WEAKLY STATIONARY PROCESSES WITH NON–POSITIVE AUTOCORRELATIONS

Šárka Došlá and Jiří Anděl

We deal with real weakly stationary processes $\{X_t,\ t \in \mathbb{Z}\}$ with non-positive autocorrelations $\{r_k\}$, i.e. it is assumed that $r_k \leq 0$ for all $k = 1, 2, \dots$. We show that such processes have some special interesting properties. In particular, it is shown that each such a process can be represented as a linear process. Sufficient conditions under which the resulting process satisfies $r_k \leq 0$ for all $k = 1, 2, \dots$ are provided as well.

Keywords: non-positive autocorrelations, linear process

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1. INTRODUCTION

Let $\{X_t,\ t \in \mathbb{Z}\}$ be a real weakly stationary process with finite second order moments and an autocorrelation function $\{r_k,\ k \in \mathbb{Z}\}$. We deal with processes with non-positive autocorrelations, i.e. it is assumed that

$$r_k \leq 0 \quad \text{holds for all} \quad k = 1, 2, \dots. \tag{1}$$

In particular, we are interested in the non-trivial cases where at least one of the inequalities in (1) is sharp, i.e. there exists k_0 such that $r_{k_0} < 0$. The condition (1) then reflects some kind of negative dependence among all the variables $\{X_t,\ t \in \mathbb{Z}\}$. The class of processes with non-positive autocorrelations involves a number of various well-known processes, namely particular ARMA models, a fractional Gaussian noise with the Hurst exponent $H \in (0, 1/2)$, see [1, Chap. 2], and others. Furthermore, processes with the correlation structure (1) might arise in some practical situations, and therefore it is desirable to investigate their properties, see [2]. Bernoulli processes satisfying (1) have some special practical applications mentioned in [6].

It is easy to characterize Gaussian processes $\{X_t\}$ with the autocorrelation structure (1). It is shown in [2] that a sequence of real numbers $\{r_k,\ k \in \mathbb{Z}\}$ such that $r_0 = 1$, $r_{-k} = r_k$ for $k \in \mathbb{Z}$, and $r_k \leq 0$ for $k \geq 1$ is the autocorrelation function of a stationary Gaussian process $\{X_t\}$ if and only if

$$\sum_{k=1}^{\infty} r_k \geq -\frac{1}{2}. \tag{2}$$

Moreover, an autocorrelation function is always positive semidefinite, and this implies that the inequality (2) holds for any weakly stationary process $\{X_t\}$ with $r_k \le 0$ for $k \ge 1$.

The inequality (2) is crucial and has several consequences. For instance, if (1) holds for $\{r_t\}$ then (2) implies $\sum_{k=1}^{\infty} |r_k| < \infty$. This means that a process with the correlation structure (1) has always a short memory (see [1, p. 6]). It further follows that processes satisfying (1) have some special properties which we derive in Section 2. In particular, it is shown that such a process can be represented as a linear process. In Section 3 we further provide some sufficient conditions for the coefficients of the MA(∞) representation under which the resulting process has the desirable property (1). A construction in the spectral domain is discussed as well.

2. GENERAL PROPERTIES

In this section we investigate properties of general weakly stationary processes $\{X_t,\ t \in \mathbb{Z}\}$ with the autocorrelation structure (1). We can assume without loss of generality that $\mathsf{E} X_t = 0$. Since the autocorrelations are well-defined, we have $\mathsf{var}\, X_t > 0$. Recall that the negativeness of the correlations $\{r_k\}$ indicate that the variables $\{X_t\}$ are somehow negatively dependent. The quantity $\sum_{k=1}^{\infty} r_k$ further reflects "the strength" of the dependence. The following proposition shows how the minimal value $-1/2$ in (2) can be attained.

