

## ON HYPERPLANES AND SEMISPACES IN MAX–MIN CONVEX GEOMETRY

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The concept of separation by hyperplanes and halfspaces is fundamental for convex geometry and its tropical (max-plus) analogue. However, analogous separation results in max-min convex geometry are based on semispaces. This paper answers the question which semispaces are hyperplanes and when it is possible to “classically” separate by hyperplanes in max-min convex geometry.

*Keywords:* tropical convexity; fuzzy algebra; separation

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### 1. INTRODUCTION

Consider the set  $\mathcal{B} = [0, 1]$  endowed with the operations  $\oplus = \max$ ,  $\wedge = \min$ . This is a well-known distributive lattice, and like any distributive lattice it can be considered as a semiring equipped with addition  $\oplus$  and multiplication  $\otimes := \wedge$ . Importantly, both operations are idempotent,  $a \oplus a = a$  and  $a \otimes a = a \wedge a = a$ , and closely related to the order:  $a \oplus b = b \Leftrightarrow a \leq b \Leftrightarrow a \wedge b = a$ . For standard literature on lattices and semirings see e.g. [1] and [9].

We consider  $\mathcal{B}^n$ , the cartesian product of  $n$  copies of  $\mathcal{B}$ , and equip this cartesian product with operations of taking componentwise  $\oplus$ :  $(x \oplus y)_i := x_i \oplus y_i$  for  $x, y \in \mathcal{B}^n$  and  $i = 1, \dots, n$ , and scalar  $\wedge$ -multiplication:  $(a \wedge x)_i := a \wedge x_i$  for  $a \in \mathcal{B}$ ,  $x \in \mathcal{B}^n$  and  $i = 1, \dots, n$ . Thus  $\mathcal{B}^n$  is considered as a semimodule over  $\mathcal{B}$  [9]. Alternatively, one may think in terms of vector lattices [1].

A subset  $C$  of  $\mathcal{B}^n$  is said to be *max-min convex*, (or briefly *convex*), if the relations  $x, y \in C, \alpha, \beta \in \mathcal{B}, \alpha \oplus \beta = 1$  imply  $\alpha \wedge x \oplus \beta \wedge y \in C$ . Here and everywhere in the paper we assume the priority of  $\wedge$  over  $\oplus$ . If  $x, y \in \mathcal{B}^n$ , the set

$$\begin{aligned} [x, y] &:= \{\alpha \wedge x \oplus \beta \wedge y \in \mathcal{B}^n \mid \alpha, \beta \in \mathcal{B}, \alpha \oplus \beta = 1\} \\ &= \{\max(\min(\alpha, x), \min(\beta, y)) \in \mathcal{B}^n \mid \alpha, \beta \in \mathcal{B}, \max(\alpha, \beta) = 1\}, \end{aligned} \quad (1)$$

is called the *max-min segment* (or briefly, the *segment*) *joining*  $x$  and  $y$ . Like in the ordinary convexity in the real space, a set is max-min convex if and only if any two points are contained in it together with the max-min segment joining them. The max-min segments have been described in [14, 16]. Other types of convex sets are max-min semispaces, halfspaces and hyperplanes.

One of the main motivations for the investigation of max-min convex sets is in the study of tropically convex sets, analogously defined over the semiring  $\mathbb{R}_{\max}$ , which is the completed set of real numbers  $\mathbb{R} \cup \{-\infty\}$  endowed with operations of idempotent addition  $a \oplus b := \max(a, b)$  and multiplication  $a \otimes b := a + b$ . Tropical convexity and its lattice-theoretic generalizations, pioneered in [17, 18], received much attention and rapidly developed over the last decades [3, 4, 5, 6, 10, 12, 13]. Another source of interest comes from the matrix algebra developed over the max-min semiring, also known as *fuzzy algebra* [2, 7, 8].

In this article we continue the study of max-min convex structures started in [11, 14, 15]. We are interested in separation of max-min convex sets by max-min hyperplanes and semispaces.

