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ON CENTRAL ATOMS OF ARCHIMEDEAN ATOMIC LATTICE EFFECT ALGEBRAS

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If element z of a lattice effect algebra $(E, \oplus, \mathbf{0}, \mathbf{1})$ is central, then the interval $[\mathbf{0}, z]$ is a lattice effect algebra with the new top element z and with inherited partial binary operation $\oplus$. It is a known fact that if the set $C(E)$ of central elements of E is an atomic Boolean algebra and the supremum of all atoms of $C(E)$ in E equals to the top element of E, then E is isomorphic to a direct product of irreducible effect algebras ([16]). In [10] Paseka and Riečanová published as open problem whether $C(E)$ is a bifull sublattice of an Archimedean atomic lattice effect algebra E. We show that there exists a lattice effect algebra $(E, \oplus, \mathbf{0}, \mathbf{1})$ with atomic $C(E)$ which is not a bifull sublattice of E. Moreover, we show that also $B(E)$, the center of compatibility, may not be a bifull sublattice of E.

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1. INTRODUCTION, BASIC DEFINITIONS AND KNOWN FACTS

Effect algebras, introduced by D. J. Foulis and M. K. Bennett [2], have their importance in the investigation of uncertainty. Lattice ordered effect algebras generalize orthomodular lattices and MV-algebras. Thus they may include non-compatible pairs of elements as well as unsharp elements.

Definition 1.1. (Foulis and Bennett [2]) An *effect algebra* is a system $(E; \oplus, \mathbf{0}, \mathbf{1})$ consisting of a set E with two different elements $\mathbf{0}$ and $\mathbf{1}$, called *zero* and *unit*, respectively and $\oplus$ is a partially defined operation satisfying the following conditions for all $p, q, r \in E$:

(E1) If $p \oplus q$ is defined, then $q \oplus p$ is defined and $p \oplus q = q \oplus p$.

(E2) If $q \oplus r$ is defined and $p \oplus (q \oplus r)$ is defined, then $p \oplus q$ and $(p \oplus q) \oplus r$ are defined and $p \oplus (q \oplus r) = (p \oplus q) \oplus r$.

(E3) For every $p \in E$ there exists a unique $q \in E$ such that $p \oplus q$ is defined and $p \oplus q = \mathbf{1}$.

(E4) If $p \oplus 1$ is defined then $p = \mathbf{0}$.

The element q in (E3) will be called the *supplement* of p, and will be denoted as p'.

Definition 1.2. A *generalized effect algebra* (GEA for brevity) is a system $(E, \oplus, \mathbf{0})$ satisfying conditions (E1) and (E2) from Definition 1.1, and the following conditions for $a, b \in E$.

(E3') If $a \oplus b = a \oplus c$, then $b = c$

(E4') If $a \oplus b = \mathbf{0}$, then $a = b = \mathbf{0}$.

(E5) $a \oplus \mathbf{0} = a$.

Definition 1.3. Let $(E, \oplus, \mathbf{0})$ be a GEA and $a, b \in E$ be arbitrary elements. A partial order $\leq$ is given by

$$b \leq a, \quad \text{if there exists } c \in E \text{ such that} \quad c \oplus b = a.$$

In the whole paper, for a GEA $(E, \oplus, \mathbf{0})$, writing of $a \oplus b$ for arbitrary $a, b \in E$ will mean that $a \oplus b$ exists. Definition 1.3 enables us to introduce another partial binary operation $\ominus$ by

$$a \ominus b = c \quad \Leftrightarrow \quad b \oplus c = a.$$

Further, in this article we often briefly write 'an effect algebra (generalized effect algebra) E' skipping the operations.

Every effect algebra E is a generalized effect algebra. Conversely, a generalized effect algebra E is an effect algebra if and only if E has a greatest element $\mathbf{1}$ (see [1], p.17).

If every pair x, y of elements of a lattice effect algebra E is *compatible*, meaning that $x \vee y = x \oplus (y \ominus (x \wedge y))$ then E is called an *MV-effect algebra* [7].

S. P. Gudder ([4, 5]) introduced the notion of sharp elements and sharply dominating lattice effect algebras. Recall that an element x of the lattice effect algebra E is called *sharp* if $x \wedge x' = \mathbf{0}$. Jenča and Riečanová in [6] proved that in every lattice effect algebra E the set $S(E) = \{x \in E; x \wedge x' = \mathbf{0}\}$ of sharp elements is an orthomodular lattice which is a *sub-effect algebra* of E, meaning that if among $x, y, z \in E$ with $x \oplus y = z$ at least two elements are in $S(E)$ then $x, y, z \in S(E)$. Moreover $S(E)$ is a *full sublattice* of E, hence supremum of any set of sharp elements, which exists in E, is again a sharp element. Further, each maximal subset M of pairwise compatible elements of E, called *block* of E, is a sub-effect algebra and a full sublattice of E and $E = \bigcup\{M \subseteq E; M \text{ is a block of } E\}$ (see [13, 14]). *Central elements* and centers of effect algebras were defined in [3]. In [11, 12] it was proved that in every lattice effect algebra E the *center*