Proposition 2.1. Let $\{X_t,\ t \in \mathbb{Z}\}$ be a weakly stationary process with the autocorrelation function $\{r_k,\ k \in \mathbb{Z}\}$ such that (1) holds. Then there exists a continuous spectral density f of the process $\{X_t\}$ and

$$\sum_{t=1}^{\infty} r_t = -1/2 \text{ holds if and only if } f(0) = 0. \tag{3}$$

Proof. Let $\{R_k\}$ be the autocovariance function of $\{X_t\}$. The inequality (2) implies that $\sum_{k=-\infty}^{\infty} |R_k| < \infty$, and the spectral density f of $\{X_t\}$ is given as $f(\lambda) = [1/(2\pi)] \sum_{k=-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}k\lambda} R_k$. This Fourier series is absolutely summable, and therefore f is continuous on the interval $[-\pi, \pi]$. We have $\sum_{k=1}^{\infty} R_k = \pi f(0) - R_0/2$ which gives $\sum_{k=1}^{\infty} r_k = \pi f(0)/R_0 - 1/2 \ge -1/2$. Obviously, the bound $-1/2$ is reached if and only if $f(0) = 0$. □

The proof of Proposition 2.1 shows that the statement (3) holds for a general autocorrelation function $\{r_t\}$ such that $\sum_{t=1}^{\infty} |r_t| < \infty$. It is then easy to derive conditions under which the equality $\sum_{t=1}^{\infty} r_t = -1/2$ holds for ARMA processes. Let X_t follow an ARMA(m, n) model $\sum_{k=0}^{m} a_k X_{t-k} = \sum_{j=0}^{n} b_j \varepsilon_{t-j}$, where $\{\varepsilon_t,\ t \in \mathbb{Z}\}$ is a white noise with $\mathsf{var}\, \varepsilon_t = \sigma^2$. Suppose that the zeros of $\sum_{k=0}^{m} a_k z^k = 0$ lie outside the unit disc. Then $\{X_t\}$ is a weakly stationary process, and its spectral density f is given as $f(\lambda) = [\sigma^2/(2\pi)] \left|\sum_{j=0}^{n} b_j \mathrm{e}^{-\mathrm{i}j\lambda}\right|^2 \left|\sum_{k=0}^{m} a_k \mathrm{e}^{-\mathrm{i}k\lambda}\right|^{-2}$. Hence, the equality in (2) is reached if and only if $\sum_{j=0}^{n} b_j = 0$. This holds if and only if 1 is a root of the

MA polynomial $\sum_{j=0}^{n} b_j z^j$. This demonstrates that the equality in (2) is reached neither for autoregressive sequences nor for an invertible ARMA process.

We have obtained conditions for the parameters of an ARMA model such that $\sum_{t=1}^{\infty} r_t = -1/2$ holds. However, $\sum_{t=1}^{\infty} r_t = -1/2$ does not imply $r_t \le 0$ for all $t \ge 1$. The process $X_t = \varepsilon_t - 2\varepsilon_{t-1} + \varepsilon_{t-2}$ is an example. It is much more difficult to specify conditions under which the autocorrelation function an ARMA(m, n) process satisfies (1). We show such restrictions for a simple ARMA$(1, 1)$ model.

Let $\{X_t,\ t \in \mathbb{Z}\}$ follow an ARMA(1,1) model $X_t + aX_{t-1} = \varepsilon_t + b\varepsilon_{t-1}$ with $a \in (-1, 1)$, $b \in [-1, 1]$, $a \ne b$, $a, b \ne 0$. Then $\{X_t\}$ is stationary, and it is well-known that $r_1 = [(1 - ab)(b - a)]/(1 - 2ab + b^2)$ and $r_k = -ar_{k-1} = (-a)^{k-1} r_1$ for $k \ge 2$. The condition $r_t < 0$ for $t \ge 1$ is satisfied if and only if $-1 \le b < a < 0$ holds. Define $g(a, b) = \sum_{t=1}^{\infty} r_t = (1 - ab)(b - a)/[(1 - 2ab + b^2)(1 + a)]$. For any given a, b satisfying $-1 \le b < a < 0$ we have $g(a, b) \ge g(a, -1) = g(0, -1) = -1/2$. This illustrates the above conclusion that the minimum $-1/2$ is reached whenever 1 is a root of the MA polynomial.

We were able to derive conditions under which $r_t \le 0$ for all $t \ge 1$ holds for a stationary ARMA$(1, 1)$ process. However, the situation becomes considerably more complicated for higher order ARMA models, and therefore this approach cannot be used. Since each stationary ARMA process can be represented as a linear process, one can equivalently investigate conditions for the MA(∞) parameters under which the resulting process satisfies (1). This approach turns out to be more convenient. In the following we justify its use for general stationary processes.

