KYBERNETIKA — VOLUME *46* (2010), NUMBER 4, PAGES 585–599

OPEN PROBLEMS POSED AT THE TENTH INTERNATIONAL CONFERENCE ON FUZZY SET THEORY AND APPLICATIONS (FSTA 2010, LIPTOVSKÝ JÁN, SLOVAKIA)

Radko Mesiar and Peter Sarkoci

Eighteen open problems posed during FSTA 2010 (Liptovský Ján, Slovakia) are presented. These problems concern copulas, triangular norms and related aggregation functions. Some open problems concerning effect algebras are also included.

Keywords: copula, effect algebra, triangular norm

Classification: 03E72, 06F25, 60E05

1. INTRODUCTION

A public announcement of open problems had a great impact on the development of several areas of science, including mathematics. It seems so that the most famous was the formulation of D. Hilbert's problems [28]. In the domain of fuzzy sets and related topics, several open problems were published in monographs [9, 40, 47, 58]. There are several papers devoted purely to open problems in triangular norms [3, 42]. Other collections of open problems are linked to problems posed at conferences; recall for example the collections summarizing the open problems posed at the 2nd and 8th FSTA conference [39, 46]. To illustrate the influence of these collections to the development of mathematics, observe that just within the field of fuzzy sets there are more than 20 papers devoted to the solution of some of the exposed problems. The aim of this paper is the presentation of open problems posed during the conference FSTA 2010 "Tenth International Conference on Fuzzy Set Theory and Applications" held from February 1 to February 5, 2010 in Liptovský Ján, Slovakia.

The paper is organized as follows. In each section a brief introduction to the area of the summarized open problems is given. In the 2nd section copulas are discussed. Section 3 is devoted to open problems in triangular norms. Finally, Section 4 deals with effect algebras.

2. PROBLEMS IN COPULAS

Given an n-ary operation $O\colon [0,1]^n \to [0,1]$ and an n-dimensional Cartesian interval $I = [u_1, v_1] \times \cdots \times [u_n, v_n] \subseteq [0,1]^n$ we define the *O-volume of I* to be

$$V_O(I) = \sum_{J \subseteq \{1,2,\ldots,n\}} (-1)^{|J|}\, O\left(x_1^J, x_2^J, \ldots, x_n^J\right)$$

where x_i^J equals u_i whenever $i \in J$ and v_i otherwise. A *semi-copula* [19] is any n-ary operation S on the unit interval which is isotone with respect to the standard order and has neutral element 1, i. e., $S(x_1, \ldots, x_n) = x_i$ whenever $x_j = 1$ holds for each $j \neq i$. A *quasi-copula* [11, 23] is any semi-copula which is 1-Lipschitz. An *n-copula* is any n-ary semi-copula C with $V_C(I) \geq 0$ for every n-dimensional Cartesian interval $I \subseteq [0,1]^n$. In the case $n = 2$ we often speak simply of *copulas.* The prototypical examples of copulas are the independence copula Π and the Fréchet-Hoeffding bounds W and M, given by

$$\begin{aligned} W(x,y) &= \max\{0, x+y-1\}\,, \\ \Pi(x,y) &= xy\,, \\ M(x,y) &= \min\{x,y\}\,. \end{aligned}$$

For more details on copulas we recommend the book by Nelsen [50] and the recent monograph on Aggregation Functions [24]. The class of all semi-copulas, quasi-copulas, and copulas will be denoted $\mathcal{S}$, $\mathcal{Q}$, and $\mathcal{C}$ respectively.

2.1. Associative copulas of higher arity

Recently, Couceiro [10] has recalled a concept concerning associativity of n-ary functions, originally introduced by Post [52]. According to Post, an n-ary operation O on the unit interval is said to be associative if

$$\begin{aligned} O(O(x_1, \ldots, x_n), x_{n+1}, \ldots, x_{2n-1}) &= O(x_1, O(x_2, \ldots, x_{n+1}), x_{n+2}, \ldots, x_{2n-1}) \\ &\;\;\vdots \\ &= O(x_1, \ldots, x_{n-1}, O(x_n, \ldots, x_{2n-1})) \end{aligned}$$

holds for every $x_1, \ldots, x_{2n-1} \in [0,1]$. As an example, consider a ternary function $O\colon [0,1]^3 \to [0,1]$ given by

$$O(x,y,z) = \frac{x(1-y)z}{x(1-y)z + (1-x)y(1-z)}$$

with the convention $\frac{0}{0} = 0$. Then O is associative in the sense of Post but there is no associative binary operation A on $[0,1]$ such that $O(x,y,z) = A(A(x,y),z)$.

The binary associative copulas play an important role in the copula theory [24, 50]. They are completely characterized as ordinal sums of *Archimedean* copulas [50]. Further, due to Moynihan [49] a copula C is Archimedean if and only if there is a continuous convex strictly decreasing function $t\colon [0,1] \to [0,\infty]$ with $t(1) = 0$ such that

$$C(x,y) = t^{-1}\big(\min\{t(0), t(x) + t(y)\}\big).$$

Problem 2.1. (R. Mesiar, `mesiar@math.sk`) Is there a representation of n-ary associative copulas (in the sense of Post) similar to that one concerning binary copulas?