When  $z \in \mathcal{B}^n$ , we call a subset  $S(z)$  of  $\mathcal{B}^n$  a *max-min semispace* (or, briefly, a *semispace*) at  $z$ , if it is a maximal (with respect to set-inclusion) max-min convex set avoiding  $z$ ; a subset  $S$  of  $\mathcal{B}^n$  is called a *semispace*, if there exists  $z \in \mathcal{B}^n$  such that  $S = S(z)$ . We recall that in  $\mathcal{B}^n$  there exist at most  $n + 1$  semispaces at each point, exactly  $n + 1$  at each finite point, and each convex set avoiding  $z$  is contained in at least one of those semispaces [15].

A *max-min hyperplane* (briefly, a *hyperplane*) is the set of all points  $x = (x_1, \dots, x_n) \in \mathcal{B}^n$  satisfying an equation of the form

$$a_1 \wedge x_1 \oplus \dots \oplus a_n \wedge x_n \oplus a_{n+1} = b_1 \wedge x_1 \oplus \dots \oplus b_n \wedge x_n \oplus b_{n+1}, \quad (2)$$

with  $a_i, b_i \in \mathcal{B}$  for  $i = 1, \dots, n + 1$ , where each side contains at least one term. The combinatorial structure of hyperplanes is described in [11]. If the equality in (2) is replaced by a strict (resp. non-strict) inequality, then we obtain an open halfspace (resp. a closed halfspace). Note that any max-min closed halfspace is a max-min hyperplane (due to  $a \oplus b = b \Leftrightarrow a \leq b$ ) but not conversely.

One of the main applications of semispaces is in separation results: the family of semispaces is the smallest intersectional basis for the family of all convex sets in  $\mathcal{B}^n$  [15]. Separation results by hyperplanes and halfspaces in the tropical convexity and lattice-theoretic generalizations are found in [3, 4, 5, 6, 10, 17, 18]. These results are very optimistic: any point can be separated from a closed tropically convex set, and even any two compact non-intersecting convex sets can be separated from each other by two closed halfspaces [6]. In contrast, [11] contains a counterexample to separation of a point and a max-min convex set by max-min hyperplanes (equivalently, by max-min halfspaces).

The main goal of this paper is to further clarify separation by hyperplanes in max-min algebra. The main result of this paper, Theorem 3.1, shows which closures of semispaces are hyperplanes and which are not. As a corollary, we obtain in what case it is possible to separate a point from a closed max-min convex set by a hyperplane.

## 2. THE STRUCTURE OF SEMISPACES

We recall the structure of the semispaces in  $\mathcal{B}^n$  at an arbitrary point  $x^0$ . We follow closely [15].

Without loss of generality we may assume that the coordinates  $(x_1^0, \dots, x_n^0)$  of the point  $x^0$  are in decreasing order, that is:

$$x_1^0 \geq \dots \geq x_n^0. \quad (3)$$

The set  $\{x_1^0, \dots, x_n^0\}$  admits a natural subdivision into ordered subsets such that the elements of each subset are either equal to each other or are in strictly decreasing order, say

$$\begin{aligned} x_1^0 &= \dots = x_{k_1}^0 > \dots > x_{k_1+l_1+1}^0 = \dots = x_{k_1+l_1+k_2}^0 \dots \\ &> x_{k_1+l_1+k_2+l_2+1}^0 = \dots = x_{k_1+l_1+k_2+l_2+k_3}^0 > \dots \\ &> x_{k_1+l_1+\dots+k_{p-1}+l_{p-1}+1}^0 = \dots = x_{k_1+l_1+\dots+k_{p-1}+l_{p-1}+k_p}^0 \\ &> \dots > x_{k_1+l_1+\dots+k_p+l_p}^0 (= x_n^0), \end{aligned} \quad (4)$$

where we make the following conventions:

1.  $k_1 = 0$  if and only if the sequence (4) starts with the strict inequality  $x_1^0 > x_2^0$ ; in this case  $l_1 \neq 0$  and the beginning of the sequence will be:

$$\begin{aligned} x_1^0 &> \dots > x_{l_1}^0 > x_{l_1+1}^0 = \dots = x_{l_1+k_2}^0 > \dots \\ &\dots > x_{l_1+k_2+l_2}^0 > x_{l_1+k_2+l_2+1}^0 = \dots; \end{aligned} \quad (5)$$

in particular, if (4) has only strict inequalities between its terms one has  $p = 1, k_1 = 0, l_1 = n$ . When (4) has only equalities between its terms, one has  $p = 1, k_1 = n, l_1 = 0$ .