Theorem 2.2. Let $\{X_t,\ t \in \mathbb{Z}\}$ be a real weakly stationary process with the autocorrelation function $\{r_k,\ k \in \mathbb{Z}\}$ such that (1) holds. Then the spectral density f of $\{X_t\}$ satisfies

$$\int_{-\pi}^{\pi} \ln f(\lambda)\,\mathrm{d}\lambda > -\infty. \tag{4}$$

P r o o f. We can assume without loss of generality that $\mathsf{var}\, X_t = 1$, and then $\{r_k\}$ is the autocovariance function of $\{X_t\}$. Recall that the inequality (2) holds. Assume first that $\sum_{t=1}^{\infty} r_t > -1/2$. The spectral density of $\{X_t\}$ is given as $f(\lambda) = [1/(2\pi)]\,[1 + 2\sum_{t=1}^{\infty} r_t \cos(t\lambda)]$ which implies that

$$2\pi f(\lambda) = 1 + 2\sum_{t=1}^{\infty} r_t \cos(t\lambda) = 1 - 2\sum_{t=1}^{\infty} |r_t| \cos(t\lambda) \ge 1 - 2\sum_{t=1}^{\infty} |r_t| > 0 \tag{5}$$

holds for all $\lambda \in [-\pi, \pi]$. In this case $f(\lambda)$ is positive and continuous on $[-\pi, \pi]$, and therefore $\ln[f(\lambda)]$ is continuous on $[-\pi, \pi]$. Hence, the integral in (4) is finite.

Now let $\sum_{t=1}^{\infty} r_t = -1/2$. Then we can write

$$2\pi f(\lambda) = 1 - 2\sum_{t=1}^{\infty} |r_t| \cos(t\lambda) = 2\sum_{t=1}^{\infty} |r_t|[1 - \cos(t\lambda)]. \tag{6}$$

Since $|r_t| \ge 0$ and $1 - \cos(t\lambda) \ge 0$ for all $t \ge 1$, the equality (6) implies that $f(\lambda) = 0$ if and only if $|r_t|[1 - \cos(t\lambda)] = 0$ for all $t \ge 1$. This is always satisfied for $\lambda = 0$, and thus $\lambda = 0$ is always a zero point of $f(\lambda)$ (cf. Proposition 2.1). Furthermore, $\lambda \in [0, \pi]$ is a root of $f(\lambda) = 0$ if and only if $\cos(t\lambda) = 1$ for all $t \geq 1$ such that $r_t \neq 0$. Recall that we assume that $\sum_{t=1}^{\infty} |r_t| = 1/2$ which eliminates the trivial case $r_t = 0$ for all $t \geq 1$.

Let $f(\lambda) > 0$ for all $\lambda \in (0, \pi]$, i.e. $\lambda = 0$ is the only root of $f(\lambda) = 0$ in the interval $[0, \pi]$ (this assumption is removed in the next step). Since f is even we can write $\int_{-\pi}^{\pi} \ln[f(\lambda)]\,\mathrm{d}\lambda = 2\int_0^{\pi} \ln[f(\lambda)]\,\mathrm{d}\lambda$. The function $\ln[f(\lambda)]$ is continuous on $(0, \pi]$, and therefore it suffices to investigate its behavior in a neighborhood of the point 0. Let $\delta > 0$ be sufficiently small and take $T \in \mathbb{N}$ such that $T\delta < \pi/2$. We can assume that there exists $r_t \neq 0$ for some $1 \leq t \leq T$ (otherwise, we would choose smaller δ). Define $A(\lambda) = \sum_{t=1}^{T} |r_t|[1 - \cos(t\lambda)]$ and $B(\lambda) = \sum_{t=T+1}^{\infty} |r_t|[1 - \cos(t\lambda)]$. Then $\pi f(\lambda) = A(\lambda) + B(\lambda)$. Obviously, $B(\lambda) \geq 0$ for all $\lambda \in [0, \pi]$. Furthermore, $(2t^2\lambda^2)/\pi^2 < 1 - \cos(t\lambda)$ holds for all $\lambda \in (0, \delta)$. Hence, we get

$$\frac{2\lambda^2}{\pi^2} \sum_{t=1}^{T} |r_t|\, t^2 < A(\lambda).$$

Denote $C = \sum_{t=1}^{T} |r_t|\, t^2$. Then $C > 0$. We have obtained the inequality $f(\lambda) > 2C\lambda^2/\pi^3$ which implies that $\ln[f(\lambda)] > 2\ln\lambda + \ln\left[(2C)/\pi^3\right]$ for all $\lambda \in (0, \delta)$. Hence, $\int_0^{\delta} \ln[f(\lambda)]\,\mathrm{d}\lambda$ converges and so does $\int_0^{\pi} \ln[f(\lambda)]\,\mathrm{d}\lambda$.

