KYBERNETIKA — VOLUME *46* (2010), NUMBER 3, PAGES 387–396

MONOTONE INTERVAL EIGENPROBLEM IN MAX–MIN ALGEBRA

Martin Gavalec and Ján Plavka

The interval eigenproblem in max-min algebra is studied. A classification of interval eigenvectors is introduced and six types of interval eigenvectors are described. Characterization of all six types is given for the case of strictly increasing eigenvectors and Hasse diagram of relations between the types is presented.

Keywords: (max, min) eigenvector, interval coefficients

Classification: 08A72, 90B35, 90C47

1. INTRODUCTION

In practice, the values of vector or matrix inputs are not exact numbers and often they are rather contained in some intervals. Considering matrices and vectors with interval coefficients is therefore of great practical importance, see [2, 3, 5, 7, 8]. This paper investigates monotone interval eigenvectors of interval matrices in max-min algebra.

By max-min algebra we understand a triple $(\mathcal{B}, \oplus, \otimes)$, where $\mathcal{B}$ is a linearly ordered set, and $\oplus = \max$, $\otimes = \min$ are binary operations on $\mathcal{B}$. The notation $\mathcal{B}(m, n)$ ($\mathcal{B}(n)$) denotes the set of all matrices (vectors) of given dimension over $\mathcal{B}$. Operations $\oplus$, $\otimes$ are extended to matrices and vectors in a formal way. The linear ordering on $\mathcal{B}$ induces partial ordering on $\mathcal{B}(m, n)$ and $\mathcal{B}(n)$, the notations $\wedge$ ($\vee$) and $\bigwedge$ ($\bigvee$) are used for the operation of meet (join) in these sets.

The eigenproblem for a given matrix $A \in \mathcal{B}(n, n)$ in max-min algebra consists of finding a value $\lambda \in \mathcal{B}$ (eigenvalue) and a vector $x \in \mathcal{B}(n)$ (eigenvector) such that the equation $A \otimes x = \lambda \otimes x$ holds true. It is well-known that the above problem in max-min algebra can be reduced to solving the equation $A \otimes x = x$. The eigenproblem in max-min algebra has been studied by many authors. Interesting results were found in describing the structure of the eigenspace (the set of all eigenvectors), and algorithms for computing the largest eigenvector of a given matrix were suggested, see e. g. [1, 4].

A classification consisting of six different types of interval eigenvectors is presented in this paper, and detailed characterization of all described types is given for strictly increasing interval eigenvectors using methods from [6].

2. INTERVAL EIGENVECTORS CLASSIFICATION

Let n be a given natural number. We shall use the notation $N = \{1, 2, \ldots, n\}$. Similarly to [3, 5, 7], we define interval matrix with bounds $\underline{A}, \overline{A} \in \mathcal{B}(n,n)$ and interval vector with bounds $\underline{x}, \overline{x} \in \mathcal{B}(n)$ as follows

$$[\underline{A}, \overline{A}] = \left\{ A \in \mathcal{B}(n,n);\, \underline{A} \le A \le \overline{A} \right\}, \quad [\underline{x}, \overline{x}] = \{ x \in \mathcal{B}(n);\, \underline{x} \le x \le \overline{x} \}.$$

We assume in this section that an interval matrix $\mathbf{A} = [\underline{A}, \overline{A}]$ and an interval vector $\mathbf{X} = [\underline{x}, \overline{x}]$ are fixed. The interval eigenproblem for $\mathbf{A}$ and $\mathbf{X}$ consists in recognizing whether $A \otimes x = x$ holds true for $A \in \mathbf{A}$, $x \in \mathbf{X}$. In dependence on the applied quantifiers, we get six types of interval eigenvectors.

Definition 2.1. If interval matrix $\mathbf{A}$ is given, then interval vector $\mathbf{X}$ is called

- *strong eigenvector* of $\mathbf{A}$ if $(\forall A \in \mathbf{A})(\forall x \in \mathbf{X})[\, A \otimes x = x\,]$,
- *strong universal eigenvector* of $\mathbf{A}$ if $(\exists x \in \mathbf{X})(\forall A \in \mathbf{A})\,[A \otimes x = x\,]$,
- *universal eigenvector* of $\mathbf{A}$ if $(\forall A \in \mathbf{A})(\exists x \in \mathbf{X})[\, A \otimes x = x\,]$,
- *strong tolerance eigenvector* of $\mathbf{A}$ if $(\exists A \in \mathbf{A})(\forall x \in \mathbf{X})[\, A \otimes x = x\,]$,
- *tolerance eigenvector* of $\mathbf{A}$ if $(\forall x \in \mathbf{X})(\exists A \in \mathbf{A})[\, A \otimes x = x\,]$,
- *weak eigenvector* of $\mathbf{A}$ if $(\exists A \in \mathbf{A})(\exists x \in \mathbf{X})[\, A \otimes x = x\,]$.