2.2. Ultramodularity of copulas and its strengthening

Among stronger concepts of monotonicity of functions, the *ultramodularity* is a genuine strengthening of the supermodularity [45]. Note that a binary operation O on the unit interval is ultramodular if

$$O(x_1+y_1+z_1, x_2+y_2+z_2) - O(x_1+y_1, x_2+y_2) - O(x_1+z_1, x_2+z_2) + O(x_1, x_2) \geq 0$$

holds whenever the arguments of O stay within $[0,1]$ and (x_1,x_2),(y_1,y_2),$(z_1,z_2) \in [0,1]^2$.

Typical examples of ultramodular copulas are Π and W, while M is not ultramodular. The concept of ultramodularity can be generalized to *n-ultramodularity* with $n > 2$. A binary operation O on the unit interval will be called n-ultramodular if

$$\sum_{J \subseteq \{1,\ldots,k\}} (-1)^{(k+|J|)}\, O\left(x + \sum_{i\in J} u_i, y + \sum_{i \in J} v_i\right) \geq 0\,.$$

holds for every $k \in \{2,\ldots,n\}$ whenever the arguments of O do not run out of $[0,1]$ and $(x,y),(u_1,v_1)\ldots,(u_k,v_k) \in [0,1]^2$. It can be checked that the product copula Π is n-ultramodular for any natural n.

Problem 2.2. (R. Mesiar, `mesiar@math.sk`) For a fixed $n \geq 2$, characterize (all, all associative, all Archimedean) 2-copulas which are n-ultramodular.

2.3. Power stability of binary operations

The problem of *power stability* of functions is important in risk management when dealing with extremal events. Especially in the multivariate probability distribution case, copulas with such a property are often considered and they are called *Extreme value copulas* (or, *EV-copulas*, for short). Recall that a binary operation O on the unit interval is power stable if for any power $p \in \,]0,\infty[$ it satisfies

$$O(x^p, y^p) = \big(O(x,y)\big)^p\,.$$

It is known that a copula C is power stable (i. e., C is an EV-copula) if and only if there exists a convex function $f\colon [0,1] \to [0,1]$ satisfying, for all $x \in [0,1]$,

$$\max\{x, 1-x\} \leq f(x) \leq 1\,,$$

and

$$C(x,y) = (xy)^{f\left(\frac{\ln(x)}{\ln(xy)}\right)} \tag{1}$$

(see for example Tawn [59]).

One can check easily that any power stable binary operation O on the unit interval – not necessarily a copula – can be written in the form (1) where the range of f need not be a subset of $[0,1]$ in general. An equivalent representation of power stable binary operations is

$$O(x,y) = (xy)^{g\left(\frac{\ln(x)}{\ln(y)}\right)} \tag{2}$$

where $g\colon [0,\infty] \to [0,\infty]$ is an appropriate function [24, Section 7]. All the properties of O are are now reflected by the function f or g. For example, O is commutative if and only if $f(x) = f(1-x)$ for all $x \in [0,1]$.

Problem 2.3. (R. Mesiar, `mesiar@math.sk`) How are the properties of power stable binary operations related to those of functions f and g? Interesting properties to be considered are monotonicity, continuity, 1-Lipschitz property, associativity, existence of a neutral element, etc.

2.4. Double conic copulas

This problem was inspired by the presentation of double conic copulas [37] and is related to the question, how to construct a copula when its behavior is prescribed on a subset of the unit square. As a prominent example of such problems let us mention the construction of diagonal copulas, that is of copulas with a given diagonal section [13, 30, 50].

Problem 2.4. (C. Sempi, `carlo.sempi@unisalento.it`) Given a fixed $x_0 \in\,]0,1[$ characterize all $\delta\colon [0,1] \to [0,1]$ such that there exists a copula C with $C(x,x) = \delta(x)$ for all $x \in [0,1]$, and C is linear on the segments connecting the diagonal point (x_0,x_0) with the corner points $(1,0)$ and $(0,1)$. Also, try to construct pointwise maximal and minimal elements of this class.

Notice that not every diagonal of a copula solves the problem. For example the function

$$\delta(x) = \min\{\max\{0, 2x-0.9\}, \max\{0.6, 2x-1\}\}$$

is a diagonal of a copula, however for the choice $x_0 = 0.75$ this δ cannot solve the problem. Indeed, for any such solution C we would have $V_C([0.6,0.8]^2) = -0.06$ thus violating solutions among copulas.

2.5. Structure of the class of associative copulas

A 2-copula C is said to be associative if it is associative as a binary operation. Associative copulas are known to be exactly the 1-Lipschitz t-norms. Whence all associative copulas are commutative. It is well known that the class $\mathcal{C}_a$ of associative copulas is not a convex set; for example, $(M+W)/2$ is a copula that is not associative. However, it was proved that every convex combination of associative copulas belongs to the class of Schur-concave and commutative copulas [2, 17]. Denote by $\tilde{\mathcal{C}_a}$ the closure (with respect to the topology of uniform convergence) of the set of all convex combinations of associative copulas.

Problem 2.5. (F. Durante, `fabrizio.durante@unibz.it`) Does $\tilde{\mathcal{C}_a}$ coincide with the class of all Schur-concave and commutative copulas?

Notice that, since $\mathcal{C}_a$ is compact with respect to the L^∞-norm [41], it is easily checked that $\tilde{\mathcal{C}_a}$ is compact as well. Moreover, $\tilde{\mathcal{C}_a}$ is also a convex set. It follows from Krein-Milman theorem [57] that each element of $\tilde{\mathcal{C}_a}$ can be represented as convex combination of the *extremal elements* in $\tilde{\mathcal{C}_a}$. We recall that $C \in \tilde{\mathcal{C}_a}$ is extremal if and only if C cannot be represented as a non-trivial convex combination of two elements in $\tilde{\mathcal{C}_a}$.