$$C(E) = \{x \in E; (\forall y \in E) y = (y \wedge x) \vee (y \wedge x')\} = S(E) \cap B(E), \tag{1}$$

where $B(E) = \bigcap\{M \subseteq E; M \text{ is a block of } E\}$. Since $S(E)$ is an orthomodular lattice and $B(E)$ is an MV-effect algebra, we obtain that $C(E)$ is a Boolean algebra. Note that E is an orthomodular lattice if and only if $E = S(E)$ and E is an MV-effect algebra if and only if $E = B(E)$. Thus E is a Boolean algebra if and only if $E = S(E) = B(E) = C(E)$.

Recall that an element p of a (generalized) effect algebra E is called an *atom* if and only if p is a minimal non-zero element of E and E is *atomic* if for each $x \in E$, $x \neq \mathbf{0}$, there exists an atom $p \leq x$.

Definition 1.4. Let $(E, \oplus, 0)$ be a GEA. To each $a \in E$ we define its *isotropic index*, notation $ord(a)$, as the maximal positive integer n such that

$$na := \underbrace{a \oplus \cdots \oplus a}_{n\text{-times}}$$

exists. We set $ord(a) = \infty$ if na exists for each positive integer n. We say that E is *Archimedean*, if for each $a \in E$, $a \neq \mathbf{0}$, $ord(a)$ is finite.

An element $u \in E$ is called *finite*, if there exists a finite system of atoms $a_1, \ldots, a_n$ (which are not necessarily distinct) such that $u = a_1 \oplus \cdots \oplus a_n$. An element $v \in E$ is called *cofinite*, if there exists a finite element $u \in E$ such that $v = u'$.

We say that for a finite system $F = (x_j)_{j=1}^k$ of not necessarily different elements of an effect algebra $(E, \oplus, \mathbf{0}, \mathbf{1})$ is $\oplus$-*orthogonal* if $x_1 \oplus x_2 \oplus \cdots \oplus x_n = (x_1 \oplus x_2 \oplus \cdots \oplus x_{n-1}) \oplus x_n$ exists in E (briefly we will write $\bigoplus_{j=1}^n x_j$). We define also $\oplus\emptyset = \mathbf{0}$.

Let $(P, \oplus, \mathbf{0})$ be a GEA. Denote by P^* a set disjoint from P and of the same cardinality. Consider a bijection $p \to p^*$ from P onto P^* and put $E = P \,\dot{\cup}\, P^*$. We define a partial commutative operation $\oplus^*$ on E by the following rules valid for $a, b \in P$.

1. $a \oplus^* b$ is defined if and only if $a \oplus b$ is defined, and $a \oplus^* b = a \oplus b$.

2. $b^* \oplus^* a$ is defined if and only if there exists a $c \in P$ such that $a \oplus c = b$ and then $c^* = b^* \oplus^* a$.

Theorem 1.5. (Dvurečenskij and Pulmannová [1]) For every GEA $(P, \oplus, \mathbf{0})$ and the above defined set $E = P \,\dot{\cup}\, P^*$ the structure $(E, \oplus^*, \mathbf{0}, \mathbf{0}^*)$ is an effect algebra. Moreover, the partial order induced by $\oplus^*$ and restricted to P, coincides with the partial order of P induced by $\oplus$. For every $a \in P$, $a \oplus^* a^* = \mathbf{0}^*$.

The structure $(E, \oplus^*, \mathbf{0}, \mathbf{0}^*)$ introduced in Theorem 1.5, where $E = P \,\dot{\cup}\, P^*$, will be called the *effect algebraic extension* of the GEA $(P, \oplus, \mathbf{0})$. Instead of $\oplus^*$ we will use the notation $\oplus$ since these two operations coincide on P.

Theorem 1.6. (Mosná [8]) Let $(P, \oplus, \mathbf{0})$ be a GEA and $(E, \oplus, \mathbf{0}, \mathbf{0}^*)$, where $E = P \,\dot{\cup}\, P^*$, be its effect algebraic extension. Then $p \in E$ is an atom of E if and only if one of the following conditions is satisfied:

(i) $p \in P$ and p is an atom of P,

(ii) $p = a^* \in P^*$, where $a \in P$ is a maximal element of P.

Theorem 1.7. (Riečanová and Marinová [18]) Let $(P, \oplus, 0)$ be a GEA and $(E, \oplus, \mathbf{0}, \mathbf{0}^*)$ be its effect algebraic extension. Then E is a lattice effect algebra preserving joins and meets existing in P if and only if the following conditions hold for $a, b \in P$.

