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CONTROL OF A CLASS OF CHAOTIC SYSTEMS BY A STOCHASTIC DELAY METHOD

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A delay stochastic method is introduced to control a certain class of chaotic systems. With the Lyapunov method, a suitable kind of controllers with multiplicative noise is designed to stabilize the chaotic state to the equilibrium point. The method is simple and can be put into practice. Numerical simulations are provided to illustrate the effectiveness of the proposed controllable conditions.

Keywords: random dynamical system, unified chaotic system, stochastic delay differential equations, multiplicative noise, maximal Lyapunov exponent

Classification: O415

1. INTRODUCTION

It is well known that output of a chaotic system is unpredictable, yet a controlled system being desirable if it meets predefined requirement. Hence, chaos control becomes an important scientific topic and many techniques have studied [1, 2, 3, 4, 5], where the dominantly majority of the studies is focused on deterministic cases. In reality, physical systems are often perturbed by noise, and it becomes important to assess how the dynamics of deterministic chaotic system is affected by noise. In physical systems, a deterministic system and its equilibrium state are the transitional result from initial chaotic state to controlled system equilibrium and state system operational modes.

A system perturbed by noise can be illustrated by a random dynamical system which consists of two basic ingredients: a noise model and a model that is disturbed by noise [6]. Currently, system stability and controllability via white noise have been studied, one example is the stochastic stability of the Hamiltonian system [17]. The stability of a system is also illustrated in [12] that if the original system is exponentially stable and stochastic perturbation is sufficient small, the controlled system will remain exponentially stable. An unstable system, $\dot{X}(t) = \hat{F}(X(t))$, can be stabilized using a stochastic method, and the function $\hat{F}(X(t))$ satisfies the condition

$$X(t)^T \hat{F}(X(t)) \leq -k|X(t)|^2 \quad (k > 0) \tag{1}$$

A study by Craud and Flandoli [8] analyzes how additive noise destroys a pitchfork bifurcation of one-dimensional dynamical system. In this study, a stochastic delay method is proposed for controlling a class of chaotic systems. The primary concerns are on the qualitative of solutions of stochastic delay differential equations (SDDEs) interpreted in the Itô sense. An Itô SDDEs is

$$\mathrm{d}X(t) = F(X(t), X(t-\tau))\,\mathrm{d}t + G(X(t), X(t-\tau))\,\mathrm{d}w(t) \tag{2}$$

where $\tau > 0$ is the delay, $F(X(t), X(t-\tau))$ is the drift term, and $G(X(t), X(t-\tau))$ is the diffusion term. If SDDEs could be solved explicitly, it would be easy to determine whether the trivial solution is stable. Nevertheless such solutions are generally not available, our study applies the Lyapunov method and achieves analytic conditions to control chaotic systems. The approximation solution of our approach is developed via a numerical method by Euler–Maruyama [7]. A special case can be found in [16]. The new approach has both noise and delay in the controlled systems, therefore it is more realistic in system engineering than that of deterministic methods. Another important feature is its simplicity and applicability, which leads to simple and pragmatic system implementations.

The paper is organized as follows. In Section 2, the relevant theoretical work is illustrated for the dynamical analysis of SDDEs; Section 3 identifies that the controlled chaotic system has a unique solution, and the conditions for controlling a class of chaotic systems are obtained thereafter. Section 4 takes the unified system as an example to simulate and verify our results.

2. THE BASIC SETUP AND PRELIMINARY RESULTS

This study is on the stability of chaotic systems by the Lyapunov method. The discussion covers the range of the Lyapunov exponents for the relevant SDDEs. In order for the controlled system to be valid, it is necessary to show the uniqueness and existence of the solution of SDDEs. In this section, the relevant theoretical results are introduced.

2.1. Existence and uniqueness theorem

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $(\mathcal{F}_t)_{t\geq 0}$ which is right-continuous and satisfies that each $(\mathcal{F}_t)(t \geq 0)$ contains all P-null sets in $\mathcal{F}$. Let $w(t)$ be the given m-dimensional Brownian motion defined on $(\Omega, \mathcal{F}, P)$. Let $\tau > 0$ and $C([-\tau, 0]; R^d)$ denote the family of continuous functions $\xi(\cdot) : [-\tau, 0] \to R^d$ with the norm $|\xi| = \sup\limits_{-\tau\leq\theta\leq 0} |\xi(\theta)|$.