2.  $l_p = 0$  if and only if the sequence  $\{x_1^0, \dots, x_n^0\}$  ends with equalities, that is, with  $x_{n-1}^0 = x_n^0$ ; in this case, if  $p \geq 2$ , the end of the sequence (4) will be

$$\dots > x_{k_1+l_1+\dots+k_{p-1}+l_{p-1}+1}^0 = \dots = x_{k_1+l_1+\dots+k_{p-1}+l_{p-1}+k_p}^0, \quad (6)$$

while if  $p = 1$ , the whole sequence will be  $x_1^0 = \dots = x_{k_1}^0 (= x_n^0)$ . In other words, we take  $l_p \neq 0$  if and only if  $x_{n-1}^0 > x_n^0$ .

Let us introduce the following notations:

$$L_0 = 0, K_1 = k_1, L_1 = K_1 + l_1 = k_1 + l_1, \quad (7)$$

$$K_j = L_{j-1} + k_j = k_1 + l_1 + \dots + k_{j-1} + l_{j-1} + k_j \quad (j = 2, \dots, p), \quad (8)$$

$$L_j = K_j + l_j = k_1 + l_1 + \dots + k_j + l_j \quad (j = 2, \dots, p). \quad (9)$$

We observe that  $l_j = 0$  if and only if  $K_j = L_j$  and  $L_p = n$ .

The following description of semispaces is taken from [15, Proposition 4.1]. We need to distinguish the case when the sequence (4) ends with zeros either/or begins with ones, since some semispaces become empty in that case.

**Proposition 2.1.** Let  $x^0 = (x_1^0, \dots, x_n^0) \in \mathcal{B}^n$ ,  $x_1^0 \geq \dots \geq x_n^0$ , and let  $k_1, l_1, \dots, k_p, l_p, p$  be non-negative integers as above.

a) If  $0 < x_i^0 < 1$  for all  $i = 1, \dots, n$ , then there are  $n + 1$  semispaces  $S_0(x^0), S_1(x^0), \dots, S_n(x^0)$  at  $x^0$ , namely:

$$S_0(x^0) = \{x \in \mathcal{B}^n \mid x_i > x_i^0 \text{ for some } 1 \leq i \leq n\}, \quad (10)$$

$$S_{K_j+q}(x^0) = \{x \in \mathcal{B}^n \mid x_{K_j+q} < x_{K_j+q}^0, \text{ or } x_i > x_i^0 \text{ for some } K_j + q + 1 \leq i \leq n\} \\ (q = 1, \dots, l_j; j = 1, \dots, p) \text{ if } l_j \neq 0, \quad (11)$$

$$S_{L_{j-1}+q}(x^0) = \{x \in \mathcal{B}^n \mid x_{L_{j-1}+q} < x_{L_{j-1}+q}^0, \text{ or } x_i > x_i^0 \text{ for some } K_j + 1 \leq i \leq n\} \\ (q = 1, \dots, k_j; j = 1, \dots, p \text{ if } k_1 \neq 0, \text{ or } j = 2, \dots, p \text{ if } k_1 = 0). \quad (12)$$

b) If there exists an index  $i \in \{1, \dots, n\}$  such that  $x_i^0 = 1$ , but no index  $j$  such that  $x_j^0 = 0$ , then the semispaces at  $x^0$  are  $S_1(x^0), \dots, S_n(x^0)$  of part a).

c) If there exists an index  $j \in \{1, \dots, n\}$  such that  $x_j^0 = 0$ , but no index  $i$  such that  $x_i^0 = 1$ , then the semispaces at  $x^0$  are  $S_0(x^0), S_1(x^0), \dots, S_{\beta-1}(x^0)$  of part a), where

$$\beta := \min\{1 \leq j \leq n \mid x_j^0 = 0\}. \quad (13)$$

d) If there exist indices  $i, j \in \{1, \dots, n\}$  such that  $x_i^0 = 1$  and  $x_j^0 = 0$ , then the semispaces at  $x^0$  are  $S_1(x^0), \dots, S_{\beta-1}(x^0)$  of part a), where  $\beta$  is the number (13).