Problem 2.6. (F. Durante, `fabrizio.durante@unibz.it`) Determine the extremal points, in the Krein-Milman's sense, of the set $\tilde{\mathcal{C}_a}$. Prove or disprove that extremal points of $\tilde{\mathcal{C}_a}$ form a subset of the extremal points of the class of all copulas.

The previous problem could help in the characterization of the class of extremal doubly stochastic measures [1, 27, 53].

2.6. Transformations of copulas

Given a 2-copula C and a strictly increasing bijection $\varphi\colon [0,1] \to [0,1]$ we define the *φ-transform* of C to be a binary operation C_φ on $[0,1]$ given by

$$C_\varphi(x,y) = \varphi^{-1}\big(C(\varphi(x),\varphi(y))\big) .$$

The class $\mathcal{C}^*$ formed by the transformations of all 2-copulas by means of all possible increasing bijections $\varphi\colon [0,1] \to [0,1]$ has turned out to be important within the framework of bivariate ageing [8, 20]. It is well known that C_φ need not be a copula [12, 18, 43] and that $\mathcal{C}^*$ is strictly contained in the class of continuous semi-copulas [6].

Problem 2.7. (F. Durante, `fabrizio.durante@unibz.it`) Describe the class $\mathcal{C}^*$.

2.7. Intermediate classes between copulas and quasi-copulas

When extending fuzzy measures to aggregation functions (utility functions) by a method based on the Möbius transform of the underlying fuzzy measure, the following constraint on an n-ary quasi-copula Q turns out to be crucial [44]:

$$V_Q(I) \geq 0$$

for each n-dimensional interval I such that at least one of its vertices is contained in the boundary $[0,1]^n \setminus]0,1[^n$ of the unit hypercube $[0,1]^n$. Let us denote the class of such quasi-copulas $\mathcal{F}$. Obviously $\mathcal{C} \subsetneq \mathcal{F} \subseteq \mathcal{Q}$. Observe that, while in the case $n=2$ the last inclusion boils down to equality, for $n>2$ the inclusion is strict.

Similarly, by $\mathcal{G}$, we denote the class of all semi-copulas $S\colon [0,1]^n \to [0,1]$ such that $V_S(I) \geq 0$ for every n-dimensional interval I such that its top-vertex coincides with the top-vertex of the unit hypercube $[0,1]^n$.

Problem 2.8. (A. Kolesárová, `anna.kolesarova@stuba.sk`) Provide a characterization of the class $\mathcal{F}$ or of its distinguished sub-classes.

Problem 2.9. (A. Kolesárová, `anna.kolesarova@stuba.sk`) Provide a characterization of the class $\mathcal{G}$ or of its distinguished sub-classes.

2.8. Properties resembling the moderate growth

Motivated by the problem of common measurability in unsharp observable approach to quantum mechanics, the notion of *witness map* has been introduced and studied [34]. Let $[0,u]$ be a unit interval in a partially ordered abelian group G, let H be a subset of $[0,u]$. Let $\mathrm{Fin}\,(H)$ be the system of all finite subsets of H. We say that $\beta\colon \mathrm{Fin}\,(H) \to G$ is a *witness map* if and only if $\beta(\emptyset) = 1$, $\beta(\{x\}) = x$ and

$$\sum_{X \subseteq Z \subseteq A} (-1)^{|X|+|Z|} \beta(Z) \geq 0\,, \tag{3}$$

for all $x \in H$ and $X, A \in \mathrm{Fin}\,(H)$ with $X \subseteq A$.

Let us now consider a commutative, associative binary operation O on the real unit interval $[0,1]$, such that 1 is its neutral element. Naturally, every such O gives rise to a mapping $\beta_O\colon \mathrm{Fin}\,([0,1]) \to [0,1]$ given by the rule

$$\beta_O(\{x_1, \dots, x_n\}) = O(x_1, O(x_2, \dots, O(x_{n-1}, x_n)))\,.$$

Problem 2.10. (G. Jenča, P. Sarkoci, `{jenca‖sarkoci}@math.sk`) Characterize all commutative and associative binary operations $O\colon\ [0,1]^2 \to [0,1]$ with neutral element 1 for which the corresponding β_O is a witness map.

Note that, since the conditions $\beta_O(\emptyset) = 1$ and $\beta_O(\{x\}) = x$ are trivially satisfied, β_O is a witness map if and only if (3) holds. To start the discussion, let us remark that both β_M and β_Π are witness maps. On the other hand β_W is not a witness map, $X = \emptyset$, $A = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\}$ being a counterexample.