Of our interest are the properties of the solutions of SDDEs of Itô type. The following is the existence and uniqueness theorem of SDDEs. The SDDEs [12] is

$$\mathrm{d}x(t) = f(x_t, t)\,\mathrm{d}t + g(x_t, t)\,\mathrm{d}w(t), \qquad t \geq t_0 \tag{3}$$

where $x_t = \{x(t+\theta) : -\tau \leq \theta \leq 0\}$, the initial data $x_{t_0} = \xi = \{\xi(\theta) : -\tau \leq \theta \leq 0\}$, $f : C([-\tau, 0]; R^d) \times [t_0, T] \to R^d$ and $g : C([-\tau, 0]; R^d) \times [t_0, T] \to R^{d\times m}$.

Lemma 1. Assume that there exist two positive constants K_1 and K_2, such that

(i) (Lipschitz condition) for all $\varphi, \psi \in C([-\tau, 0]; R^d)$ and $t \in [t_0, T]$

$$|f(\varphi,t) - f(\psi,t)|^2 \vee |g(\varphi,t) - g(\psi,t)|^2 \leq K_1|\varphi - \psi|^2, \tag{4}$$

(ii) (Linear growth condition) for all $(\varphi, t) \in C([-\tau, 0); R^d) \times [t_0, T]$,

$$|f(\varphi,t)|^2 \vee |g(\varphi,t)|^2 \leq K_2(|\varphi|^2 + 1). \tag{5}$$

Then there exists a unique solution $x(t)$ of the Eq. (3) and the solution belongs to $\mathcal{M}^2([t_0 - \tau, T]; R^d)$, which denotes all real-value measurable $\{\mathcal{F}_t\}-$adapted process $\{f(t)\}_{t_0-\tau \leq t \leq T}$ with $E\int_{t_0-\tau}^{T} |f(t)|^2 \, \mathrm{d}t < \infty$. In this paper, the norms of vector $X(t) \in R^d$ and matrix $A \in R^{d\times d}$ are respectively defined as

$$|X(t)| = \sqrt{x_1^2(t) + \cdots + x_d^2(t)}, \qquad |A| = \sqrt{\mathrm{trace}(A^T A)} \tag{6}$$

With Lemma 1, it is identifiable whether the SDDEs have a unique solution.

2.2. Lyapunov exponent

For a deterministic system, the Lyapunov exponent is an important index to verify whether the system is chaotic or stable. For a random dynamical system, a compatible approach is applied. Hale [9] shows that the infinite-dimensional properties are inherent in the delay equation; that is, for each solution trajectory based on an arbitrary initial function Φ, the least Lyapunov exponent trends to be $-\infty$. Many studies on the stochastically stable in probability employ the maximal Lyapunov exponent based on the Oseledets multiplicative ergodic theory [14]. Therefore, the cases with the maximal Lyapunov exponent are considered [7].

Definition 2. For a solution $Y(\Phi, t)$ of an SDDE on $[t_0, +\infty)$,

$$\Lambda(Y(\Phi,t)) = \lim_{t\to+\infty} \tfrac{1}{t} \log |Y(\Phi,t)|$$

provided the right-hand side is well-defined, and it is termed as the Lyapunov exponent for the solution Y generated by Φ (an initial function). The maximal Lyapunov exponent associated with the SDDE is

$$\Lambda^* = \sup_{\Phi} \lim_{t\to+\infty} \tfrac{1}{t} \log |Y(\Phi,t)|$$

provided the right-hand side exists. Then, with probability one, we shall have

$$\lim_{t\to+\infty} \frac{1}{t} \log |Y(\Phi,t)| = \Lambda^* \tag{7}$$

Many important results of stability can be derived using Lyapunov exponents for SDDEs. When the maximal Lyapunov exponent is positive, the state of the system is unstable. When the maximal Lyapunov exponent is negative, the state of the system is exponentially stable.