Analogously as in [6], we denote the set of all strictly increasing vectors of dimension n as

$$\mathcal{B}^{<}(n) = \{ x \in \mathcal{B}(n);\, (\forall\, i, j \in N)[i < j \Rightarrow x_i < x_j] \},$$

and the set of all increasing vectors as

$$\mathcal{B}^{\le}(n) = \{ x \in \mathcal{B}(n);\, (\forall\, i, j \in N)[i \le j \Rightarrow x_i \le x_j] \}.$$

Further we denote the eigenspace of a matrix $A \in \mathcal{B}(n,n)$ as

$$\mathcal{F}(A) = \{ x \in \mathcal{B}(n);\, A \otimes x = x \},$$

and the eigenspaces of all strictly increasing eigenvectors (increasing eigenvectors) as

$$\mathcal{F}^{<}(A) = \mathcal{F}(A) \cap \mathcal{B}^{<}(n), \quad \mathcal{F}^{\le}(A) = \mathcal{F}(A) \cap \mathcal{B}^{\le}(n).$$

It is clear that any vector $x \in \mathcal{B}(n)$ can be permuted to an increasing vector. Therefore, in view of the next theorem, the structure of the eigenspace $\mathcal{F}(A)$ of a given $n \times n$ max-min matrix A can be described by investigating the structure of monotone eigenspaces $\mathcal{F}^{<}(A_{\varphi\varphi})$ and $\mathcal{F}^{\le}(A_{\varphi\varphi})$, for all permutations φ on N.

Theorem 2.2. (Gavalec [6]) Let $A \in \mathcal{B}(n,n)$, $x \in \mathcal{B}(n)$ and let φ be a permutation on N. Then $x \in \mathcal{F}(A)$ if and only if $x_\varphi \in \mathcal{F}(A_{\varphi\varphi})$.

For $A \in \mathcal{B}(n,n)$, the structure of $\mathcal{F}^<(A)$ has been described in [6] as an interval of strictly increasing vectors. Vectors $m^\star(A)$, $M^\star(A) \in \mathcal{B}(n)$ are defined as follows. For any $i \in N$, we put

$$m_i^\star(A) := \max_{j \le i} \max_{k > j} a_{jk} \qquad\qquad M_i^\star(A) := \min_{j \ge i} \max_{k \ge j} a_{jk}$$

Remark 2.3. If a maximum of an empty set should be computed in the above definition of $m^\star(A)$, then we use the fact that, by usual definition, $\max \emptyset$ is the least element in $\mathcal{B}$.

Theorem 2.4. [6] Let $A \in \mathcal{B}(n,n)$ and let $x \in \mathcal{B}(n)$ be a strictly increasing vector. Then $x \in \mathcal{F}(A)$ if and only if $m^\star(A) \le x \le M^\star(A)$. In formal notation,

$$\mathcal{F}^<(A) = \langle m^\star(A), M^\star(A)\rangle \cap \mathcal{B}^<(n)\,.$$

Our considerations in this paper will be restricted to strictly increasing eigenvectors in $\mathbf{X}$. The restricted interval eigenvectors will be called monotone interval eigenvectors, and similarly, the names of the restricted types will be extended by word 'monotone', i. e. monotone strong eigenvector, ..., monotone weak eigenvector.

Formally, we denote by $\mathbf{X}^< = [\underline{x}, \overline{x}] \cap \mathcal{B}^<(n)$ the set of all strictly increasing vectors in $[\underline{x}, \overline{x}]$. Then all the above general definitions concerning interval eigenvectors and their types are modified by substituting $\mathbf{X}$ by $\mathbf{X}^<$.

Using Theorem 2.4, we describe necessary and sufficient conditions for six types of monotone interval eigenvectors defined in Definition 2.1, which will be referred to as T1, T2, T3, T4, T5, T6.

Theorem 2.5. (T1) Let interval matrix $\mathbf{A} = [\underline{A}, \overline{A}]$ and monotone interval vector $\mathbf{X}^<$ with strictly increasing bounds $\underline{x}$, $\overline{x}$ be given. Then $\mathbf{X}^<$ is a monotone strong eigenvector of $\mathbf{A}$ if and only if

$$m^\star(\overline{A}) \le \underline{x}\,, \quad \overline{x} \le M^\star(\underline{A})\,. \tag{1}$$

P r o o f. Let $\mathbf{X}^<$ be a monotone strong eigenvector of $\mathbf{A}$. Then $A \otimes x = x$ holds for every $A \in \mathbf{A}$ and every strictly increasing vector $x \in \mathbf{X}^<$. In particular,

$$\overline{A} \otimes \underline{x} = \underline{x}\,, \quad \underline{A} \otimes \overline{x} = \overline{x}\,.$$

In view of Theorem 2.4 we immediately get the inequalities in (1).