Finally, let $\lambda_0 \neq 0$ be a root of $f(\lambda) = 0$ in the interval $[0, \pi]$. It follows from (6) that $\cos(t\lambda_0) = 1$ for all t such that $r_t \neq 0$. In other words, $r_t = 0$ for all t such that $t\lambda_0$ is not a multiple of 2π. Obviously, λ_0 must be of the form $\lambda_0 = l\pi/s$ for some $l, s \in \mathbb{N}$, $1 \leq l \leq s$ such that the greatest common divisor of l and s equals 1. Otherwise we would get $r_t = 0$ for all $t \geq 1$, and this is not possible. It further follows that if l is odd then $r_t = 0$ for all t such that $t \neq 2sk$ for all $k \in \mathbb{N}$, $k \geq 1$. Similarly, if l is even then $r_t = 0$ for all t such that $t \neq sk$ for all $k \in \mathbb{N}$, $k \geq 1$. Assume that l is odd (the situation for even l is analogous). Then we can write

$$2\pi f(\lambda) = 2\sum_{k=1}^{\infty} |r_{2sk}|[1 - \cos(2sk\lambda)].$$

Since $\cos[2sk(l\pi/s - \lambda)] = \cos(2sk\lambda) = \cos[2sk(l\pi/s + \lambda)]$ the behavior of $f(\lambda)$ in a neighborhood of the point $l\pi/s$ is the same as in a neighborhood of the point 0. The previous conclusions imply that $\int_0^{\pi} \ln[f(\lambda)]\,\mathrm{d}\lambda$ converges. □

Remark 2.3. Let us point out some properties of the spectral density f that are derived in the foregoing proof. We have shown that if $\sum_{t=1}^{\infty} r_t > -1/2$ then f has no zero points. Furthermore, if $\sum_{t=1}^{\infty} r_t = -1/2$ and $r_1 \neq 0$ then $\lambda = 0$ is the only root of $f(\lambda) = 0$ on the interval $[-\pi, \pi]$. On the other hand, if $\lambda_0 \in (0, \pi]$ is a root of $f(\lambda) = 0$ then $r_t = 0$ for all t such that $t\lambda_0$ is not a multiple of 2π. In particular, if $\sum_{t=1}^{\infty} r_t = -1/2$ and $f(\pi/s) = 0$ for some $s \geq 1$, $s \in \mathbb{N}$, then $f(l\pi/s) = 0$ for all $l = 1, \ldots, s$ and $r_t = 0$ for all $t \geq 1$ such that $t \neq 2ks$ for all $k \in \mathbb{Z}$. For $s = 1$ it follows that if $f(\pi) = 0$ then $r_t = 0$ for all odd $t \in \mathbb{Z}$.

Theorem 2.4. Let $\{X_t,\ t \in \mathbb{Z}\}$ be a real weakly stationary process with the autocorrelation function $\{r_k, k \in \mathbb{Z}\}$ such that $r_k \le 0$ for all $|k| \ge 1$. Then $\{X_t\}$ can be represented as a linear process

$$X_t = \sum_{j=0}^{\infty} c_j \varepsilon_{t-j}, \tag{7}$$

where $\{\varepsilon_t,\ t \in \mathbb{Z}\}$ is a white noise with $\operatorname{var} \varepsilon_t = 1$ and $\{c_k\}_0^\infty$ is a sequence of real constants such that $\sum_{k=0}^\infty c_k^2 < \infty$. The white noise $\{\varepsilon_t,\ t \in \mathbb{Z}\}$ belongs to the space generated by all the values of $\{X_t,\ t \in \mathbb{Z}\}$.