1. $a \wedge_P b$ exists.
2. If there is $d \in P$ such that $a, b \leq d$ then $a \vee_P b$ exists.
3. For all $c \in P$ the existence of $a \vee_P b$, $a \oplus c$, and $b \oplus c$, implies the existence of $(a \vee_P b) \oplus c$.
4. Either $a \vee_P b$ exists or $\bigvee\{c \in P; a \oplus c$ and $b \oplus c$ are defined$\}$ exists in P.
5. $\bigvee\{c \in P; c \leq b$ and $a \oplus c$ is defined$\}$ exists in P.

A GEA P satisfying the properties $1-5$ of Theorem 1.7 is called a *prelattice effect algebra*.

Definition 1.8. For a lattice $(L, \wedge, \vee)$ and a subset $D \subseteq L$ we say that D is a *bifull sublattice* of L, if and only if for any $X \subseteq D$, $\bigvee_L X$ exists if and only if $\bigvee_D X$ exists and $\bigwedge_L X$ exists if and only if $\bigwedge_D X$ exists, in which case $\bigvee_L X = \bigvee_D X$ and $\bigwedge_L X = \bigwedge_D X$.

Recall that an element $a \in L$, where $(L, \wedge, \vee)$ is a lattice, is called a *compact element* if for arbitrary $D \subset L$ with $\bigvee D \in L$, if $a \leq \bigvee D$ then $a \leq \bigvee F$ for some finite set $F \subseteq D$. The lattice L is called *compactly generated* if every element of L is a join of compact elements.

Lemma 1.9. Let $(E, \oplus, \vee, \wedge, \mathbf{0}, \mathbf{1})$ be an atomic Archimedean lattice effect algebra. Then

(i) (see [8]) a block M of E is atomic if there exists a maximal pairwise compatible set A of atoms of E such that $A \subseteq M$ and if M_1 is a block of E with $A \subseteq M_1$, then $M_1 = M$. Moreover for all $x \in E$ and all $a \in A$ the following holds

$$x \in M \quad \Leftrightarrow \quad x \leftrightarrow a,$$

(ii) (see [15]) to every nonzero element $x \in E$ there exist mutually distinct atoms $a_\alpha \in E$ and positive integers k_α for $\alpha \in \mathcal{I}$ such that

$$x = \bigoplus_{\alpha \in \mathcal{I}} (k_\alpha a_\alpha) = \bigvee_{\alpha \in \mathcal{I}} (k_\alpha a_\alpha).$$

It is known that if E is a distributive effect algebra (i. e., the effect algebra E is a distributive lattice – e. g., if E is an MV-effect algebra) then $C(E) = S(E)$. If moreover E is Archimedean and atomic then the set of atoms of $C(E) = S(E)$ is the set $\{n_a a; a \in E$ is an atom of $E\}$, where $n_a = ord(a)$ (see [17]). Since $S(E)$ is a bifull sublattice of E if E is an Archimedean atomic lattice effect algebra (see [10]), we obtain that

$$1 = \bigvee_{C(E)} \{p \in C(E); p \text{ is an atom of } C(E)\} = \bigvee_{E} \{p \in C(E); p \text{ is an atom of } C(E)\}$$

for every Archimedean atomic distributive lattice effect algebra E. We are going to show that there are Archimedean atomic lattice effect algebras with atomic center where this property fails to be true.

2. EXAMPLE OF A LATTICE EFFECT ALGEBRA WITH NON-BIFULL SUBLATTICE OF CENTRAL ELEMENTS

Theorem 2.1. There exists an atomic Archimedean lattice effect algebra E such that $C(E)$ is not a bifull sublattice of E. More precisely, $\bigvee_{C(E)} A_C = \mathbf{1}$ and $\bigvee_E A_C$ does not exist, where A_C are atoms of $C(E)$.

Example 2.2. Let us give a detailed construction of an atomic Archimedean lattice effect algebra E whose center $C(E)$ is not a bifull sublattice of E.

Let us assume that there exist the following sequences of mutually different elements (that will play the role of atoms in the constructed effect algebra E):

$$(a_i)_{i=0}^{\infty}, \quad (b_i)_{i=0}^{\infty}, \quad (c_j)_{j=1}^{\infty}, \quad (d_j)_{j=1}^{\infty}, \quad (p_j)_{j=1}^{\infty} \tag{2}$$

fulfilling the following binary relation $\not\leftrightarrow$ (that will play the role of the non-compatibility relation of atoms in the constructed effect algebra E)

$$\begin{array}{llll} c_j \not\leftrightarrow a_i, \quad c_j \not\leftrightarrow b_i & \text{for all} \quad j = 1, 2, \dots & \text{and} & i = 0, \dots, j-1, \\ d_j \not\leftrightarrow a_i, \quad d_j \not\leftrightarrow b_i & \text{for all} \quad j = 1, 2, \dots & \text{and} & i = 0, \dots, j-1, \\ c_j \not\leftrightarrow d_i & \text{for all} \quad i, j = 1, 2, \dots & \text{such that} & i \neq j, \\ c_j \not\leftrightarrow c_i, \quad d_j \not\leftrightarrow d_i & \text{for all} \quad i, j = 1, 2, \dots & \text{such that} & i \neq j. \end{array} \tag{3}$$

All other pairs of elements from the above sequences fulfil the complementary binary relation $\leftrightarrow$ (that will play the role of compatibility relation of atoms in the constructed effect algebra E) and we will call such pairs of atoms *compatible*.