To prove the converse implication, let us assume that the inequalities in (1) hold true. Then for every $A \in \mathbf{A}$ and every strictly increasing vector $x \in \mathbf{X}^<$ we get by the monotonicity rules

$$m^\star(A) \le m^\star(\overline{A}) \le \underline{x} \le x \le \overline{x} \le M^\star(\underline{A}) \le M^\star(A)\,.$$

Hence, $m^\star(A) \le x \le M^\star(A)$ and $A \otimes x = x$, in view of Theorem 2.4. In other words, $\mathbf{X}^<$ is a monotone strong eigenvector of $\mathbf{A}$. □

Theorem 2.6. (T2) Let interval matrix $\mathbf{A} = [\underline{A}, \overline{A}]$ and monotone interval vector $\mathbf{X}^< = [\underline{x}, \overline{x}]$ with strictly increasing bounds $\underline{x}$, $\overline{x}$ be given. Then $\mathbf{X}^<$ is a monotone strong universal eigenvector of $\mathbf{A}$ if and only if

$$m^\star(\overline{A}) \le \overline{x}\,, \quad \underline{x} \le M^\star(\underline{A})\,, \quad \langle m^\star(\overline{A}), M^\star(\underline{A})\rangle \cap \mathcal{B}^<(n) \neq \emptyset\,. \tag{2}$$

P r o o f. Let us assume that $\mathbf{X}^<$ is a monotone strong universal eigenvector of $\mathbf{A}$, i.e. there exists strictly increasing vector $x \in \mathbf{X}^<$ such that $A \otimes x = x$ holds for every $A \in \mathbf{A}$, in particular, $\underline{A} \otimes x = x$ and $\overline{A} \otimes x = x$. In view of Theorem 2.4, we get the inequalities $m^\star(\overline{A}) \le x \le M^\star(\underline{A})$, which directly imply all three conditions in (2).

To prove the converse implication, let us assume that the conditions in (2) hold true, i.e. $m^\star(\overline{A}) \le \overline{x}$, $\underline{x} \le M^\star(\underline{A})$ and there is a strictly increasing vector x' with $m^\star(\overline{A}) \le x' \le M^\star(\underline{A})$. Let us denote $x = (x' \vee \underline{x}) \wedge \overline{x}$. Using distributivity of operations $\wedge$, $\vee$, it is easy to show that $x = (x' \wedge \overline{x}) \vee \underline{x}$, and it is also clear that $\underline{x} \le x \le \overline{x}$. By assumption, vectors x', $\underline{x}$, $\overline{x}$ are strictly increasing. As the operations $\wedge$, $\vee$ preserve strict monotonicity, vector x is strictly increasing, too. Further, we have

$$m^\star(\overline{A}) \le x' \wedge \overline{x} \le x \le x' \vee \underline{x} \le M^\star(\underline{A})\,.$$

As a consequence, for any matrix $A \in \mathbf{A}$ we get

$$m^\star(A) \le m^\star(\overline{A}) \le x \le M^\star(\underline{A}) \le M^\star(A)\,,$$

i.e. $A \otimes x = x$. Thus, $\mathbf{X}^<$ is a monotone strong universal eigenvector of $\mathbf{A}$. □

Theorem 2.7. (T3) Let interval matrix $\mathbf{A} = [\underline{A}, \overline{A}]$ and monotone interval vector $\mathbf{X}^< = [\underline{x}, \overline{x}]$ with strictly increasing bounds $\underline{x}$, $\overline{x}$ be given. Then $\mathbf{X}^<$ is a monotone universal eigenvector of $\mathbf{A}$ if and only if

$$m^\star(\overline{A}) \le \overline{x}\,, \quad \underline{x} \le M^\star(\underline{A})\,, \quad (\forall\, A \in \mathbf{A})\left[\langle m^\star(A), M^\star(A)\rangle \cap \mathcal{B}^<(n) \neq \emptyset\right]\,. \tag{3}$$

P r o o f. Let us assume that $\mathbf{X}^<$ is a monotone universal eigenvector of $\mathbf{A}$, i.e. for every $A \in \mathbf{A}$ there exists strictly increasing vector $x \in \mathbf{X}^<$ such that $A \otimes x = x$ holds. In particular, there exist x', $x'' \in \mathbf{X}^<$ such that $\underline{A} \otimes x' = x'$ and $\overline{A} \otimes x'' = x''$. In view of Theorem 2.4, we get the inequalities $\underline{x} \le x' \le M^\star(\underline{A})$, $m^\star(\overline{A}) \le x'' \le \overline{x}$. The third condition in (3) follows directly from the assumption and from Theorem 2.4.