3. CONTROLLING A CLASS OF CHAOTIC SYSTEMS

Considering the following chaotic system [15]

$$\dot{X}(t) = AX(t) + \hat{X}(t)h(X(t)) \tag{8}$$

where A is a $n \times n$ matrix, $X(t) = (x_1(t), \ldots, x_n(t))^T \in R^n$, the diagonal matrix $\hat{X}(t)$ is $\text{diag}\{x_{i1}(t), \ldots, x_{in}(t)\}$, where $(i1, \ldots, in)$ is some permutation of $(1, \ldots, n)$. The vector function $h(\cdot) : R^n \to R^n$ is continuous and satisfies the Lipschitz condition, i. e. there exists a positive constant L, such that

$$|h(X(t)) - h(Y(t))|^2 \le L|X(t) - Y(t)|^2, \qquad X(t), Y(t) \in R^n \tag{9}$$

3.1. Problem statement

A controlled chaotic system can be illustrated by inserting an additive noise to a deterministic system as follows

$$\mathrm{d}X(t) = (AX(t) + \hat{X}(t)h(X(t)))\,\mathrm{d}t + G(X(t), X(t-\tau))\,\mathrm{d}w(t) \tag{10}$$

where $w(t)$ is a one-dimensional Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}, P)$ with a filtration $(\mathcal{F}_t)_{t\ge 0}$, the undetermined function $G : R^n \times R^n \to R^n$ is assumed to be Borel measurable and satisfies the following conditions

$$\begin{aligned} &|G(X(t), X(t-\tau)) - G(Y(t), Y(t-\tau))|^2 \\ &\le \alpha_1|X(t) - Y(t)|^2 + \alpha_2|X(t-\tau) - Y(t-\tau)|^2 \end{aligned} \tag{11}$$

$$\text{trace}[G(X(t), X(t-\tau))^T G(X(t), X(t-\tau))] \le \gamma_1|X(t)|^2 + \gamma_2|X(t-\tau)|^2 \tag{12}$$

$$|X(t)^T G(X(t), X(t-\tau))|^2 \ge \beta_1|X(t)|^4 + \beta_2|X(t-\tau)|^4 \tag{13}$$

where $\alpha_1, \alpha_2, \gamma_1, \gamma_2, \beta_1, \beta_2$ are nonnegative constants. When the controlled terms, $G(X(t), X(t-\tau))\,\mathrm{d}\omega(t)$, are added, the deterministic chaotic system becomes stochastic. It is necessary to show that this kind of control methods is meaningful, i. e., Eq. (10) has a solution.

If we define $d = n, m = 1$ and

$$f(\varphi) = F(\varphi(0), \varphi(-\tau)) = AX(t) + \hat{X}(t)h(X(t))$$

$$g(\varphi) = G(\varphi(0), \varphi(-\tau)) = G(X(t), X(t-\tau))$$

for $\varphi \in C([-\tau, 0]; R^d)$, then Eq. (10) can be written as Eq. (3), therefore Lemma 1. can be applied to Eq. (10).

With the definition in Eq. (6) and combined with Eqs. (11) and (12), the function $G(X(t), X(t-\tau))$ satisfies the Lipschitz condition and the linear growth condition. Before doing the similar analysis of the function $F(\varphi(0), \varphi(-\tau))$, it is worth to notice some facts of the chaotic system (8). Based on the chaotic property, the state $X(t)$

of system (8) is bounded, i. e. $|X(t)| \leq L_1$ for some positive constant L_1. Since the function $h(X(t))$ is continuous, $h(X(t))$ is also bounded, i. e. $|h(X(t))| \leq L_2$ for some positive constant L_2. The following is to prove that the function $F(\varphi(0), \varphi(-\tau))$ satisfies the Lipschitz condition and the linear growth condition.