3. PROBLEMS IN TRIANGULAR NORMS

Recall that a *triangular norm* [5, 40] (or a *t-norm*, for short) is any commutative, associative, and non-decreasing binary operation $T\colon\ [0,1]^2 \to [0,1]$ with neutral element 1. Similarly, a t-conorm $S\colon\ [0,1]^2 \to [0,1]$ is a commutative, associative, and increasing mapping with neutral element 0. Particular examples of t-norms that will appear within this collection are the minimum $T_{\mathbf{M}}$, the product $T_{\mathbf{P}}$, and the Łukasiewicz t-norm $T_{\mathbf{L}}$ which coincide with the copulas M, Π, and W, respectively. In addition to these three important examples we will consider also the drastic t-norm $T_{\mathbf{D}}$ and the nilpotent minimum $T_{\mathbf{nM}}$ defined by

$$T_{\mathbf{D}}(x,y) = \begin{cases} \min\{x,y\} & \text{if } \max\{x,y\} = 1\,, \\ 0 & \text{otherwise,} \end{cases}$$

$$T_{\mathbf{nM}}(x,y) = \begin{cases} \min\{x,y\} & \text{if } x+y > 1\,, \\ 0 & \text{otherwise.} \end{cases}$$

Given a t-norm T, a particular t-conorm S is defined via

$$S(x,y) = 1 - T(1-x, 1-y)\,, \tag{4}$$

and vice-versa. Triangular norms and conorms linked to each other via (4) (which is actually, a version of De Morgan law) are called dual to each other. The duals to $T_{\mathbf{M}}$, $T_{\mathbf{L}}$, $T_{\mathbf{D}}$, $T_{\mathbf{P}}$, and $T_{\mathbf{nM}}$ are denoted $S_{\mathbf{M}}$, $S_{\mathbf{L}}$, $S_{\mathbf{D}}$, $S_{\mathbf{P}}$, and $S_{\mathbf{nM}}$, respectively.

3.1. T-based α-cuts of IF-sets

Atanassov [7] introduced the notion of *intuitionistic fuzzy set* or, shortly, an *IF-set*. An IF-set A of the universe X is a pair of maps $M_A, N_A\colon X \to [0,1]$ such that $M_A + N_A \le 1$. The value $M_A(x)$ is interpreted as a membership degree of the element $x \in X$ to the IF-set A while N_A is understood as a non-membership degree of x to A. Due to Atanassov [7] the *α-cut* of an IF-set A, for $\alpha \in \,]0,1]$, is defined to be the set

$$A_\alpha = \{x \in X \mid M_A(x) \ge \alpha \text{ and } N_A(x) \le 1-\alpha\}.$$

A generalization of this definition was offered by Vasilev [60]. If we consider this statement as a conjunction of two statements on cuts (in the later case cuts of the fuzzy set $1-N_A$), then it is natural to use a triangular norm as a model for conjunction in fuzzy logic. More precisely, we consider the conjunction of the statements: "The element x belongs to A" and "The element x does not belong to A". In order to obtain good properties of cuts, we restrict ourselves to the left-continuous t-norms. Let $\alpha \in \,]0,1]$ and let T be an arbitrary left-continuous triangular norm. We define the *T-based α-cut* of an IF-set A to be the set [29]

$$A_{T,\alpha} = \{x \in X \mid T(M_A(x), 1-N_A(x)) \ge \alpha\}.$$

Observe that each standard fuzzy set with a membership function M can be represented as an IF-set described by $(M, 1-M)$. The problem of coincidence of standard alpha-cuts of fuzzy sets with the T-based alpha-cuts of the corresponding IF-sets gives raise to a constraint according to which T has to satisfy

$$T(x, 1-T(1-x,x)) = x \tag{5}$$

for every $x \in [0,1]$.

Problem 3.1. (V. Janiš, `janis@fpv.umb.sk`) Describe all, not necessarily left-continuous, triangular norms satisfying (5).

Observe that $T_{\mathbf{D}}$, $T_{\mathbf{M}}$, and $T_{\mathbf{nM}}$ solve the problem. Moreover, every t-norm T with $T \le T_{\mathbf{nM}}$ is a solution to the problem (for example $T_{\mathbf{L}}$). As an example of a t-norm violating the property let us mention $T_{\mathbf{P}}$. So far, only a partial answer was given [29].

3.2. Opposite diagonal splice of triangular norms

Given two t-norms T_1, T_2 *their opposite diagonal splice* $[T_1, T_2]$ is a binary operation on the unit interval defined by

$$[T_1, T_2](x, y) = \begin{cases} T_2(x, y) & \text{if } x + y - 1 > 0\,, \\ T_1(x, y) & \text{otherwise.} \end{cases}$$

Note that, in the framework of copulas, an analogous notion was already considered by Durante et al. [15]. Clearly, the resulting operation is commutative and has neutral element 1. Moreover, if $T_1 \leq T_2$ then even the non-decreasingness of $[T_1, T_2]$ follows for free. On the other hand, the splice need not be associative. As an example when the opposite diagonal splice is a t-norm let us mention $[T_{\mathbf{D}}, T_{\mathbf{M}}] = [T_{\mathbf{L}}, T_{\mathbf{M}}] = T_{\mathbf{nM}}$.

Problem 3.2. (J. Fodor, `fodor@bmf.hu`) Characterize, possibly all, couples T_1, T_2 of different t-norms with $T_1 \leq T_2$, such that their opposite diagonal splice $[T_1, T_2]$ is again a t-norm.

3.3. Properties reflecting set equalities

In the fuzzy set theory the t-norms and t-conorms are used to interpret intersections and unions of fuzzy sets, respectively. In the classical set theory the union of two sets is always decomposable in the following way:

$$A \cup B = (A \cap B) \cup (B \setminus A) \cup (A \setminus B)\,.$$

What follows is a problem related to fuzzification of the above property and stems in the framework of fuzzy preference structures.