To prove the converse implication, let us assume that the conditions in (3) hold true, i.e. $m^\star(\overline{A}) \le \overline{x}$, $\underline{x} \le M^\star(\underline{A})$ and for every $A \in \mathbf{A}$ there is a strictly increasing vector x' with $m^\star(A) \le x' \le M^\star(A)$. Let matrix A and vector x' be fixed, and let us denote $x = (x' \vee \underline{x}) \wedge \overline{x}$. Similarly as in the above proof, we show easily that $x = (x' \wedge \overline{x}) \vee \underline{x}$, and $x \in \mathbf{X}^<$. By the assumption we have $m^\star(A) \le m^\star(\overline{A}) \le \overline{x}$ and $\underline{x} \le M^\star(\underline{A}) \le M^\star(A)$, which implies

$$m^\star(A) \le x' \wedge \overline{x} \le x \le x' \vee \underline{x} \le M^\star(\underline{A})\,.$$

As a consequence, we get $m^\star(A) \le x \le M^\star(A)$, i.e. $A \otimes x = x$. As the fixed matrix $A \in \mathbf{A}$ is arbitrary, we have proved that $\mathbf{X}^<$ is a monotone universal eigenvector of $\mathbf{A}$. □

Theorem 2.8. (T4) Let interval matrix $\mathbf{A} = [\underline{A}, \overline{A}]$ and monotone interval vector $\mathbf{X}^< = [\underline{x}, \overline{x}]$ with strictly increasing bounds $\underline{x}$, $\overline{x}$ be given. Further let us denote

$$\tilde{\mathbf{A}} = \{ A \in \mathbf{A};\, m^\star(A) \le \underline{x} \} \,, \quad \tilde{A} = \bigvee \tilde{\mathbf{A}} \,.$$

Then $\mathbf{X}^<$ is a monotone strong tolerance eigenvector of $\mathbf{A}$ if and only if

$$m^\star(\underline{A}) \le \underline{x} \,, \quad \overline{x} \le M^\star(\tilde{A}) \,. \tag{4}$$

Proof. Let $\mathbf{X}^<$ be monotone strong tolerance eigenvector of $\mathbf{A}$. Then there exists matrix $A \in \mathbf{A}$ such that $A \otimes x = x$ holds for every vector $x \in \mathbf{X}^<$. In particular we have $A \otimes \underline{x} = \underline{x}$, $A \otimes \overline{x} = \overline{x}$, and $m^\star(A) \le \underline{x} \le \overline{x} \le M^\star(A)$. Hence $A \in \tilde{\mathbf{A}}$, which implies $\underline{A} \le A \le \tilde{A}$, $m^\star(\underline{A}) \le m^\star(A) \le \underline{x}$, and $\overline{x} \le M^\star(A) \le M^\star(\tilde{A})$.

Conversely, let $m^\star(\underline{A}) \le \underline{x}$ (i.e. $\underline{A} \in \tilde{\mathbf{A}}$), and $\overline{x} \le M^\star(\tilde{A})$. It is easy to verify that for any two matrices $A', A'' \in \tilde{\mathbf{A}}$ also their join $A' \vee A''$ belongs to $\tilde{\mathbf{A}}$. In other words, $\tilde{\mathbf{A}}$ is closed under the operation $\vee$. As a consequence, $\tilde{A} \in \tilde{\mathbf{A}}$ and

$$m^\star(\tilde{A}) \le \underline{x} \le x \le \overline{x} \le M^\star(\tilde{A}) \,, \quad \tilde{A} \otimes x = x$$

holds for every $x \in \mathbf{X}^<$. Hence, $\mathbf{X}^<$ is monotone strong tolerance eigenvector of $\mathbf{A}$. □

Theorem 2.9. (T5) Let interval matrix $\mathbf{A} = [\underline{A}, \overline{A}]$ and monotone interval vector $\mathbf{X}^< = [\underline{x}, \overline{x}]$ with strictly increasing bounds $\underline{x}$, $\overline{x}$ be given. Then $\mathbf{X}^<$ is a monotone tolerance eigenvector of $\mathbf{A}$ if and only if

$$m^\star(\underline{A}) \le \underline{x} \,, \quad \overline{x} \le M^\star(\overline{A}) \,. \tag{5}$$

Proof. Let $\mathbf{X}^<$ be monotone tolerance eigenvector of $\mathbf{A}$. Then for every vector $x \in \mathbf{X}^<$ there exists matrix $A \in \mathbf{A}$ such that $A \otimes x = x$. In particular, there are matrices $A', A'' \in \mathbf{A}$ with $A' \otimes \underline{x} = \underline{x}$, $A'' \otimes \overline{x} = \overline{x}$, which gives $m^\star(\underline{A}) \le m^\star(A') \le \underline{x}$ and $\overline{x} \le M^\star(A'') \le M^\star(\overline{A})$.