For any $X(t) = (x_1(t), \ldots, x_n(t))^T \in R^n, Y(t) = (y_1(t), \ldots, y_n(t))^T \in R^n$

$$\begin{aligned} |AX(t) + \hat{X}(t)h(X(t))| &\leq |A||X(t)| + |\hat{X}(t)||h(X(t))| \\ &\leq (|A| + L_2)|X(t)| \end{aligned} \tag{14}$$

$$\begin{aligned} &|AX(t) + \hat{X}(t)h(X(t)) - AY(t) - \hat{Y}(t)h(Y(t))| \\ &\leq |A||X(t) - Y(t)| + |\hat{X}(t)h(X(t)) - \hat{Y}(t)h(Y(t))| \\ &\leq |A||X(t) - Y(t)| + |\hat{X}(t)||h(X(t)) - h(Y(t))| + |h(Y(t))||\hat{X}(t) - \hat{Y}(t)| \\ &\leq (|A| + L_1\sqrt{L} + L_2)|X(t) - Y(t)| \end{aligned} \tag{15}$$

A combination of Eqs. (14) and (15) indicates that the function $F(\varphi(0), \varphi(-\tau))$ satisfies the Lipschitz condition and linear growth condition. By Lemma 1, system (10) has a unique solution, and this solution has a continuous path and its every moment is finite.

Remark: From the following conditions:

1. $\hat{X}(t) = \mathrm{diag}\{x_{i1}(t), \ldots, x_{in}(t)\}, \hat{Y}(t) = \mathrm{diag}\{y_{i1}(t), \ldots, y_{in}(t)\}$;
2. $X(t)$ and $Y(t)$ are the states of the same kind of chaotic systems;
3. Based on 1, 2 and Eq. (6), $|X(t)| = |\hat{X}(t)|$ and $|X(t) - Y(t)| = |\hat{X}(t) - \hat{Y}(t)|$.

To control the chaotic system to the equilibrium point, it is assumed that $G(0, 0) = 0$, and $F(0, 0) \equiv 0$. Hence system (10) has the solution $X(t) \equiv 0$ corresponding to the initial value $X(t_0) = 0$, and white noise is the multiplicative noise. The set $\{0\}$ is considered to be the attractor of this random dynamical system, thus leading to the conclusive result for the controlled chaotic systems.

One of the applications of the proposed controlled chaotic system is in designing the initial system conditions for the local oscillation of radio receivers, which white noise is considered to be a necessary input to the positive feedback oscillator. In such a system, white noise is generated at start, where selected frequency component is supplied with energy and the rest is resided into the origin, or zero energy state.

3.2. Designing the suitable controllers

The control of chaotic systems is proposed in Theorem 4. To prove the theorem, the following Lemma is a necessary step [12].

Lemma 3. For all $x_0 \neq 0$ in R^n, $P\{x(t; t_0, x_0) \neq 0 \textit{ on } t \geq t_0\} = 1$. That is, almost all the sample path of any solution starting from a non-zero state will never reach the origin.

Now, we can state the main result as follows.

Theorem 4. Suppose that the conditions Eqs. (4) and (11) – (13) hold, $\tau > 0$ is arbitrary delay, and

$$\gamma_2 \le 4\beta_2, \tag{16}$$

$$2(|A| + L_2) + \gamma_1 + \frac{\gamma_2}{2} - 2\beta_1 < 0. \tag{17}$$

Then the controlled chaotic system (10), with the initial value date $\xi = \{\xi(\theta) : \theta \in [-\tau, 0]\}$, can be stabilized to the equilibrium point almost surely, and

$$\Lambda^* = |A| + L_2 + \frac{\gamma_1}{2} + \frac{\gamma_2}{4} - \beta_1 \qquad \text{a.s.} \tag{18}$$

P r o o f. Since $X(t; t_0, 0) \equiv 0$, the conclusion is valid for $X(t_0) = 0$. Next, it needs to prove the result for $X(t_0) \neq 0$. A combination of the Lyapunov function $V(t, X(t)) = \log |X(t)|^2$ and the Itô formula leads