Pictures of all types of semispaces in  $\mathcal{B}^2$  are shown in Figure 1. The figure is taken from [15].

### 3. MAIN RESULTS

If we take the topological closure of semispaces, all inequalities in (10)–(12) become non-strict. We denote such closures by  $\overline{S}_i(x^0)$ .

We will also denote

$$\mathcal{D}_n = \{\overbrace{(a, \dots, a)}^n \mid a \in \mathcal{B}\}. \quad (14)$$

This set will be called the *diagonal* of  $\mathcal{B}^n$ .

Next we investigate when the closures of semispaces are hyperplanes.

**Theorem 3.1. (Semispaces and Hyperplanes)** Assume that  $x^0 \in \mathcal{B}^n$  satisfies  $x_1^0 \geq x_2^0 \geq \dots \geq x_n^0$  and let  $\mathcal{H} := \overline{S}_i(x^0)$  for  $i = 0, 1, \dots, n$ . The following statements are equivalent.

a)  $\mathcal{H}$  takes either of the following forms for some  $a \in \mathcal{B}$ :

$$\mathcal{H}^+(a) = \{x \mid x_i \geq a \text{ for some } i = 1, \dots, n\}, \quad \text{for } a < 1, \\ \mathcal{H}_i^-(a) = \{x \mid x_i \leq a\}, \quad \text{for } a > 0, \quad i = 1, 2, \dots, n. \quad (15)$$

b)  $\mathcal{H} = \overline{S}_i(y)$  for some  $y \in \mathcal{D}_n$ .

c)  $\mathcal{H}$  is a hyperplane.