Denote

$$A_0 = (a_i)_{i=0}^{\infty} \cup (b_i)_{i=0}^{\infty} \cup (p_j)_{j=1}^{\infty}, \tag{4}$$

and for $j = 1, 2, \dots$ let

$$A_j = (a_i)_{i=j}^{\infty} \cup (b_i)_{i=j}^{\infty} \cup (p_j)_{j=1}^{\infty} \cup \{c_j, d_j\}. \tag{5}$$

Property (3) is equivalent with the fact that A_0 and A_j $(j = 1, 2, \dots)$ are unique maximal sets of pairwise compatible atoms.

Let us represent elements of (2) by the following subsets of $\mathbb{R}^2$ and elements of the set $\mathbb{N} = \{1, 2, \dots\}$:

$$\begin{array}{rcll} a_0 & = & \{(x, y) \in \mathbb{R}^2; 0 \le x \le 1,\ y \in \mathbb{R}\}, & \\ a_l & = & \{(x, y) \in \mathbb{R}^2; l < x \le l+1,\ y \in \mathbb{R}\}, & \text{for } l = 1, 2, \dots, \\ b_0 & = & \{(x, y) \in \mathbb{R}^2; -1 \le x < 0,\ y \in \mathbb{R}\}, & \\ b_l & = & \{(x, y) \in \mathbb{R}^2; -l-1 \le x < -l,\ y \in \mathbb{R}\}, & \text{for } l = 1, 2, \dots, \\ c_j & = & \{(x, y) \in \mathbb{R}^2; -j \le x \le j,\ y \le j \cdot x\}, & \text{for } j = 1, 2, \dots, \\ d_j & = & \{(x, y) \in \mathbb{R}^2; -j \le x \le j,\ y > j \cdot x\}, & \text{for } j = 1, 2, \dots, \\ p_j & = & \{j\}, \quad \text{for } j = 1, 2, \dots. & \end{array} \tag{6}$$

For such a choice of elements, the elements $q_1 \neq q_2$ are compatible if and only if $q_1 \cap q_2 = \emptyset$. From this we get that they fulfil the condition (3).

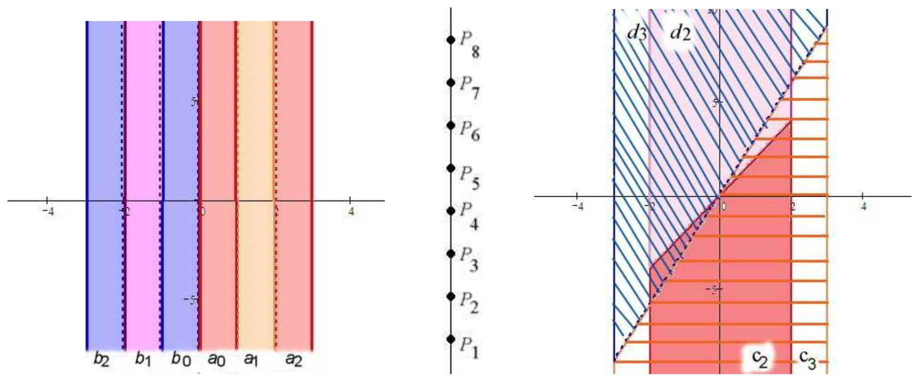

Fig. 1. Illustration of sequences of elements $(a_l)_l$, $(b_l)_l$, $(p_j)_j$, $(c_j)_j$, $(d_j)_j$.

Recall that, for any non-empty set X and any partition $\{Y_\alpha; \alpha \in \Lambda\}$ of X, the complete Boolean sub-algebra B of the set of all subsets of X generated by $\{Y_\alpha; \alpha \in \Lambda\}$ is atomic and its atoms are sets Y_α, $\alpha \in \Lambda$. Moreover, the complement of an element $Z \in B$ is the usual set-theoretic complement $X \setminus Z$. Note also that the sets A_0, A_j $(j = 1, 2, \dots)$ are partitions of the set $\mathbb{R}^2 \cup \mathbb{N}$.

Denote B_0, B_j (for $j = 1, 2, \dots$) complete atomic Boolean algebras with the corresponding sets of atoms A_0, A_j $(j = 1, 2, \dots)$. For elements $u_1, u_2 \in B_l$, $l = 0, 1, 2, \dots$, such that $u_1 \cap u_2 = \emptyset$ we introduce the partial operation $\oplus$ by

$$u_1 \oplus u_2 = u_1 \cup u_2.$$

Note that in the complete Boolean algebras B_0, B_j (for $j = 1, 2, \dots$) the orthogonal sum of an orthogonal system corresponds to the disjoint union of such system.