$$\begin{aligned} V(t, X(t)) &= \log |X(t)|^2 = \log(X^T(t)X(t)) \\ &= \log(\xi^T(t_0)\xi(t_0)) + 2\int_{t_0}^{t} \frac{X^T(s)(AX(s) + \hat{X}(t)h(X(s)))}{X^T(s)X(s)} d\,\mathrm{d}s \\ &\quad + \int_{t_0}^{t} \frac{\mathrm{trace}(G^T(X(s), X(s-\tau))G(X(s), X(s-\tau)))}{X^T(s)X(s)}\,\mathrm{d}s \\ &\quad -2\int_{t_0}^{t} \frac{|X^T G(X(s), X(s-\tau))|^2}{[X^T(s)X(s)]^2}\,\mathrm{d}s + M(t) \end{aligned} \tag{19}$$

where

$$M(t) = 2\int_{t_0}^{t} \frac{X^T G(X(s), X(s-\tau))}{X^T(s)X(s)}\,\mathrm{d}w(s)$$

is a continuous martingale with initial value $M(t_0) = 0$. By the exponential martingale inequality, there exists arbitrary positive constants T, ε, β such that

$$P\{\omega : \sup_{t_0 \le t \le T} \left[M(t) - 2\varepsilon \int_{t_0}^{t} \frac{|X^T G(X(s), X(s-\tau))|^2}{|X(s)|^4}\,\mathrm{d}s \right] > \beta\} < e^{-\varepsilon\beta} \tag{20}$$

Equation (20) implies that, for almost every $\omega \in \Omega$,

$$M(t) \le \beta + 2\varepsilon \int_{t_0}^{t} \frac{|X^T G(X(s), X(s-\tau))|^2}{|X(s)|^4}\,\mathrm{d}s, \qquad \forall\, t \in [t_0, T]. \tag{21}$$

Since the trajectory $X(t)$ is bounded for all $t \geq t_0$, the inequality (21) can be extended to the case of $T \to \infty$. Substituting inequality (21) into Eq. (19) and using Eq. (12) and Eq. (13), it yields that, for almost every $\omega \in \Omega$,

$$
\begin{aligned}
& V(t, X(t)) = \log |X(t)|^2 = \log(X^T(t)X(t)) \\
\leq\ & \log(\xi^T(t_0)\xi(t_0)) + 2\int_{t_0}^{t} \frac{|X^T(s)(AX(s) + \hat{X}(s)h(X(s)))|}{|X(s)|^2}\,\mathrm{d}s \\
& + \int_{t_0}^{t} \frac{\mathrm{trace}(G^T(X(s), X(s-\tau))G(X(s), X(s-\tau))}{X^T(s)X(s)}\,\mathrm{d}s + \beta \\
& -2(1-\varepsilon)\int_{t_0}^{t} \frac{|X^T G(X(s), X(s-\tau))|^2}{\left[X^T(s)X(s)\right]^2}\,\mathrm{d}s \\
\leq\ & \log(\xi^T(t_0)\xi(t_0)) + 2\int_{t_0}^{t} \frac{|A||\hat{X}(t)| + |\hat{X}(t)||h(X(s))|}{|X(s)|}\,\mathrm{d}s + \beta \\
& + \int_{t_0}^{t} \frac{\gamma_1|X(s)|^2 + \gamma_2|X(t-\tau)|^2}{|X(s)|^2}\,\mathrm{d}s - 2(1-\varepsilon)\int_{t_0}^{t} \frac{\beta_1|X(s)|^4 + \beta_2|X(t-\tau)|^4}{|X(s)|^4}\,\mathrm{d}s \\
\leq\ & \log(\xi^T(t_0)\xi(t_0)) + 2(|A| + L_2 + \tfrac{\gamma_1}{2} - (1-\varepsilon)\beta_1)(t-t_0) \\
& +\gamma_2\int_{t_0}^{t} \frac{|X(s-\tau)|^2}{|X(s)|^2}\,\mathrm{d}s - 2(1-\varepsilon)\beta_2\int_{t_0}^{t} \frac{|X(s-\tau)|^4}{|X(s)|^4}\,\mathrm{d}s\Big] + \beta
\end{aligned} \tag{22}
$$