P r o o f. It is shown in [4, p. 288] that a weakly stationary process can be represented as a linear process (7) if and only if its spectral distribution function is absolutely continuous and its spectral density f satisfies the condition (4). Hence, the assertion follows from Theorem 2.2. □

The representation (7) of the process $\{X_t\}$ is not unique in general. However, there exists a unique linear process (7) such that $c_0 > 0$ and the zeros of the function $\sum_{j=0}^\infty c_j z^j$ do not lie in the interior of the unit disc, see [4, p. 289]. In the following we always consider such representation due to a possible invertibility of the linear process. The coefficients c_k from this representation can be obtained as follows (see [4, p. 289] for more details). Define

$$\begin{aligned} d_k &= \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}k\lambda} \ln[f(\lambda)]\,\mathrm{d}\lambda = 2\int_0^\pi \ln[f(\lambda)]\cos[k\lambda]\,\mathrm{d}\lambda, \quad k = 0, 1, 2\ldots, \\ P &= \exp\Big\{\frac{d_0}{4\pi}\Big\} = \exp\Big\{\frac{1}{4\pi}\int_{-\pi}^{\pi} \ln f(\lambda)\,\mathrm{d}\lambda\Big\}, \\ h(z) &= \exp\Big\{\frac{1}{2\pi}\sum_{k=1}^{\infty} d_k z^k\Big\}. \end{aligned} \tag{8}$$

The function $h(z)$ belongs to the Hardy space H_2, and therefore $h(z) = \sum_{k=0}^\infty a_k z^k$ with $\sum_{k=0}^\infty a_k^2 < \infty$ and $a_0 = h(0) = 1$, see [5, Sec. III.3]. The coefficients $\{c_k\}$ from (7) are then given as $c_k = \sqrt{2\pi} P a_k$ for $k \ge 0$.

The linear process (7) in Theorem 2.4 is standardized in the way that $\operatorname{var} \varepsilon_t = 1$. However, it is sometimes useful to work with the representation (7) where c_0 is set to be 1. The following theorem presents some properties of the coefficients from such representation.

Theorem 2.5. Let $\{X_t,\ t \in \mathbb{Z}\}$ satisfy the assumptions of Theorem 2.4. Then $\{X_t\}$ can be represented as a linear process $X_t = \sum_{k=0}^\infty a_k \varepsilon_{t-k}$ such that $a_0 = 1$, $\sum_{k=0}^\infty a_k^2 < \infty$, and the zeros of $\sum_{k=0}^\infty a_k z^k$ do not lie in the interior of the unit disc. The process $\{\varepsilon_t,\ t \in \mathbb{Z}\}$ is a white noise that belongs to the space generated by the values of $\{X_t,\ t \in \mathbb{Z}\}$, and $\operatorname{var} \varepsilon_t = \sigma^2 = 2\pi \exp\Big\{[1/(2\pi)]\int_{-\pi}^{\pi} \ln f(\lambda)\,\mathrm{d}\lambda\Big\}$, $0 < \sigma^2 < \infty$.

The coefficients $\{a_k\}_{k=0}^{\infty}$ satisfy $0 \le \sum_{k=0}^{\infty} a_k \le 1$, and the equality

$$\left(\sum_{k=0}^{\infty} a_k\right)^2 = \left(\sum_{k=0}^{\infty} a_k^2\right)\left(1 + 2\sum_{k=1}^{\infty} r_k\right) \tag{9}$$

holds. In particular, $\sum_{k=0}^{\infty} a_k = 0$ if and only if $\sum_{k=1}^{\infty} r_k = -1/2$, and $\sum_{k=0}^{\infty} a_k = 1$ if and only if $r_k = 0$ for all $k \ge 1$ (and so $a_k = 0$ for all $k \ge 1$).

P r o o f. Recall that $\{X_t\}$ can be represented as (7) with $c_k = \sqrt{2\pi} P a_k$, where a_k, P are defined in (8), and the zeros of $\sum_{k=0}^{\infty} c_k z^k$ do not lie in the interior of the unit disc. Equivalently, we can write $X_t = \sum_{k=0}^{\infty} a_k \tilde{\varepsilon}_{t-k}$, where $\{\tilde{\varepsilon}_t\}$ is a white noise such that $\tilde{\varepsilon}_t = \sqrt{2\pi} P \varepsilon_t$. Then $\operatorname{var} \tilde{\varepsilon}_t = \sigma^2 = 2\pi P^2 = 2\pi \exp\left\{[1/(2\pi)] \int_{-\pi}^{\pi} \ln f(\lambda)\, \mathrm{d}\lambda\right\}$.