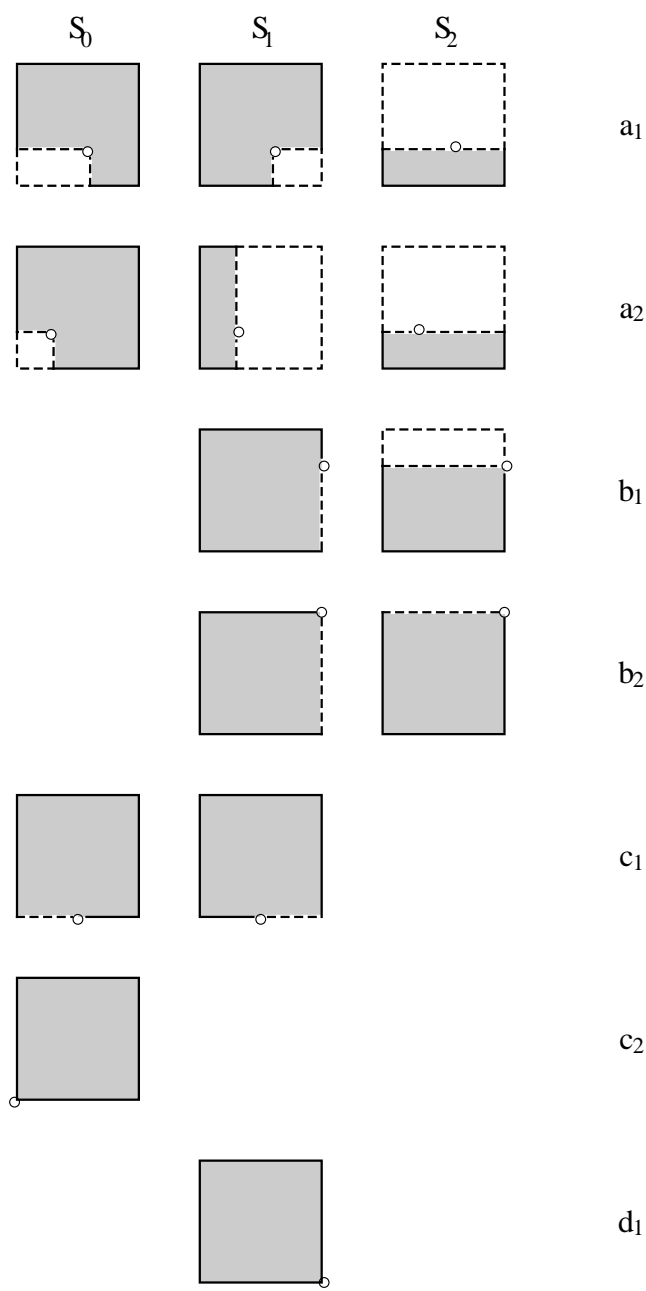


Fig. 1. Semispaces in dimension 2.

**Proof.** First we observe that a) and b) are equivalent. Indeed,  $\mathcal{H}^+(a) = \overline{S}_0(x)$  and  $\mathcal{H}_i^-(a) = \overline{S}_i(x)$  where  $x = (a, \dots, a)$ .

We can represent

$$\begin{aligned}\mathcal{H}^+(a) &= \{x \mid \bigoplus_{i=1}^n x_i = a \oplus \bigoplus_{i=1}^n x_i\}, \\ \mathcal{H}_i^-(a) &= \{x \mid x_i = a \wedge x_i\}.\end{aligned}\tag{16}$$

which shows a)  $\Rightarrow$  c).

It remains to show that the closure of a semispace that is not of the form (15) cannot be a hyperplane.

*Case 1.* Consider  $\overline{S}_0(x^0)$  where  $x^0 \notin \mathcal{D}_n$ .

If  $x_i^0 = 0$  for some  $i$ , then  $\overline{S}_0(x^0) = \mathcal{B}^n = \mathcal{H}^+(0)$ . Hence we can assume  $x_i^0 > 0$  for all  $i$ .

Let  $y \in \mathcal{B}^n$  be such that  $y_i < x_i^0$  for all  $i$ , and

$$x_1^0 > y_1 > x_n^0 > y_n.\tag{17}$$

We define  $z^1$  and  $z^2$  by

$$z_i^1 = \begin{cases} x_1^0, & \text{if } i = 1, \\ y_i, & \text{otherwise,} \end{cases} \quad z_i^2 = \begin{cases} x_n^0, & \text{if } i = n, \\ y_i, & \text{otherwise.} \end{cases}\tag{18}$$

Obviously  $\overline{S}_0(x^0)$  contains both  $z^1$  and  $z^2$ , but it does not contain  $y = z^1 \wedge z^2$ . Our goal is to show that any hyperplane defined by (2) that contains  $z^1$  and  $z^2$  will also contain  $y$ . Equation (2) for  $z^1$ ,  $z^2$  and  $y$  reduces to, respectively,

$$a_1 \wedge x_1^0 \oplus a_n \wedge y_n \oplus \alpha = b_1 \wedge x_1^0 \oplus b_n \wedge y_n \oplus \beta,\tag{19}$$

$$a_1 \wedge y_1 \oplus a_n \wedge x_n^0 \oplus \alpha = b_1 \wedge y_1 \oplus b_n \wedge x_n^0 \oplus \beta,\tag{20}$$

$$a_1 \wedge y_1 \oplus a_n \wedge y_n \oplus \alpha = b_1 \wedge y_1 \oplus b_n \wedge y_n \oplus \beta,\tag{21}$$

where

$$\alpha = \bigoplus_{i \neq 1, n} a_i \wedge y_i, \quad \beta = \bigoplus_{i \neq 1, n} b_i \wedge y_i.\tag{22}$$

We need to show that (19) and (20) together imply (21). We do this by showing that the minimum of left hand sides of (19) and (20) is always equal to the left hand side of (21). By analogy, the same holds for the right hand sides.