Based on Eq. (16), the Hölder inequality and the inequality $2ab \leq a^2 + b^2$, for almost every $\omega \in \Omega$, the Lyapunov function $V(t, X(t))$ is bounded by

$$
\begin{aligned}
& V(t, X(t)) = \log |X(t)|^2 = \log(X^T(t)X(t)) \\
\leq\ & \log(\xi^T(t_0)\xi(t_0)) + 2(|A| + L_2 + \frac{\gamma_1}{2} - (1-\varepsilon)\beta_1)(t-t_0) \\
& +\gamma_2\left(\int_{t_0}^{t} \frac{|X(s-\tau)|^4}{|X(s)|^4}\,\mathrm{d}s)^{\frac{1}{2}}(t-t_0)^{\frac{1}{2}} - 2(1-\varepsilon\right)\beta_2\int_{t_0}^{t} \frac{|X(s-\tau)|^4}{|X(s)|^4}\,\mathrm{d}s + \beta
\end{aligned} \tag{23}
$$

$$
\begin{aligned}
\leq\ & \log\left(\xi^T(t_0)\xi(t_0)\right) + 2\left(|A| + L_2 + \frac{\gamma_1}{2} - (1-\varepsilon)\beta_1 + \frac{\gamma_2}{4}\right)(t-t_0) \\
& + \left(\frac{\gamma_2}{2} - 2(1-\varepsilon)\beta_2\right)\int_{t_0}^{t} \frac{|X(s-\tau)|^4}{|X(s)|^4}\,\mathrm{d}s + \beta \\
\leq\ & \log(\xi^T(t_0)\xi(t_0)) + 2(|A| + L_2 + \frac{\gamma_1}{2} - (1-\varepsilon)\beta_1 \\
& +\frac{\gamma_2}{4})(t-t_0) + 2\varepsilon\beta_2\int_{t_0}^{t} \frac{|X(s-\tau)|^4}{|X(s)|^4}\,\mathrm{d}s + \beta
\end{aligned}
$$

Let the both sides of Eq. (23) be divided by $t > 0$, and let $t \to \infty$, and $\varepsilon \to 0$. Then,

$$\lim_{t\to+\infty} \frac{1}{t} \log |X(t)|^2 = 2\left(|A| + L_2 + \frac{\gamma_1}{2} - \beta_1 + \frac{\gamma_2}{4}\right) \qquad \text{a.s.} \tag{24}$$

A combination of Eqs. (17) and (24) leads to

$$\lim_{t\to+\infty} \frac{1}{t} \log |X(t)| = |A| + L_2 + \frac{\gamma_1}{2} + \frac{\gamma_2}{4} - \beta_1 < 0. \qquad \text{a.s.} \tag{25}$$

Hence, the maximal Lyapunov exponent is negative, and as a result, the controlled chaotic system converges to the equilibrium point almost surely. □

In the proof above, function $AX(t)+\hat{X}(t)h(X(t))$ is estimated, i. e. $|X^T(t)(AX(t)+\hat{X}(t)h(X(t)))| \leq (|A| + L_2)|X(t)|^2$. The exact bound and positively invariance set of a chaotic system can be technically challenging. Rigid bounds for some specific chaotic systems can be identified [10], and this approach helps in deriving more rigid bounds for L_1, L_2 that are useful in designing the controllers.

4. EXAMPLE

In this section, the unified chaotic system is employed as an example to verify the usefulness of the theoretical results in Section 3.

4.1. Controlling the unified chaotic system

The unified system is described as [11]

$$\left\{\begin{array}{lcl} \dot{x}_1(t) & = & (25\alpha + 10)(x_2(t) - x_1(t)), \\ \dot{x}_2(t) & = & (28 - 35\alpha)x_1(t) - x_1(t)x_3(t) + (29\alpha - 1)x_2(t), \\ \dot{x}_3(t) & = & x_1(t)x_2(t) - \frac{8+\alpha}{3}x_3(t), \end{array}\right. \tag{26}$$

which has the equilibrium points

$$\begin{array}{lcl} O & = & (0,0,0); \\ P_1 & = & \left(\sqrt{(8+\alpha)(9-2\alpha)}, \sqrt{(8+\alpha)(9-2\alpha)}, 27-6\alpha\right); \\ P_2 & = & \left(-\sqrt{(8+\alpha)(9-2\alpha)}, -\sqrt{(8+\alpha)(9-2\alpha)}, 27-6\alpha\right). \end{array}$$