Problem 3.3. (J. Fodor, `fodor@bmf.hu`) Characterize all triples (S_1, S_2, T), where S_1, S_2 are t-conorms and T a t-norm, so that the equality

$$S_1(x, y) = S_2\big(T(x, y), T(1-x, y), T(x, 1-y)\big) \tag{6}$$

is satisfied for all $x, y \in [0, 1]$.

One possible solution to the problem is the triple $(S_{\mathbf{P}}, S_{\mathbf{L}}, T_{\mathbf{P}})$. Supposing $S_1 = S_2$ a modified version of the problem is obtained; one solution in such a case is the triple $(S_{\mathbf{nM}}, S_{\mathbf{nM}}, T_{\mathbf{nM}})$.

The next problem generalizes the famous Frank functional equation [22] which is related to the classical valuation property of characteristic functions of sets,

$$\mathbf{1}_{\mathrm{A}\cup\mathrm{B}} + \mathbf{1}_{\mathrm{A}\cap\mathrm{B}} = \mathbf{1}_{\mathrm{A}} + \mathbf{1}_{\mathrm{B}}$$

where $\mathbf{1}_{\mathrm{A}}$ stands for the indicator of the standard set A.

Problem 3.4. (J. Fodor, `fodor@bmf.hu`) For a fixed t-norm T_0 and a t-conorm S_0, characterize all couples (T, S), where T is a t-norm and S is a t-conorm, such that the equality

$$T(x,y) + S(x,y) = T_0(x,y) + S_0(x,y) \tag{7}$$

holds for all $x, y \in [0,1]$.

Observe that if $T_0 \in \{T_{\mathbf{L}}, T_{\mathbf{P}}, T_{\mathbf{M}}\}$ and S_0 is dual to T_0, then (7) boils down to the Frank functional equation

$$T(x,y) + S(x,y) = x + y\,.$$

In this case there exists an infinitude of solutions [22]. On the other hand, if $T_0 = T_{\mathbf{D}}$ and $S_0 = S_{\mathbf{D}}$ the monotonicity of t-norms and t-conorms exclude any nontrivial solution of (7).

3.4. Quasi triangular norms

A *track* is any set of the form

$$B = \{\big(f(t), g(t)\big) \mid t \in [0,1]\}$$

where $f, g\colon [0,1] \to [0,1]$ are nondecreasing continuous surjections. According to the original definition [4], a *quasi-copula* is any binary operation Q on the unit interval such that for every track B there exists a copula C_B with $Q(x,y) = C_B(x,y)$ for each $(x,y) \in B$. Later it was shown that quasi-copulas are exactly 1-Lipschitz semicopulas [23]. By an alternative characterization the class of all quasi-copulas is exactly the sup(inf)-closure of the class of all copulas [51].

In analogy with the definition of quasi-copula one can formally define the concept of *quasi triangular norm* (or a *quasi-t-norm* for short). A binary operation U on the unit interval is a quasi-t-norm if for any track B there exists a t-norm T_B that coincides with U on B, that is $U(x,y) = T_B(x,y)$ holds for every $(x,y) \in B$.

Recall that the class of quasi-t-norms is strictly included in the class of commutative semi-copulas. The strictness of this inclusion is demonstrated by the following example: let δ and S_δ be a unary and a binary operation, respectively, on $[0,1]$ defined via

$$\begin{aligned}
\delta(t) &= \begin{cases} \frac{t}{2} & \text{if } t \in \left[0, \frac{1}{2}\right]\,, \\ \frac{1}{4} & \text{if } t \in \left]\frac{1}{2}, \frac{3}{4}\right]\,, \\ 3t - 2 & \text{otherwise,} \end{cases} \\
S_\delta(x,y) &= \begin{cases} \min\{\delta(x), \delta(y)\} & \text{if } (x,y) \in [0,1[^2\,, \\ \min\{x,y\} & \text{otherwise.} \end{cases}
\end{aligned}$$

Then S_δ is a commutative continuous semi-copula [19], but it is not a quasi-t-norm, because there does not exist a t-norm that coincides with S_δ on the diagonal track $\{(t,t) \mid t \in [0,1]\}$; such a t-norm would have δ as its *diagonal* which is not possible [47, Remark 17].

Problem 3.5. (F. Durante, `fabrizio.durante@unibz.it`) Provide an alternative characterization of the class of all quasi-t-norms.

Notice that the above problem could have consequences on the extension of a triangular norm from a given (affine) section to the whole unit square.

4. PROBLEMS IN EFFECT ALGEBRAS

A partial algebra $(E; \oplus, 0, 1)$ is called an *effect algebra* [21] if 0, 1 are two distinct elements and $\oplus$ is a partially defined binary operation on E which satisfy the following conditions for any $x, y, z \in E$:

(Ei) $x \oplus y = y \oplus x$ if $x \oplus y$ is defined,

(Eii) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ if one side is defined,

(Eiii) for every $x \in E$ there exists a unique $y \in E$ such that $x \oplus y = 1$ (we write $x' = y$),

(Eiv) if $1 \oplus x$ is defined then $x = 0$.

We often denote the effect algebra $(E; \oplus, 0, 1)$ briefly by E. On every effect algebra E the partial order $\leq$ and a partial binary operation $\ominus$ can be introduced as follows:

$$x \leq y \text{ and } y \ominus x = z \text{ iff } x \oplus z \text{ is defined and } x \oplus z = y.$$

If E with the defined partial order is a lattice (a complete lattice) then $(E; \oplus, 0, 1)$ is called a *lattice effect algebra* (a *complete lattice effect algebra*).