We first pull  $\alpha$  out of the brackets using the distributivity law  $(b \oplus a) \wedge (c \oplus a) = (b \wedge c) \oplus a$ :

$$\begin{aligned}(a_1 \wedge x_1^0 \oplus a_n \wedge y_n \oplus \alpha) \wedge (a_1 \wedge y_1 \oplus a_n \wedge x_n^0 \oplus \alpha) \\ = ((a_1 \wedge x_1^0 \oplus a_n \wedge y_n) \wedge (a_1 \wedge y_1 \oplus a_n \wedge x_n^0)) \oplus \alpha.\end{aligned}\tag{23}$$

It remains to show that

$$\begin{aligned}(a_1 \wedge x_1^0 \oplus a_n \wedge y_n) \wedge (a_1 \wedge y_1 \oplus a_n \wedge x_n^0) \\ = a_1 \wedge y_1 \oplus a_n \wedge y_n.\end{aligned}\tag{24}$$

If  $a_1, a_n$  are large enough then (17) implies

$$a_1 \wedge x_1^0 \geq a_1 \wedge y_1 \geq a_n \wedge x_n^0 \geq a_n \wedge y_n, \quad (25)$$

and in this case it is easy to see that (24) holds, both sides being equal to  $a_1 \wedge y_1$ . Note that the first and the third inequalities always hold. The second inequality may not hold true, but then  $a_1 \leq y_1$ , in which case  $a_1 \wedge x_1^0 = a_1 \wedge y_1$ . In this case we use the distributivity again, and this transforms the left hand side of (24) to

$$(a_1 \wedge y_1) \oplus (a_n \wedge y_n \wedge x_n^0) = (a_1 \wedge y_1) \oplus (a_n \wedge y_n), \quad (26)$$

which proves (24) and hence the claim for Case 1.

*Case 2.* Consider  $\overline{S}_i(x^0)$  where  $x^0 \notin \mathcal{D}_n$ .

Denote

$$n(i) = \begin{cases} i+1, & \text{if } K_s + 1 \leq i \leq L_s, \\ K_{s+1} + 1, & \text{if } L_s + 1 \leq i \leq K_{s+1}, \end{cases} \quad (27)$$

where  $K_s, L_s$  are defined by (7)–(9).

If  $n(i) = n+1$  then  $\overline{S}_i(x^0) = \mathcal{H}_i^-(x_i^0)$ . If  $x_i^0 = 1$  or  $x_j^0 = 0$  for some  $j \geq n(i)$  then  $\overline{S}_i(x^0) = \mathcal{B}^n$ . Otherwise, we construct points  $y, z$  and  $v$  defined by

$$\begin{aligned} y_j &= \begin{cases} 1, & \text{if } j = i, \\ x_j^0, & \text{otherwise,} \end{cases} \quad z_j = \begin{cases} 0, & \text{if } j \geq n(i), \\ x_j^0, & \text{otherwise,} \end{cases} \\ v_j &= \begin{cases} 1, & \text{if } j = i, \\ 0, & \text{if } j \geq n(i), \\ x_j^0, & \text{otherwise.} \end{cases} \end{aligned} \quad (28)$$

It is clear that  $y$  and  $z$  belong to  $\overline{S}_i(x^0)$  but  $v$  does not. Our goal will be to show that if a hyperplane defined by (2) contains  $y$  and  $z$  then it also contains  $v$ . Equality (2) reduces in the cases of  $y, z$  and  $v$  respectively to

$$a_i \oplus \bigoplus_{s \geq n(i)} (a_s \wedge x_s^0) \oplus \alpha = b_i \oplus \bigoplus_{s \geq n(i)} (b_s \wedge x_s^0) \oplus \beta, \quad (29)$$

$$a_i \wedge x_i^0 \oplus \alpha = b_i \wedge x_i^0 \oplus \beta, \quad (30)$$

$$a_i \oplus \alpha = b_i \oplus \beta, \quad (31)$$

where

$$\alpha = \bigoplus_{s \neq i, s < n(i)} (a_s \wedge x_s^0) \oplus a_{n+1}, \quad \beta = \bigoplus_{s \neq i, s < n(i)} (b_s \wedge x_s^0) \oplus b_{n+1}. \quad (32)$$