Conversely, let conditions (5) be fulfilled and let $x \in \mathbf{X}^<$ be arbitrary, but fixed. Define matrix $A \in \mathcal{B}(n,n)$ by putting, for every $j,k \in N$,

$$a_{jk} = \begin{cases} \overline{a}_{jk} \wedge x_j & j < k \,, \\ \overline{a}_{jk} & j \ge k \,. \end{cases}$$

For $i \in N$ we have, in view of the strict monotonicity of x,

$$m_i^\star(A) = \max_{j \le i} \max_{k > j} \big(\overline{a}_{jk} \wedge x_j \big) \le \max_{j \le i} \max_{k > j} \big(\overline{a}_{jk} \wedge x_i \big) = \Big(\max_{j \le i} \max_{k > j} \overline{a}_{jk} \Big) \wedge x_i$$
$$= m_i^\star(\overline{A}) \wedge x_i \,.$$

Hence $m^\star(A) \le m^\star(\overline{A}) \wedge x \le x$. Further we have

$$M_i^\star(A) = \min_{j \ge i} \max_{k \ge j} a_{jk} = \min_{j \ge i} \Big(\overline{a}_{kk} \vee \max_{k > j} \big(\overline{a}_{jk} \wedge x_j \big) \Big) \ge \min_{j \ge i} \Big(\overline{a}_{kk} \vee \max_{k > j} \big(\overline{a}_{jk} \wedge x_i \big) \Big)$$

$$\geq \min_{j\geq i} \max_{k\geq j} \big(\overline{a}_{jk} \wedge x_i\big) = \Big(\min_{j\geq i} \max_{k\geq j} \overline{a}_{jk}\Big) \wedge x_i = M^\star(\overline{A}) \wedge x_i \,.$$

Hence $M^\star(A) \geq M^\star(\overline{A}) \wedge x \geq \overline{x} \wedge x = x$, and the inequalities $m^\star(A) \leq x \leq M^\star(A)$ imply $A \otimes x = x$, in view of Theorem 2.4. We have shown that $\mathbf{X}^<$ is a monotone tolerance eigenvector of $\mathbf{A}$. □

Theorem 2.10. (T6) Let interval matrix $\mathbf{A} = [\underline{A}, \overline{A}]$ and monotone interval vector $\mathbf{X}^< = [\underline{x}, \overline{x}]$ with strictly increasing bounds $\underline{x}$, $\overline{x}$ be given. Then $\mathbf{X}^<$ is a monotone weak eigenvector of $\mathbf{A}$ if and only if

$$\underline{x} \leq M^\star(\overline{A})\,, \quad m^\star(\underline{A}) \leq \overline{x}\,, \quad \langle m^\star(\underline{A}), M^\star(\overline{A})\rangle \cap \mathcal{B}^<(n) \neq \emptyset\,. \tag{6}$$

P r o o f. Let us assume that $\mathbf{X}^<$ is a monotone weak eigenvector of $\mathbf{A}$, i. e. there exists a matrix $A \in \mathbf{A}$ and a strictly increasing vector $x \in \mathbf{X}^<$ such that $A \otimes x = x$ holds. In view of Theorem 2.4, we have $m^\star(\underline{A}) \leq m^\star(A) \leq x \leq M^\star(A) \leq M^\star(\overline{A})$, hence $\langle m^\star(\underline{A}), M^\star(\overline{A})\rangle \cap \mathcal{B}^<(n) \neq \emptyset$. Moreover, the above inequalities imply

$$\underline{x} \leq x \leq M^\star(A) \leq M^\star(\overline{A})\,, \quad m^\star(\underline{A}) \leq m^\star(A) \leq x \leq \overline{x}\,.$$

To prove the converse implication, let us assume that the conditions in (6) hold true, i. e. $\underline{x} \leq M^\star(\overline{A})$, $m^\star(\underline{A}) \leq \overline{x}$, and there is a strictly increasing vector x' with $m^\star(\underline{A}) \leq x' \leq M^\star(\overline{A})$. We denote $x = (x' \vee \underline{x}) \wedge \overline{x}$. Similarly as in the proof of Theorem 2.6, it can be easily shown that $x = (x' \wedge \overline{x}) \vee \underline{x}$, and $x \in \mathbf{X}^<$. Moreover, the inequalities $m^\star(\underline{A}) \leq x' \leq M^\star(\overline{A})$ together with assumption $\underline{x} \leq M^\star(\overline{A})$, $m^\star(\underline{A}) \leq \overline{x}$ imply

$$m^\star(\underline{A}) \leq (x' \wedge \overline{x}) \leq (x' \wedge \overline{x}) \vee \underline{x} = x = (x' \vee \underline{x}) \wedge \overline{x} \leq x' \vee \underline{x} \leq M^\star(\overline{A})\,.$$