The Boolean algebras B_0, B_j, $j = 1, 2, \dots$, have the following top elements:

$$\mathbb{R}^2 \cup \mathbb{N} = \mathbf{1} = 1_0 \;=\; a_0 \oplus b_0 \oplus \bigoplus_{i=1}^{\infty} (a_i \oplus b_i \oplus p_i) \tag{7}$$

$$\mathbb{R}^2 \cup \mathbb{N} = \mathbf{1} = 1_1 \;=\; (c_1 \oplus d_1) \oplus \bigoplus_{i=1}^{\infty} (a_i \oplus b_i \oplus p_i) \tag{8}$$

$$\mathbb{R}^2 \cup \mathbb{N} = \mathbf{1} = 1_j \;=\; (c_j \oplus d_j) \oplus \bigoplus_{i=j}^{\infty} (a_i \oplus b_i \oplus p_i) \oplus \bigoplus_{i=1}^{j-1} p_i, \text{ for all } j = 2, 3, \dots \tag{9}$$

An element $u \in B_l$ is a finite element of Boolean algebra B_l if and only if $u = q_1 \oplus q_2 \oplus \dots \oplus q_n$ for an $n \in \mathbb{N}$ and $q_1, q_2, \dots, q_n \in A_l$. Put $Q_l = \{u \in B_l; u \text{ is finite}\}$, $l = 0, 1, 2, \dots$. Then Q_l is a generalized Boolean algebra, since $M_l = Q_l \dot\cup Q_l^*$ is a Boolean algebra, where $Q_l^* = \{u^*; u^* = 1_l \ominus u \text{ and } u \in Q_l\}$ (see [18], or [1, pp. 18–19]). Note that any element of M_l is an orthogonal sum (disjoint union) of atoms from B_l and hence from M_l. This means that M_l is a Boolean subalgebra of finite and cofinite elements of B_l $(l = 0, 1, 2, \dots)$.

Let us put $E = \bigcup_{l=0}^{\infty} M_l$ and let the partial operation $\oplus$ be defined for disjoint elements of M_l $(l = 0, 1, 2, \dots)$ as a disjoint union. Also, let us put $\mathbf{1} = \mathbb{R}^2 \cup \mathbb{N}$ and $\mathbf{0} = \emptyset$. Let us show that $(E, \oplus, \vee, \wedge, \mathbf{0}, \mathbf{1})$ is a lattice effect algebra with the family $(M_l)_{l=0}^{\infty}$ of atomic blocks of E. We have the following equality

$$c_j \oplus d_j = \bigoplus_{i=0}^{j-1}(a_i \oplus b_i) = \{(x, y) \in \mathbb{R}^2; -j \leq x \leq j\}, \quad \text{for all } j = 1, 2, \dots. \tag{10}$$

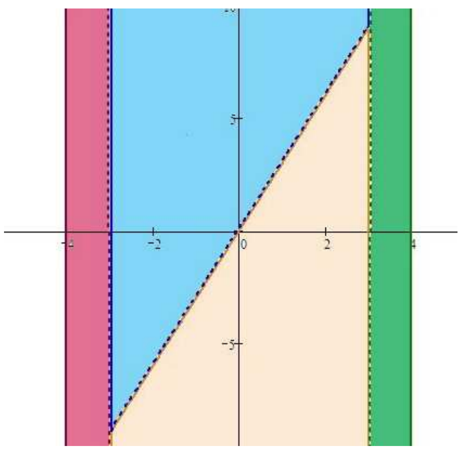

Fig. 2. Illustration of the element $a_3 \oplus b_3 \oplus c_3 \oplus d_3$.

To each $x \in E$ there is a unique x^* since $x^* = (\mathbb{R}^2 \cup \mathbb{N}) \setminus x$ in each Boolean algebra M_l $(l = 0, 1, 2, \dots)$. Due to equalities (7−9) we get the coincidence of top elements (bottom elements of blocks are of course identical to $\emptyset$). It is easy to check that E is an effect algebra, since the commutativity and associativity of the partial operation $\oplus$ is due to the commutativity and associativity of disjoint union. Moreover, E, where $E = P \dot{\cup} P^*$, is an effect algebraic extension of GEA $(P, \oplus, \mathbf{0})$, where $P = \bigcup_{l=0}^{\infty} Q_l$ and $P^* = \bigcup_{l=0}^{\infty} Q_l^*$ (see [1] pp. 18–19). To prove that E is a lattice effect algebra it is enough to show that P is a prelattice GEA (Theorem 1.7, or [18]).

First we show the following proposition.

Proposition 2.3. The GEA $(P, \oplus, \wedge, \vee, \mathbf{0})$ from Example 2.2 is a lattice with the bottom element $\mathbf{0} = \emptyset$.