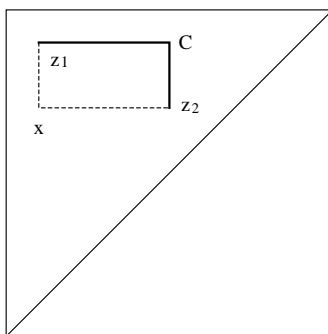
We need to show that (29) and (30) imply (31). Assume by contradiction that  $a_i \oplus \alpha \neq b_i \oplus \beta$ . Then there exists  $s \geq n(i)$  such that (29) equals  $a_s \wedge x_s^0$  or  $b_s \wedge x_s^0$  implying that  $x_s^0 \geq a_i \oplus b_i \oplus \alpha \oplus \beta$ . But then  $x_i^0 \geq x_s^0 \geq a_i \oplus b_i \oplus \alpha \oplus \beta$ , and equation (30), which is assumed to hold, is the same as (31), a contradiction. The proof of Case 2 is complete and the theorem is proved.  $\square$

**Remark 3.2.** We recall that in the tropical (max-plus) convex geometry the closure of *any* semispace is a hyperplane [12].

The key ingredient in the proof of Theorem 3.1 is the construction of examples where a point cannot be separated from a closed semispace by a hyperplane. The proof of Theorem 3.1 shows that such examples can be constructed for any dimension and for any semispace except for (15) which are precisely the hyperplanes. The proof for the case of  $\overline{S}_0(x^0)$  (Case 1) also shows that such examples can be constructed for any point outside the diagonal. We conclude the following.

**Corollary 3.3. (Non-separation by Hyperplanes)** Let  $x \in \mathcal{B}^n$  and  $x \notin \mathcal{D}_n$ . Then there exists a closed max-min convex set  $C \subseteq \mathcal{B}^n$  such that  $x$  cannot be separated from  $C$  by a hyperplane.

Simple counterexamples to separation by hyperplanes in dimension two have been obtained by one of the authors [11]: as shown on Figure 2, it actually suffices to take certain max-min segments [14, 16]. The convex set  $C = [z_1, z_2]$  cannot be separated by hyperplanes from the point  $x = z_1 \wedge z_2$ .



**Fig. 2.** Forbidden separation.

Such examples can be extended cylindrically to any dimension, which is precisely the geometric idea of the proof of Theorem 3.1.

On the other hand, as the semispaces taken at a diagonal point are hyperplanes, it is possible to separate a point on the diagonal from a closed convex set “classically”.

**Corollary 3.4. (Diagonal Separation by Hyperplanes)** Let  $x \in \mathcal{B}^n$  and  $x \in \mathcal{D}_n$ . Then any closed max-min convex set  $C \subseteq \mathcal{B}^n$  such that  $x \notin C$ , can be separated from  $x$  by a hyperplane.

**Proof.** Since  $x \notin C$ , there is a semispace  $S$  at  $x$  containing  $C$  [15, Theorem 5.1].

If  $S = S_0(x)$ , then for any  $y \in C$  there exists  $1 \leq i \leq n$  such that  $y_i > x_i$ . Due to compactness of  $C$ , there is  $\delta > 0$  such that above inequalities can be replaced by  $y_i \geq x_i + \delta$ . For  $x + \delta = (x_1 + \delta, \dots, x_n + \delta)$ , this implies  $C$  is included in  $\overline{S}_0(x + \delta)$ . By Theorem 3.1  $\overline{S}_0(x + \delta)$  is a hyperplane. Moreover it does not contain  $x$ .



If  $S = S_i(x)$ , then  $y_i < x_i$  for any  $y \in C$ . Due to compactness of  $C$ , there is  $\delta > 0$  such that above inequality can be replaced by  $y_i \leq x_i - \delta$ . This implies  $C$  is included in  $\overline{S}_i(x - \delta)$ . By Theorem 3.1  $\overline{S}_i(x - \delta)$  is a hyperplane. It avoids  $x$ .  $\square$

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