempty upper-eigenspace $S^a(f)$ is projectively bounded, so that f is indecomposable, by Theorem 6.1. Thus, for each $f \in \mathcal{MH}_n$,

$$f \text{ satsifies Assumption 3.1} \implies f \text{ is indecomposable.}$$

As it is shown by an example below, the reverse implication is not generally true, so that the communication requirement in Assumption 3.1 is (strictly) stronger than the indecomposability property.

Example 6.1. Let $f \in \mathcal{MH}_3$ be specified as follows:

$$f(\mathbf{x}) := \begin{bmatrix} x_3 \wedge x_2 \\ x_3 \vee x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{x} \in \mathcal{P}_3. \tag{6.2}$$

For this function, direct calculations yield that, for each $t \geq 0$,

$$f(t\mathbf{e}_i + \mathbb{1}) = t\mathbf{e}_{i+1} + \mathbb{1}, \quad i = 1, 2, \quad \text{and} \quad f(t\mathbf{e}_3 + \mathbb{1}) = t\mathbf{e}_2 + \mathbb{1} \tag{6.3}$$

expressions that immediately yield

$$\lim_{t\to\infty} f(t\mathbf{e}_1 + \mathbb{1}) = \begin{bmatrix} 1 \\ \infty \\ 1 \end{bmatrix}, \tag{6.4}$$

$$\lim_{t\to\infty} f(t\mathbf{e}_2 + \mathbb{1}) = \begin{bmatrix} 1 \\ 1 \\ \infty \end{bmatrix}, \qquad \lim_{t\to\infty} f(t\mathbf{e}_3 + \mathbb{1}) = \begin{bmatrix} 1 \\ \infty \\ 1 \end{bmatrix}, \tag{6.5}$$

as well as $\sum_{k=0}^{2} f^k(t\mathbf{e}_2 + \mathbb{1}) = 3\mathbb{1} + 2t\mathbf{e}_2 + t\mathbf{e}_3$. This last equality implies that

$$\lim_{t\to\infty} \sum_{k=0}^{3} f_1^k(t\mathbf{e}_2 + \mathbb{1}) = 3 < \infty.$$

Consequently,

(a) The function f does not satisfy Assumption 3.1.

Next, it is claimed that

(b) f is indecomposable.

To verify this assertion, it will be shown that (6.1) fails for each of the six possible partitions of $S = \{1, 2, 3\}$. Let (I, J) be a partition of S.

(i) Consider the cases in which I is a singleton:

$I = \{1\}$ and $J = \{2, 3\}$.
In this case, $t\mathbf{e}_J + \mathbb{1} = (1, t+1, t+1)'$ and (6.2) yields that $f_1(t\mathbf{e}_J + \mathbb{1}) = t + 1$.

$I = \{2\}$ and $J = \{1, 3\}$.
For this partition $t\mathbf{e}_J + \mathbb{1} = (t+1, 1, t+1)'$, and then $f_2(t\mathbf{e}_J + \mathbb{1}) = t + 1$;

$I = \{3\}$ and $J = \{1, 2\}$.
In this context $t\mathbf{e}_J + \mathbb{1} = (t+1, t+1, 1)$ and $f_3(t\mathbf{e}_J + \mathbb{1}) = t + 1$.

Therefore, if $I = \{i\}$ is a singleton, $f_i(t\mathbf{e}_J + \mathbb{1}) = t + 1 \to \infty$ as $t \to \infty$, and (6.1) does not hold.

(ii) Consider now the three possibilities in which I contains two elements.

$I = \{1, 2\}$ and $J = \{3\}$.
In this case $t\mathbf{e}_J + \mathbb{1} = t\mathbf{e}_3 + \mathbb{1}$, and the second convergence in (6.5) yields that

$$\lim_{t\to\infty} f_2(t\mathbf{e}_J + \mathbb{1}) = \infty$$

$I = \{1, 3\}$ and $J = \{2\}$.
In this context $t\mathbf{e}_J + \mathbb{1} = t\mathbf{e}_2 + \mathbb{1}$ and then

$$\lim_{t\to\infty} f_3(t\mathbf{e}_J + \mathbb{1}) = \infty,$$

by the first convergence in (6.5).

$I = \{2, 3\}$ and $J = \{1\}$.
For this partition, $t\mathbf{e}_J + \mathbb{1} = t\mathbf{e}_1 + \mathbb{1}$ and $\lim_{t\to\infty} f_2(t\mathbf{e}_J + \mathbb{1}) = \infty$; see (6.4).

Consequently, when I contains two elements, there exists $i \in I$ such that

$$\lim_{t\to\infty} f_i(t\mathbf{e}_J + \mathbb{1}) = \infty,$$

and then (6.1) fails. In all, the function f in (6.2) is indecomposable, but does not satisfy Assumption 3.1.

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REFERENCES

[1] M. Akian and S. Gaubert: (2003). Spectral theorem for convex monotone homogeneous maps, and ergodic control. Nonlinear Anal. *52* (2003), 2, 637–679.

[2] M. Akian, S. Gaubert, B. Lemmens, and R. Nussbaum: Iteration of order preserving subhomogeneous maps on a cone. Math. Proc. Camb. Phil. Soc. *140* (2006), 157–176.

[3] R. Cavazos–Cadena and D. Hernández–Hernández: Poisson equations associated with a homogeneous and monotone function: necessary and sufficient conditions for a solution in a weakly convex case. Nonlinear Anal.: Theory, Methods and Appl. *72* (2010), 3303–3313.

[4] C. Dellacherie: Modèles simples de la théorie du potentiel non-linéaire. Lecture Notes in Math. *1426*, pp. 52–104, Springer 1990.

[5] S. Gaubert and J. Gunawardena: The Perron–Frobenius theorem for homogeneous, monotone functions. Trans. Amer. Math. Soc. *356* (2004), 12, 4931–4950.

[6] J. Gunawardena: From max-plus algebra to nonexpansive maps: a nonlinear theory for discrete event systems. Theoret. Comput. Sci. *293* (2003), 141–167.

[7] V. N. Kolokoltsov: Nonexpansive maps and option pricing theory. Kybernetika *34* (1998), 713–724.

[8] B. Lemmens and M. Scheutzow: On the dynamics of sup-norm nonexpansive maps. Ergodic Theory Dynam. Systems *25* (2005), 3, 861–871.

[9] H. Minc: Nonnegative Matrices, Wiley, New York 1988.

[10] R. D. Nussbaum: Hilbert's projective metric and iterated nonlinear maps. Memoirs of the AMS *75* (1988), 391.

[11] R. D. Nussbaum: Iterated nonlinear maps and Hilbert's projective metric. Memoirs of the AMS *79* (1989), 401.

[12] E. Seneta: Non-negative Matrices and Markov Chains. Springer, New York 1980.

[13] W. H. M. Zijm: Generalized eigenvectors and sets of nonnegative matrices. inear Alg. Appl. *59* (1984), 91–113.

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