The system (26) is chaotic for all $\alpha \in [0, 1]$. In particular, when $\alpha = 0$ or 1, it becomes the Lorenz system or the Chen system respectively [13]. For the chaotic unified system, we have

$$X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}, \qquad \hat{X} = \begin{pmatrix} x_1(t) & 0 & 0 \\ 0 & x_3(t) & 0 \\ 0 & 0 & x_2(t) \end{pmatrix},$$

$$h(X) = \begin{pmatrix} 0 \\ -x_1(t) \\ x_1(t) \end{pmatrix}, A = \begin{pmatrix} -(25\alpha+10) & 25\alpha+10 & 0 \\ 28-35\alpha & 29\alpha-1 & 0 \\ 0 & 0 & -\frac{8+\alpha}{3} \end{pmatrix}.$$

It is shown that the function $h(X(t))$ satisfies inequality (9). Next, the bound is derived for $|X^T(t)(AX(t)+\hat{X}(t)h(X(t)))|$, where

$$\begin{aligned}
&|X(t)^T[AX(t)+\hat{X}(t)h(X(t))]| \\
&= \left|-(25\alpha+10)x_1^2(t)+(38-10\alpha)x_2(t)x_1(t)+(29\alpha-1)x_2^2(t)-\tfrac{8+\alpha}{3}x_3^2(t)\right| \\
&\le \left|-(25\alpha+10)x_1^2(t)+(38-10\alpha)x_1(t)x_2(t)+(29\alpha-1)x_2^2(t)\right|+\tfrac{8+\alpha}{3}x_3^2(t) \\
&\le (25\alpha+10)x_1^2(t)+(38-10\alpha)|x_1(t)x_2(t)|+(1-29\alpha)x_2^2(t)+\tfrac{8+\alpha}{3}x_3^2(t) \\
&\le (-30\alpha+29)x_1^2(t)+(20-34\alpha)x_2^2(t)+\tfrac{8+\alpha}{3}x_3^2(t) \\
&\le (29-30\alpha)|X(t)|^2 = K(\alpha)|X(t)|^2
\end{aligned} \tag{27}$$

The function $K(\alpha)$, instead of $|A|+L_2$, is used in the following control theorem of the unified chaotic system.

Corollary 5. If Eqs. (11) – (13) hold, and

$$2K(\alpha)+\gamma_1+\gamma_2 < 2(\beta_1+\beta_2), \qquad \text{for all } \alpha \in [0, 1/29)$$

the controlled unified system with the initial value $\xi = \{\xi(\theta), \theta \in [-\tau, 0]\}$ can be stabilized to the equilibrium point almost surely.

4.2. Numerical simulation

Next the original Lorenz system, i.e. $\alpha = 0$, is taken as an example to show the effectiveness of this stochastic method. Let

$$G(X(t), X(t-\tau)) = (11.5 - \sin x_1(t-\tau)) \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} \tag{28}$$

Correspondingly, the controlled Lorenz system is described as follows

$$\begin{cases} dx_1(t) &= 10(x_2(t)-x_1(t))\,dt + (11.5-\sin x_1(t-\tau))x_1(t)\,dw(t) \\ dx_2(t) &= (28x_1(t)-x_2(t)-x_1(t)x_3(t))\,dt + (11.5-\sin x_1(t-\tau))x_2(t)\,dw(t) \\ dx_3(t) &= (x_1(t)x_2(t)-\frac{8}{3}x_3(t))\,dt + (11.5-\sin x_1(t-\tau))x_3(t)dw(t) \end{cases} \tag{29}$$

and

$$\begin{aligned}
&tr(G(X(t), X(t-\tau))^T G(X(t), X(t-\tau))) \\
&= (11.5-\sin x_1(t-\tau))^2|X(t)|^2 \le 12.5^2|x(t)|^2 = \gamma_1|X(t)|^2 \\
&|X(t)^T G(t, x(t), x(t-\tau))|^2 \\
&= (11.5-\sin x_1(t-\tau))^2|X(t)|^4 \ge 10.5^2|X(t)|^4 = \beta_1|X(t)|^4
\end{aligned} \tag{30}$$