Proof. Let $h_1, h_2 \in P$ be arbitrary elements. This means that there surely exists an $n \in \mathbb{N}$ such that for all $m = 1, 2, \dots,$ all $l = 0, 1, 2, \dots$ and all $s = 1, 2, \dots$ we have that $(p_m \leq h_1$ or $p_m \leq h_2)$ or $(a_l \leq h_1$ or $a_l \leq h_2)$ or $(b_l \leq h_1$ or $b_l \leq h_2)$ or $c_s \leq h_1$ or $c_s \leq h_2)$ or $(d_s \leq h_1$ or $d_s \leq h_2)$ implies that $m \leq n$, $l \leq n$ and $s \leq n$ (since both h_1 and h_2 are finite elements).

First assume that there is an $i \in \{0,1,2,\dots\}$ such that $h_1, h_2 \in Q_i$. Then h_1, h_2 are expressible in the form

$$h_1 = \begin{cases} \bigoplus_{l=0}^{n} (\alpha_l a_l \oplus \beta_l b_l) \oplus \bigoplus_{l=1}^{n} \pi_l p_l, & \text{if } i = 0, \\ \gamma_i c_i \oplus \delta_i d_i \oplus \bigoplus_{l=i}^{n} (\alpha_l a_l \oplus \beta_l b_l) \oplus \bigoplus_{l=1}^{n} \pi_l p_l, & \text{if } i \neq 0, \end{cases} \tag{11}$$

$$h_2 = \begin{cases} \bigoplus_{l=0}^{n} (\alpha'_l a_l \oplus \beta'_l b_l) \oplus \bigoplus_{l=1}^{n} \pi'_l p_l, & \text{if } i = 0, \\ \gamma'_i c_i \oplus \delta'_i d_i \oplus \bigoplus_{l=i}^{n} (\alpha'_l a_l \oplus \beta'_l b_l) \oplus \bigoplus_{l=1}^{n} \pi'_l p_l, & \text{if } i \neq 0, \end{cases} \tag{12}$$

where $\alpha_l, \alpha'_l, \beta_l, \beta'_l, \pi_j, \pi'_j \in \{0,1\}$ for $l = 0,1,2,\dots,n$ and $j = 1,2,\dots,n$, and $\gamma_i, \gamma'_i, \delta_i, \delta'_i \in \{0,1\}$, $i = 1,2,\dots$. Denote $\tilde{\alpha}_l = \min\{\alpha_l, \alpha'_l\}$, $\tilde{\beta}_l = \min\{\beta_l, \beta'_l\}$, $\tilde{\gamma}_i = \min\{\gamma_i, \gamma'_i\}$, $\tilde{\delta}_i = \min\{\delta_i, \delta'_i\}$ and $\tilde{\pi}_j = \min\{\pi_j, \pi'_j\}$, and put

$$v = \begin{cases} \bigoplus_{l=0}^{n} \left(\tilde{\alpha}_l a_l \oplus \tilde{\beta}_l b_l\right) \oplus \bigoplus_{m=1}^{n} \tilde{\pi}_m p_m, & \text{if } i = 0, \\ \tilde{\gamma}_i c_i \oplus \tilde{\delta}_i d_i \oplus \bigoplus_{l=i}^{n} \left(\tilde{\alpha}_l a_l \oplus \tilde{\beta}_l b_l\right) \oplus \bigoplus_{m=1}^{n} \tilde{\pi}_m p_m, & \text{if } i \neq 0. \end{cases} \tag{13}$$

Putting $\hat{\alpha}_l = \max\{\alpha_l, \alpha'_l\}$, $\hat{\beta}_l = \max\{\beta_l, \beta'_l\}$, $\hat{\gamma}_i = \max\{\gamma_i, \gamma'_i\}$, $\hat{\delta}_i = \max\{\delta_i, \delta'_i\}$ and $\hat{\pi}_j = \max\{\pi_j, \pi'_j\}$, we get

$$u = \begin{cases} \bigoplus_{l=0}^{n} \left(\hat{\alpha}_l a_l \oplus \hat{\beta}_l b_l\right) \oplus \bigoplus_{m=1}^{n} \hat{\pi}_m p_m, & \text{if } i = 0, \\ \hat{\gamma}_i c_i \oplus \hat{\delta}_i d_i \oplus \bigoplus_{l=i}^{n} \left(\hat{\alpha}_l a_l \oplus \hat{\beta}_l b_l\right) \oplus \bigoplus_{m=1}^{n} \hat{\pi}_m p_m, & \text{if } i \neq 0. \end{cases} \tag{14}$$

Elements h_1 and h_2 are sets. Since they are from the same block Q_i, we have that $v = h_1 \cap h_2$ and $u = h_1 \cup h_2$. This shows that $v = h_1 \wedge h_2$ and $u = h_1 \vee h_2$.