Hence, $m^\star(\underline{A}) \leq x \leq M^\star(\overline{A})$. Denoting

$$\underline{x}' = \bigwedge \{\, y \in \mathbf{X}^<;\, m^\star(\underline{A}) \leq y \,\}\,, \quad \overline{x}' = \bigvee \{\, y \in \mathbf{X}^<;\, y \leq M^\star(\overline{A}) \,\}\,,$$

we get $\underline{x}' \leq x \leq \overline{x}'$. Clearly, the inequalities $m^\star(\underline{A}) \leq \underline{x}'$ and $\overline{x}' \leq M^\star(\overline{A})$ hold true, which implies that the interval vector $\mathbf{X}' = [\underline{x}', \overline{x}']$ is monotone tolerance eigenvector of $\mathbf{A}$, in view of Theorem 2.9. As we have shown above, x belongs to $\mathbf{X}'$, hence there exists $A \in \mathbf{A}$ with $m^\star(A) \leq x \leq M^\star(A)$, i. e. $A \otimes x = x$. □

3. RELATIONS BETWEEN VARIOUS TYPES OF MONOTONE EIGENVECTORS

Theorem 3.1. Let interval matrix $\mathbf{A} = [\underline{A}, \overline{A}]$ and monotone interval vector $\mathbf{X}^< = [\underline{x}, \overline{x}]$ with strictly increasing bounds $\underline{x}$, $\overline{x}$ be given. Then the following implications hold true.

- $(T1 \Rightarrow T2)$ If $\mathbf{X}^<$ is a monotone strong eigenvector of $\mathbf{A}$, then $\mathbf{X}^<$ is a monotone strong universal eigenvector of $\mathbf{A}$,

- ($T1 \Rightarrow T4$) If $\mathbf{X}^<$ is a monotone strong eigenvector of $\mathbf{A}$, then $\mathbf{X}^<$ is a monotone strong tolerance eigenvector of $\mathbf{A}$,
- ($T3 \Rightarrow T6$) If $\mathbf{X}^<$ is a monotone universal eigenvector of $\mathbf{A}$, then $\mathbf{X}^<$ is a monotone weak eigenvector of $\mathbf{A}$,
- ($T5 \Rightarrow T6$) If $\mathbf{X}^<$ is a monotone tolerance eigenvector of $\mathbf{A}$, then $\mathbf{X}^<$ is a monotone weak eigenvector of $\mathbf{A}$.

Proof. The implications follow directly from Definition 2.1. □

Remark 3.2. It is easy to show that the converse implications to those in Theorem 3.1 do not hold true.

Theorem 3.3. ($T2 \Rightarrow T3$) Let interval matrix $\mathbf{A} = [\underline{A}, \overline{A}]$ and monotone interval vector $\mathbf{X}^< = [\underline{x}, \overline{x}]$ with strictly increasing bounds $\underline{x}$, $\overline{x}$ be given. If $\mathbf{X}^<$ is a monotone strong universal eigenvector of $\mathbf{A}$, then $\mathbf{X}^<$ is a monotone universal eigenvector of $\mathbf{A}$.

Proof. The implication follows directly from Definition 2.1. □

The converse implication is not true as the next example shows.

Example 3.4. ($T3 \not\Rightarrow T2$) Let $\underline{A}$, $\overline{A}$ and $\underline{x}$, $\overline{x}$ have the form

$$\underline{A} = \begin{pmatrix} 2 & 2 \\ 3 & 4 \end{pmatrix}, \quad \overline{A} = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}, \quad \underline{x} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad \overline{x} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

Then for every matrix $A \in \mathbf{A}$ of the form

$$A = \begin{pmatrix} 2 & k \\ 3 & s \end{pmatrix}, \quad 2 \le k \le 3, \quad 4 \le s \le 5,$$

strictly increasing vector $\begin{pmatrix} k \\ 4 \end{pmatrix}$ belongs to $\left\langle \begin{pmatrix} k \\ k \end{pmatrix}, \begin{pmatrix} k \\ s \end{pmatrix} \right\rangle = \langle m^\star(A), M^\star(A) \rangle$, what means that $\mathbf{X}^<$ is a monotone universal eigenvector of $\mathbf{A}$ but $\mathbf{X}^<$ is not a monotone strong universal eigenvector of $\mathbf{A}$ because

$$\langle m^\star(\overline{A}), M^\star(\underline{A}) \rangle = \left\langle \begin{pmatrix} 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix} \right\rangle \cap \mathcal{B}^<(n) = \emptyset.$$

Remark 3.5. Monotone interval vector $\mathbf{X}^<$ in the previous example fails to be a monotone strong universal eigenvector of $\mathbf{A}$ because the third condition in (2) is not fulfilled. We may note that in the example the first two conditions in (2) are satisfied, hence these two conditions together are not sufficient for the monotone strong universality of $\mathbf{X}^<$.