An element $x \in E$ is called *sharp* if $x \wedge x' = 0$. If E is lattice ordered then $S(E) = \{x \in E \mid x \wedge x' = 0\}$ is an orthomodular lattice [36]. An effect algebra E is called *sharply dominating* if for every $x \in E$ there is $w^* \in S(E), x \leq w^*$ such that if $w \in S(E)$ and $x \leq w$ then $w^* \leq w$ [25].

In every sharply dominating Archimedean atomic lattice effect algebra E to any $x \in E, x \neq 0$ there exist the unique $w_x \in S(E)$, set of atoms $\{a_\kappa | \kappa \in H\}$ and positive integers $k_\kappa \neq \operatorname{ord}(a_\kappa)$ such that

$$x = w_x \oplus \left(\bigoplus \{k_\kappa a_\kappa | \kappa \in H\} \right) . \tag{8}$$

The above equality with the unique $w_x, a_\kappa, k_\kappa (\kappa \in H)$ is called a *basic decomposition of the given element* x (*BDE of* x for brevity).

We say that an Archimedean atomic lattice effect algebra E has a *BDE-property* if every $x \in E$, $x \neq 0$ has a BDE of x. It was proved [56] that for an Archimedean atomic lattice effect algebra E the following conditions are equivalent:

(i) E is sharply dominating.

(ii) E has a BDE-property.

Consequently in such a case the existence of an (o)-continuous state on $S(E)$ implies the existence of a state on E (see the “State Smearing Theorem” [56, Theorem 4.3]). Jenča has proved [33] that in every sharply dominating effect algebra E any $x \in E, x \neq 0$ has a unique decomposition $x = w_x \oplus e_x$, where $w_x \in S(E)$ and $e_x \in M(E) = \{z \in E \mid y \in S(E), y \leq z \text{ implies } y = 0\}$. Elements of $M(E)$ are called *meager*. In [25] it was proved that the subset $S(E)$ of sharp elements of a sharply dominating effect algebra E is an orthoalgebra being a sub-effect algebra of E. Moreover, if E is S-dominating (i. e., sharply dominating and $a \wedge p$ exists in E for every $a \in E$ and $p \in S(E)$) then $S(E)$ is an orthomodular lattice [26].

It was proved [56] that if E is a sharply dominating Archimedean atomic effect algebra E (not lattice ordered) then the existence of an (o)-continuous state on $S(E)$ does not imply the existence of a state on E.

It was proved that every orthoalgebra and every lattice effect algebra E are homogeneous [32]. Recall that an effect algebra E is called *homogeneous* iff for any $u, v_1, v_2 \in E$ with $u \leq v_1 \oplus v_2 \leq u'$ there exist $u_1 \leq v_1, u_2 \leq v_2$ such that $u = u_1 \oplus u_2$ (equivalently $u \leq v_1 \oplus v_2 \oplus \ldots v_n \leq u'$ implies that there exist $u_1, \ldots, u_n \in E$ such that $u_i \leq v_i$ and $u = u_1 \oplus u_2 \oplus \ldots u_n$) [31].

Problem 4.1.
(Z. Riečanová, J. Paseka, `zdenka.riecanova@stuba.sk, paseka@math.muni.cz`)
One can ask whether every sharply dominating homogeneous Archimedean atomic effect algebra E has some kind of decomposition property of elements by atoms of E.

Problem 4.2.
(Z. Riečanová, J. Paseka, `zdenka.riecanova@stuba.sk, paseka@math.muni.cz`)
The unanswered question is whether for every sharply dominating homogeneous Archimedean atomic effect algebra E the existence of an (o)-continuous state on $S(E)$ implies the existence of a state on E.

If E is a complete atomic lattice effect algebra then the center $C(E) = \{x \in E \mid y = (y \wedge x) \vee (y \wedge x') \text{ for all } y \in E\}$ of E is a complete atomic Boolean algebra [54, Theorem 2.8]. Hence $\bigvee_{C(E)} A_{C(E)} = 1$, where $A_{C(E)}$ is the set of all atoms of $C(E)$. Moreover, since $C(E)$ is a complete sublattice of E, we have also that $\bigvee_E A_{C(E)} = 1$. Recently, M. Kalina [38] proved that there are Archimedean atomic lattice effect algebras E with atomic centers $C(E)$ for which $\bigvee_E A_{C(E)}$ does not exist. Moreover, Riečanová [55] has shown that for Archimedean atomic lattice effect algebras E with atomic centers $C(E)$ the condition $\bigvee_E A_{C(E)} = 1$ is equivalent to the condition $\bigvee_E D = \bigvee_{C(E)} D$ for every $D \subseteq C(E)$ for which at least one of these suprema exists (note that some other equivalent conditions can be found).

If an effect algebra E is not lattice ordered but atomic and *orthocomplete* (meaning that $\bigoplus G$ exists for every $G \subseteq E$ such that $x \leq y'$ for every pair of different elements $x, y \in G$ and $\bigoplus F$ exists for every finite $F = \{x_1, \ldots, x_k\} \subseteq G$, in which case we set $\bigoplus_E G = \bigvee_E \{\bigoplus F \mid F \subseteq G \text{ is finite}\}$) then $C(E)$ is an atomic Boolean algebra [35]. Here again $\bigoplus_E A_{C(E)} = 1 = \bigoplus_{C(E)} A_{C(E)}$.