Assume that are some $0 \le i < s$ such that $h_1 \in Q_i$ and $h_2 \in Q_s$, and that there is no t such that $h_1 \in Q_t$ and $h_2 \in Q_t$. The element h_1 can be written via formula (11), and for h_2 we get

$$h_2 = \gamma'_s c_s \oplus \delta'_s d_s \oplus \bigoplus_{l=s}^{n} (\alpha'_l a_l \oplus \beta'_l b_l) \oplus \bigoplus_{m=1}^{n} \pi'_m p_m. \tag{15}$$

Because of formula (10), if we denote by Γ_i all atoms of A_i which are non-compatible with c_s (or equivalently, which are non-compatible with d_s), for h_1 we get that there exists a $q \in \Gamma_i$ such that $q \le h_1$ and at the same time

$$c_s \oplus d_s = \bigoplus_{l=0}^{s-1} (a_l \oplus b_l) \not\le h_1.$$

For h_2 we get that either $c_s \le h_2$ or $d_s \le h_2$, and $c_s \oplus d_s \not\le h_2$.
In all other cases we would get that there is a t such that $h_1 \in Q_t$ and $h_2 \in Q_t$. We

put

$$\tilde{v} = \bigoplus_{l=s}^{n}\left(\tilde{\alpha}_l a_l \oplus \tilde{\beta}_l b_l\right) \oplus \bigoplus_{m=1}^{n} \tilde{\pi}_m p_m, \tag{16}$$

$$\begin{aligned}\hat{u} &= c_s \oplus d_s \oplus \bigoplus_{l=s}^{n}\left(\hat{\alpha}_l a_l \oplus \hat{\beta}_l b_l\right) \oplus \bigoplus_{m=1}^{n} \hat{\pi}_m p_m \\ &= \bigoplus_{l=0}^{s-1}(a_l \oplus b_l) \oplus \bigoplus_{l=s}^{n}\left(\hat{\alpha}_l a_l \oplus \hat{\beta}_l b_l\right) \oplus \bigoplus_{m=1}^{n} \hat{\pi}_m p_m, \end{aligned} \tag{17}$$

where $\tilde{\alpha}_l = \min\{\alpha_l, \alpha'_l\}$, $\tilde{\beta}_l = \min\{\beta_l, \beta'_l\}$, $\hat{\alpha}_l = \max\{\alpha_l, \alpha'_l\}$, $\hat{\beta}_l = \max\{\beta_l, \beta'_l\}$ for $l \in \{s, 2s+1, \dots, n\}$, and $\tilde{\pi}_m = \min\{\pi_m, \pi'_m\}$, $\hat{\pi}_m = \max\{\pi_m, \pi'_m\}$ for $m \in \{1, 2, \dots, n\}$. We show that $\tilde{v} = h_1 \wedge h_2$. We can put $h_2 = x \oplus y$, where

$$\begin{aligned} x &= \gamma'_s c_s \oplus \delta'_s d_s, \\ y &= \bigoplus_{l=s}^{n}(\alpha'_l a_l \oplus \beta'_l b_l) \oplus \bigoplus_{m=1}^{n} \pi'_m p_m. \end{aligned}$$

We have that $y \in Q_s \cap Q_i$ and by formula (13) we get $y \wedge h_1 = \bigoplus_{l=s}^{n}\left(\tilde{\alpha}_l a_l \oplus \tilde{\beta}_l b_l\right) \oplus \bigoplus_{m=1}^{n} \tilde{\pi}_m p_m$. Since $x = c_s$ or $x = d_s$, and $h_1 \notin Q_i$, we have that $x \wedge h_1 = \mathbf{0}$. If we take any element $\tilde{x}$ such that $\tilde{x} \le h_1$ and $\tilde{x} \le h_2$, then surely $c_s \not\le \tilde{x}$ and we may conclude that $\tilde{v} = h_1 \wedge h_2$. By a dual analysis we get that $u = h_1 \vee h_2$.

The fact that $\mathbf{0} = \emptyset$ is the bottom element, is trivial. □

Proposition 2.4. The GEA P from Example 2.2 is a prelattice generalized effect algebra.

P r o o f. Since P is a lattice, the only property from Theorem 1.7 that is left to prove, is the property 3. We have elements $h_1, h_2, h_3 \in P$ such that $h_1 \vee h_2$ exists, and $h_1 \oplus h_3$ and $h_2 \oplus h_3$ exist. From these, since the isotropic index of each atom is 1, for all atoms q we get the following

$$q \le h_3 \quad \Rightarrow \quad q \not\le h_1 \,\&\, q \not\le h_2. \tag{18}$$

This means that $q \not\le (h_1 \vee h_2)$. We have to distinguish two cases: there is an $l \in \{0, 1, 2, \dots\}$ such that $h_1 \in Q_l$ and $h_2 \in Q_l$, or there is no $l \in \{0, 1, 2, \dots\}$ such that $h_1 \in Q_l$ and $h_2 \in Q_l$. In the first case (18) implies the existence of $(h_1 \vee h_2) \oplus h_3$.