Theorem 3.6. ($T4 \Rightarrow T5$) Let interval matrix $\mathbf{A} = [\underline{A}, \overline{A}]$ and monotone interval vector $\mathbf{X}^< = [\underline{x}, \overline{x}]$ with strictly increasing bounds $\underline{x}$, $\overline{x}$ be given. If $\mathbf{X}^<$ is a monotone strong tolerance eigenvector of $\mathbf{A}$, then $\mathbf{X}^<$ is a monotone tolerance eigenvector of $\mathbf{A}$.

P r o o f. The implication follows directly from Definition 2.1. □

The next example shows that the converse implication is not true.

Example 3.7. ($T5 \not\Rightarrow T4$) Let $\underline{A}$, $\overline{A}$ and $\underline{x}$, $\overline{x}$ have the form

$$\underline{A} = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}, \quad \overline{A} = \begin{pmatrix} 2 & 4 \\ 3 & 4 \end{pmatrix}, \quad \underline{x} = \begin{pmatrix} 3 \\ 3.5 \end{pmatrix}, \quad \overline{x} = \begin{pmatrix} 3.5 \\ 4 \end{pmatrix},$$

with

$$m^\star(\underline{A}) = \begin{pmatrix} 3 \\ 3 \end{pmatrix}, \quad M^\star(\overline{A}) = \begin{pmatrix} 4 \\ 4 \end{pmatrix}.$$

The following inequalities

$$m^\star(\underline{A}) = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \leq \underline{x} = \begin{pmatrix} 3 \\ 3.5 \end{pmatrix}, \quad \overline{x} = \begin{pmatrix} 3.5 \\ 4 \end{pmatrix} \leq M^\star(\overline{A}) = \begin{pmatrix} 4 \\ 4 \end{pmatrix},$$

mean that $\mathbf{X}^<$ is a monotone tolerance eigenvector of $\mathbf{A}$ but $\mathbf{X}^<$ is not a monotone strong tolerance eigenvector of $\mathbf{A}$ because

$$\tilde{A} = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix} \text{ and } \quad \overline{x} = \begin{pmatrix} 3.5 \\ 4 \end{pmatrix} \not\leq M^\star(\tilde{A}) = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

Next two examples indicate further non-implication between types of monotone interval eigenvectors.

Example 3.8. ($T4 \not\Rightarrow T3$) Let $\underline{A}$, $\overline{A}$ and $\underline{x}$, $\overline{x}$ have the form

$$\underline{A} = \begin{pmatrix} 3 & 2 \\ 3 & 3 \end{pmatrix}, \quad \overline{A} = \begin{pmatrix} 3 & 3 \\ 3 & 5 \end{pmatrix}, \quad \underline{x} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \quad \overline{x} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}.$$

Then we have

$$m^\star(\underline{A}) = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \leq \underline{x}, \quad \tilde{A} = \begin{pmatrix} 3 & 3 \\ 3 & 5 \end{pmatrix}, \quad \overline{x} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \leq M^\star(\tilde{A}) = \begin{pmatrix} 3 \\ 5 \end{pmatrix},$$

which means that $\mathbf{X}^<$ is a monotone strong tolerance eigenvector of $\mathbf{A}$, but $\mathbf{X}^<$ is not a monotone universal eigenvector of $\mathbf{A}$ because

$$\underline{x} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \not\leq M^\star(\underline{A}) = \begin{pmatrix} 3 \\ 3 \end{pmatrix}.$$

Example 3.9. ($T2 \not\Rightarrow T5$) Let $\underline{A}$, $\overline{A}$ and $\underline{x}$, $\overline{x}$ have the form

$$\underline{A} = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}, \quad \overline{A} = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}, \quad \underline{x} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \quad \overline{x} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}.$$