Problem 4.3.
(Z. Riečanová, J. Paseka, zdenka.riecanova@stuba.sk, paseka@math.muni.cz)
For an Archimedean atomic effect algebra E with atomic center $C(E)$ find some conditions equivalent to the condition $\bigoplus_E A_{C(E)} = 1$, resp. to the condition $\bigoplus_E D = \bigoplus_{C(E)} D$ for every $D \subseteq C(E)$ for which at least one of these sums exists.

ACKNOWLEDGMENT

This work was supported by the grants VEGA 1/0080/10 and APVV-0012-07.

(Received March 18, 2010)

REFERENCES

[1] N. Ahmad, H. K. Kim, and R. J. McCann: Extremal doubly stochastic measures and optimal transportation. Working paper, 2009.

[2] C. Alsina: On Schur-concave t-norms and triangle functions. In: General inequalities, 4 (Oberwolfach, 1983), volume 71 of Internat. Schriftenreihe Numer. Math., pp. 241–248. Birkhäuser, Basel 1984.

[3] C. Alsina, M. J. Frank, and B. Schweizer: Problems on associative functions. Aequationes Math. *66* (2003), 128–140.

[4] C. Alsina, R. B. Nelsen, and B. Schweizer: On the characterization of a class of binary operations on distribution functions. Statist. Probab. Lett. *17* (1993), 85–89.

[5] C. Alsina, B. Schweizer, and M. J. Frank: Associative Functions. World Scientific Publishing Company, Singapore 2006.

[6] E. Alvoni, P. L. Papini, and F. Spizzichino: On a class of transformations of copulas and quasi-copulas. Fuzzy Sets and Systems *160* (2009), 334–343.

[7] K. Atanassov: Intuitionistic Fuzzy Sets, Theory and Applications. Physica-Verlag, Heidelberg 1999.

[8] B. Bassan and F. Spizzichino: Relations among univariate aging, bivariate aging and dependence for exchangeable lifetimes. J. Multivariate Anal. *93* (2005), 313–339.

[9] D. Butnariu and E. P. Klement: Triangular Norm-Based Measures and Games with Fuzzy Coalitions. Kluwer Academic Publishers, Dordrecht 1993.

[10] M. Couceiro: On two generalizations of associativity. In: Proc. FSTA 2010 (E. P. Klement et al., eds.), p. 47, Liptovský Ján 2010.

[11] I. Cuculescu and R. Theodorescu: Extreme value attractors for star unimodal copulas. CR Math. Acad. Sci. Paris *334* (2002) 689–692.

[12] F. Durante, R. Foschi, and P. Sarkoci: Distorted copulas: constructions and tail dependence. Comm. Statist. Theory Methods 2010. In press.

[13] F. Durante, A. Kolesárová, R. Mesiar, and C. Sempi: Copulas with given diagonal sections: Novel constructions and applications. Internat. J. Uncertain. Fuzziness and Knowledge-Based Systems *18* (2007), 397–410.

[14] F. Durante, R. Mesiar, and P. L. Papini: The lattice-theoretic structure of the sets of triangular norms and semi-copulas. Nonlinear Analysis, Theory, Methods and Applications *69* (2008), 46–52.

[15] F. Durante, R. Mesiar, and C. Sempi: Copulas with given diagonal section: some new results. In: Proc. EUSFLAT-LFA Conference (E. Montseny and P. Sobrevilla, eds.), pp. 932–936, Barcelona 2005.

[16] F. Durante, J. A. Rodríguez-Lallena, and M. Úbeda-Flores: New constructions of diagonal patchwork copulas. Inform. Sci. *179* (2009), 3383–3391.

[17] F. Durante and C. Sempi: Copulæ, and Schur-concavity. Internat. Math. J. *3* (2003), 893–905.

[18] F. Durante and C. Sempi: Copula and semicopula transforms. Internat. J. Math. Math. Sci. (2005), 645–655.

[19] F. Durante and C. Sempi: Semicopulæ. Kybernetika *41* (2005), 315–328.

[20] F. Durante and F. Spizzichino: Semi-copulas, capacities and families of level curves. Fuzzy Sets and Systems *161* (2009), 2009.

[21] D. J. Foulis and M. K. Bennett: Effect algebras and unsharp quantum logics. Found. Phys. *24* (1994), 1331–1352.

[22] M. J. Frank: On the simultaneous associativity of $F(x, y)$ and $x + y - F(x, y)$. Aeqationes Math. *19* (1979), 194–226.

[23] C. Genest, J. J. Quesada-Molina, J. A. Rodríguez-Lallena, and C. Sempi: A characterization of quasi-copulas. J. Multivariate Anal. *69* (1999), 193–205.

[24] M. Grabisch, J.-L. Marichal, R. Mesiar, and E. Pap: Aggregation Functions. Cambridge University Press, Cambridge 2009.

[25] S. P. Gudder: Sharply dominating effect algebras. Tatra Mt. Math. Publ. *15* (1998), 23–30.

[26] S. P. Gudder: S-dominating effect algebras. Inter. J. Theor. Phys. *37* (1998), 915–923.

[27] K. Hestir and S. C. Williams: Supports of doubly stochastic measures. Bernoulli *1* (1995), 217–243.

[28] D. Hilbert: Mathematical problems. Bull. Amer. Math. Soc. *8* (1901/02), 437–479.

[29] V. Janiš: T-Norm based cuts of intuitionistic fuzzy sets. Inform. Sci. *180* (2010), 1134–1137.