Assume that there is no $l \in \{0, 1, 2, \dots\}$ such that $h_1 \in Q_l$ and $h_2 \in Q_l$. Then $h_1 \in Q_i$ and $h_2 \in Q_j$ for some $i < j$. For $h_1 \vee h_2$ we can use formula (17), and further we can do the same considerations for h_1 and h_2 as we did just before formulas (16) and (17). Since h_1 is orthogonal to h_3 and also h_2 is orthogonal to h_3, we get $a_l \not\le h_3$, $b_l \not\le h_3$, $c_t \not\le h_3$, and $d_t \not\le h_3$, for $l \in \{0, 1, \dots, j-1\}$ and $t \in \{1, 2, \dots\}$. This means that $h_1 \vee h_2 \in Q_0$ and also $h_3 \in Q_0$. Hence formula (18) implies the existence of $(h_1 \vee h_2) \oplus h_3$, and the proof of the statement that P is a prelattice generalized effect algebra is finished. □

Observe that the GEA P has no maximal elements. Hence, Theorems 1.6, 1.7 and Proposition 2.4 imply the following.

Proposition 2.5. Let $E = P \dot{\cup} P^*$ be an effect algebraic extension of the GEA P from Example 2.2. Then E is a lattice effect algebra whose system of atoms coincides with the system (6).

We prove now the main result of the paper.

Proposition 2.6. The center $C(E)$ of the lattice effect algebra E from Proposition 2.5 is not a bifull sublattice of E. More precisely, $\bigvee_{C(E)} A_C = \mathbf{1}$ and $\bigvee_E A_C$ does not exist, where A_C are atoms of $C(E)$. Moreover, neither the center of compatibility, $B(E)$, is a bifull sublattice of E.

P r o o f. Assume that $\bigvee_E A_C = z$. Then $z \in C(E)$ since $C(E)$ is a full sublattice of E (see [14]). This implies $z = \mathbf{1}$ and hence $z' = \mathbf{0}$. But atoms a_l, b_l, c_j, d_j for $l = 0, 1, 2, \dots$ and $j = 1, 2, \dots$ are orthogonal to z. This is a contradiction which proves that $C(E)$ is not a bifull sublattice of E.
Since $S(E) = E$, E is in fact an orthomodular lattice. This gives $B(E) = C(E)$ and it follows that neither $B(E)$ is a bifull sublattice of E. □

Finally observe that *there exists a faithful σ-additive state on E whose restriction onto $C(E)$ is not σ-additive.* An example of such a state is the following:
Let us put

$$\begin{aligned} &\mu(p_j) = \tfrac{1}{3}2^{-(j+1)}, && \text{for } j = 1, 2, \dots, \\ &\mu(a_i) = \mu(b_i) = \tfrac{1}{3}2^{-(i+2)}, && \text{for } i = 0, 1, 2, \dots, \\ &\mu(c_j) = \mu(d_j) = \tfrac{1}{2}\sum_{i=0}^{j-1}(\mu(a_i) + \mu(b_i)), && \text{for } j = 1, 2, \dots. \end{aligned}$$

Then obviously it is possible to extend the function μ to a σ-additive state $\overline{\mu} : E \to [0, 1]$.
However, the restriction of $\overline{\mu}$ to $C(E)$ gives an additive state, which is not σ-additive, since $C(E)$ consists of atoms p_j, $j = 1, 2, \dots$, only and we have the equation

$$\bigvee_{C(E)} A_C = 1, \quad \text{but} \quad \sum_{j=1}^{\infty} \mu(p_j) = \frac{1}{3},$$

where A_C is the system of all central atoms.

Example 2.7. (Example of a lattice effect algebra containing unsharp elements whose center is not a bifull sublattice.) Note that the effect algebraic extension $E = P \dot{\cup} P^*$ is in fact an orthomodular lattice. Let E_1 be a direct product of the effect algebra E from Example 2.2 and of the chain $\{0, p_0, 2p_0\}$. Then E_1 is a lattice effect algebra which is neither an orthomodular lattice nor an MV-effect algebra, since E_1 includes unsharp elements as well as non-compatible pair of elements. The center $C(E_1)$ (the center of compatibility $B(E_1)$) is the direct product of centers (of centers of compatibility) of the factors of E_1. This implies that $C(E_1)$ $(B(E_1))$ is not a bifull sublattice of E_1.

3. CONCLUSIONS

In [10] Paseka and Riečanová published as open problem wether $C(E)$ is a bifull sublattice of an Archimedean atomic lattice effect algebra E. This paper shows that we have no guarantee that $C(E)$ $(B(E))$ is a bifull sublattice of E even if $C(E)$ is atomic. Moreover, we have presented the Archimedean atomic lattice effect algebra E which has a faithful σ-additive state $\overline{\mu}$ on E, whose restriction to $C(E)$ is not σ-additive.

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