The following inequalities

$$m^\star(\overline{A}) = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \leq \overline{x} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}, \quad \underline{x} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \leq M^\star(\underline{A}) = \begin{pmatrix} 3 \\ 4 \end{pmatrix},$$

$$\langle m^\star(\overline{A}), M^\star(\underline{A})\rangle \cap \mathcal{B}^<(n) = \left\langle \begin{pmatrix} 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right\rangle \cap \mathcal{B}^<(n) \neq \emptyset\,,$$

mean that $\mathbf{X}^<$ is a monotone strong universal eigenvector of $\mathbf{A}$, but $\mathbf{X}^<$ is not a monotone tolerance eigenvector of $\mathbf{A}$ because

$$m^\star(\underline{A}) = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \not\leq \underline{x} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}.$$

Remark 3.10. The previous two examples show that implications $T4 \Rightarrow T3$ and $T2 \Rightarrow T5$ are not fulfilled. It can be easily seen that if $T4 \not\Rightarrow T3$ and $T2 \not\Rightarrow T5$, then also $T2 \not\Rightarrow T4$, $T3 \not\Rightarrow T4$, $T3 \not\Rightarrow T5$, $T4 \not\Rightarrow T2$, $T5 \not\Rightarrow T2$, $T5 \not\Rightarrow T3$.

E. g., if the implication $T2 \Rightarrow T4$ holds true, then by $T4 \Rightarrow T5$ (Theorem 3.6) we get $T2 \Rightarrow T5$, a contradiction. The proofs of the remaining five non-implications are analogous.

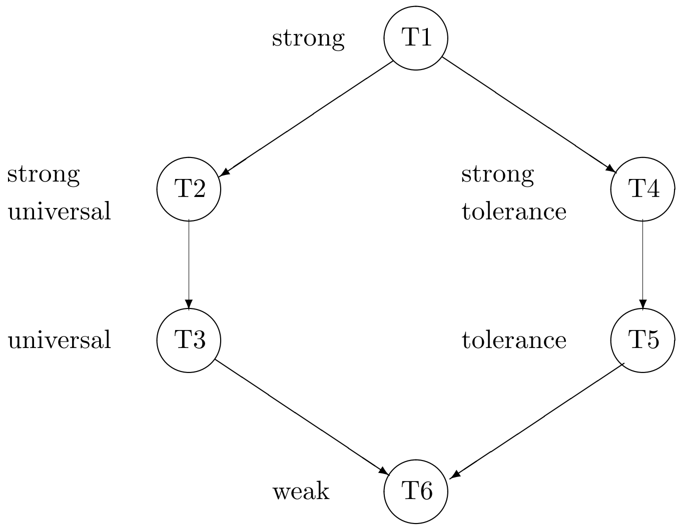

Fig. Hasse diagram of the relations between different types of monotone interval eigenvectors.

The results from this section are summarized by the Hasse diagram in Figure where every arrow indicates an implication between corresponding types, and a non-existing arrow indicates that there is no implication between corresponding types of monotone interval eigenvectors.

ACKNOWLEDGEMENT

The support of the Czech Science Foundation under Grant 402/09/0405 and APVV under project 0073-07 is kindly acknowledged.

(Received April 14, 2010)

REFERENCES

[1] K. Cechlárová: Eigenvectors in bottleneck algebra. Lin. Algebra Appl. *175* (1992), 63–73.

[2] K. Cechlárová: Solutions of interval linear systems in (max, +)-algebra. In: Proc. 6th Internat. Symposium on Operational Research, Preddvor, Slovenia 2001, pp. 321–326.

[3] K. Cechlárová and R. A. Cuninghame-Green: Interval systems of max-separable linear equations. Lin. Algebra Appl. *340* (2002), 215–224.

[4] R. A. Cuninghame-Green: Minimax Algebra. (Lecture Notes in Economics and Mathematical Systems *166*.) Springer–Verlag, Berlin 1979.

[5] M. Fiedler, J. Nedoma, J. Ramík, J. Rohn, and K. Zimmermann: Linear Optimization Problems with Inexact Data. Springer–Verlag, Berlin 2006.

[6] M. Gavalec: Monotone eigenspace structure in max-min algebra. Lin. Algebra Appl. *345* (2002), 149–167.

[7] M. Gavalec and K. Zimmermann: Classification of solutions to systems of two-sided equations with interval coefficients. Internat. J. Pure Applied Math. *45* (2008), 533–542.

[8] J. Rohn: Systems of linear interval equations. Lin. Algebra Appl. *126* (1989), 39–78.

Martin Gavalec, University of Hradec Králové, Faculty of Informatics and Management, Rokitanského 62, 500 03 Hradec Králové. Czech Republic.
e-mail: martin.gavalec@uhk.cz

Ján Plavka, Technical University in Košice, Faculty of Electrical Engineering and Informatics, B. Němcovej 32, 042 00 Košice. Slovak Republic.
e-mail: jan.plavka@tuke.sk