[30] P. Jaworski: On copulas and their diagonals. Inform. Sci. *179* (2009), 2863–2871.

[31] G. Jenča: Blocks of homogeneous effect algebras. Bull. Austr. Math. Soc. *64* (2001), 81–98.

[32] G. Jenča: Finite homogeneous and lattice ordered effect algebras. Discrete Mathematics *272* (2003), 197–214.

[33] G. Jenča: Sharp and meager elements in orthocomplete homogeneous effect algebras. Order *27* (2010), 41–61.

[34] G. Jenča: Coexistence in interval effect algebras. Proc. Amer. Math. Soc. To appear.

[35] G. Jenča and S. Pulmannová: Orthocomplete effect algebras. Proc. Amer. Math. Soc. *131* (2003), 2663–2671.

[36] G. Jenča and Z. Riečanová: On sharp elements in lattice ordered effect algebras. BUSEFAL *80* (1999), 24–29.

[37] T. Jwaid and B. De Baets: Double conic copulas. In: Proc. FSTA 2010 (E. P. Klement et al., eds.), p. 73, Liptovský Ján 2010.

[38] M. Kalina: On central atoms of Archimedean atomic lattice effect algebras. Kybernetika *46* (2010), 609–620.

[39] E. P. Klement and R. Mesiar: Open problems posed at the Eight International Conference on Fuzzy Set Theory and Applications (FSTA 2006, Liptovský Ján, Slovakia). Kybernetika *42* (2006), 225–235.

[40] E. P. Klement, R. Mesiar, and E. Pap: Triangular Norms. Kluwer Academic Publishers, Dordrecht 2000.

[41] E. P. Klement, R. Mesiar, and E. Pap: Uniform approximation of associative copulas by strict and non-strict copulas. Illinois J. Math. *45* (2001), 1393–1400.

[42] E. P. Klement, R. Mesiar, and E. Pap: Problems on triangular norms and related operators. Fuzzy Sets and Systems *145* (2004), 471–479.

[43] E. P. Klement, R. Mesiar, and E. Pap: Archimax copulas and invariance under transformations. C R Math. Acad. Sci. Paris *340* (2005), 755–758.

[44] A. Kolesárová, J. Mordelová, and A. Stupňanová: Aggregation functions as extensions of fuzzy measures. In: Proc. FSTA 2010 (E. P. Klement et al., eds.), p. 80, Liptovský Ján 2010.

[45] M. Marinacci and L. Montrucchio: Ultramodular functions. Math. Oper. Res. *30* (2005), 311–332.

[46] R. Mesiar and V. Novák: Open problems from the 2nd International Conference on Fuzzy Sets Theory and Its Applications. Fuzzy Sets and Systems *81* (1996), 185–190.

[47] A. Mesiarová: Triangular Norms and their Diagonal Functions. Master Thesis, Comenius University, Bratislava 2002.

[48] A. J. McNeil and J. Nešlehová: Multivariate Archimedean copulas, d-monotone functions and L_1-norm symmetric distributions. Ann. Statist. *37* (2009), 3059–3097.

[49] R. Moynihan: Infinite τ_T products of distribution functions. J. Austral. Math. Soc. Ser. A *26* (1978), 227–240.

[50] R. B. Nelsen: An Introduction to Copulas. Second edition. Springer Science+Business Media, New York 2006.

[51] R. B. Nelsen and M. Úbeda-Flores: The lattice-theoretic structure of sets of bivariate copulas and quasi-copulas. CR Math. Acad. Sci. Paris *341* (2005), 583–586.

[52] E. L. Post: Polyadic groups. Trans. Amer. Math. Soc. *48* (1940), 208–350.

[53] J. J. Quesada-Molina and J.-A. Rodríguez-Lallena: Some remarks on the existence of doubly stochastic measures with latticework hairpin support. Aequationes Math. *47* (1994), 164–174.

[54] Z. Riečanová: Orthogonal sets in effect algebras. Demonstratio Math. *34* (2001), 525–532.

[55] Z. Riečanová: MacNeille completion of Archimedean lattice effect algebras and their sublattice effect algebras. Preprint.

[56] Z. Riečanová and Wu Junde: States on sharply dominating effect algebras. Science in China Series A: Mathematics *51* (2008), 907–914.

[57] H. L. Royden: Real Analysis. Third edition. Macmillan Publishing Company, New York 1988.

[58] B. Schweizer and A. Sklar: Probabilistic Metric Spaces. North-Holland, New York 1983.

[59] J. Tawn: Bivariate extreme value theory: Models and estimation. Biometrika *75* (1988), 397–415.

[60] T. Vasiliev: Four extended level operators of membership/non-membership over intuitionistic fuzzy sets. In: Proc. Twelfth International Conference on Intuitionistic Fuzzy Sets, Vol. 2 (J. Kacprzyk and K. Atanassov, eds.), Sofia 2008. Notes on Intuitionistic Fuzzy Sets *14* (2008), 100–107.

Radko Mesiar, Department of Mathematics and Descriptive Geometry, Slovak University of Technology, Radlinského 11, 813 68 Bratislava. Slovak Republic.
e-mail: mesiar@math.sk

Peter Sarkoci, Department of Mathematics and Descriptive Geometry, Slovak University of Technology, Radlinského 11, 813 68 Bratislava. Slovak Republic.
e-mail: sarkoci@math.sk, peter.sarkoci@gmail.com