Corollary 5. is specified by

$$\Lambda^* = K(0) + \frac{1}{2} \times 12.5^2 - 10.5^2 = -3.125 < 0 \tag{31}$$

With the Euler–Maruyama method [7]

$$X_{n+1} = X_n + hF(X_n, X_{n-m}) + G(X_n, X_{n-m})\triangle\omega_n \tag{32}$$

where h is the step size, $t_n = t_0 + nh$, $\tau = mh$, $m, n \in N$. The increment $\triangle w = w(t_{n+1}) - w(t_n)$ is an $N(0, h)$-distributed Gaussian random variable. Let the initial value $\xi = 1$, Therefore, equation * has numerical solution. The state means of $x_1(t), x_2(t)$ and $x_3(t)$ are shown in Figure. Our experiment shows that with various value of τ, the equilibrium state is repeatedly reached.

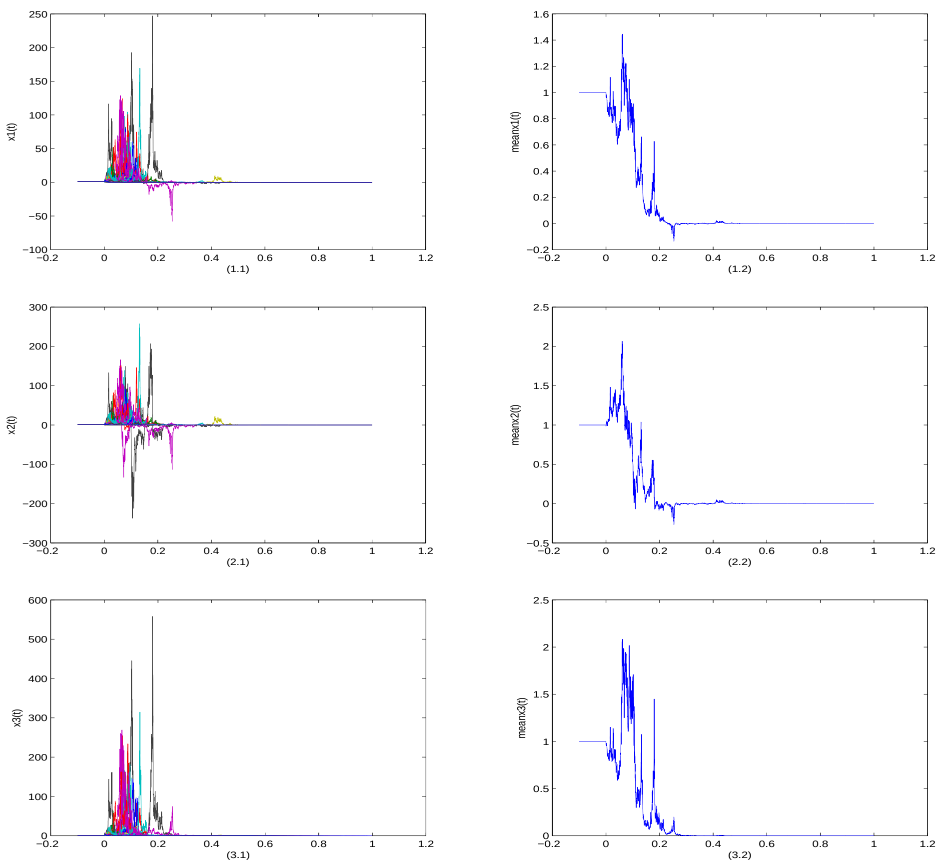

Fig. When $\alpha = 0$, $\tau = 0.1$, the left figures show 400 trajectories of the state $x_1(t), x_2(t), x_3(t)$ respectively; and the right ones show the corresponding mean of state $x_1(t), x_2(t), x_3(t)$ respectively.

5. CONCLUSION

A stochastic delay method of controlling a class of chaotic systems is presented in this paper. Provided the suitable controllers, the attractor of the random dynamical system is a one-point set $\{0\}$ almost surely, and, the chaotic states converge to the origin exponentially almost surely. The value of delay τ does not affect the convergence of the chaotic system to its equilibrium state.

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