The property $\sum_{k=0}^{\infty} a_k \ge 0$ follows from the fact that $a_0 = 1$ and the zeros of $\sum_{k=0}^{\infty} a_k z^k$ do not lie in the interior of the unit disc. The continuous spectral density f of the process $\{X_t\}$ satisfies $f(\lambda) = [\sigma^2/(2\pi)] \left|\sum_{k=0}^{\infty} a_k \mathrm{e}^{-\mathrm{i}k\lambda}\right|^2$ for all $\lambda \in [-\pi, \pi]$. Since $\operatorname{var} X_t = \sigma^2 \left(\sum_{k=0}^{\infty} a_k^2\right)$, we get

$$f(\lambda) = \frac{\sigma^2 \left(\sum_{k=0}^{\infty} a_k^2\right)}{2\pi}\left(1 + 2\sum_{t=1}^{\infty} r_t \cos(t\lambda)\right) = \frac{\sigma^2}{2\pi}\left|\sum_{k=0}^{\infty} a_k \mathrm{e}^{-\mathrm{i}k\lambda}\right|^2$$

for all $\lambda \in [-\pi, \pi]$. The equality (9) is obtained for $\lambda = 0$. Since $\sum_{k=0}^{\infty} a_k^2 = 1 + \sum_{k=1}^{\infty} a_k^2 \ge 1$, it then follows from (9) that $\sum_{k=0}^{\infty} a_k = 0$ if and only if $\sum_{k=1}^{\infty} r_k = -1/2$.

To see that $\sum_{k=0}^{\infty} a_k \le 1$ holds we show that $f(0) \le \sigma^2/(2\pi)$. If $\sum_{k=1}^{\infty} r_k = -1/2$ then $f(0) = 0$, and the inequality holds trivially. If $\sum_{k=1}^{\infty} r_k > -1/2$ it follows from (5) that $f(\lambda) \ge f(0) > 0$ holds for all $\lambda \in [-\pi, \pi]$, and

$$\frac{\sigma^2}{2\pi} = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln f(\lambda)\, \mathrm{d}\lambda\right\} \ge \exp\{\ln f(0)\} = f(0).$$

Hence, $f(0) = [\sigma^2/(2\pi)] \left(\sum_{k=0}^{\infty} a_k\right)^2 \le \sigma^2/(2\pi)$, and thus $\sum_{k=0}^{\infty} a_k \le 1$. The equality holds if and only if $f(\lambda) = f(0)$ for all $\lambda \in [-\pi, \pi]$. This is the case only for $r_t = 0$ for all $t \ge 1$ (the process $\{X_t\}$ is a white noise). $\square$

Obviously, the relationship between the autocorrelations r_k and the coefficients c_k (or a_k) is not trivial, and it is not possible to express c_k (or a_k) directly from r_k only using algebraic operations.

Let us present an example. Let $\{X_t,\ t \in \mathbb{Z}\}$ be a stationary process with $\operatorname{var} X_t = 4$ and $r_t = -1/(t^2 - 1)$ for even $|t| \ge 2$ and $r_t = 0$ for odd t. The spectral density f of $\{X_t\}$ is given as $f(\lambda) = |\sin(\lambda)|$ for $\lambda \in [-\pi, \pi]$. Let us compute the coefficients $\{a_k\}_{k=1}^{\infty}$ from the representation in Theorem 2.5. Calculations give $\sigma^2 = \pi$ and $h(z) = \sqrt{1 - z^2} = \sum_{n=0}^{\infty} (-1)^n \binom{1/2}{n} z^{2n}$ for all $|z| \le 1$. Define

$$a_k = \begin{cases} 0, & k \text{ odd}, \\ (-1)^{k/2}\dbinom{1/2}{k/2}, & k \text{ even}. \end{cases}$$

The process $\{X_t\}$ can be represented as a linear process $X_t = \sum_{k=0}^{\infty} a_k \varepsilon_{t-k}$ with $\mathsf{var}\, \varepsilon_t = \pi$. Notice that $a_k \le 0$ for all $k \ge 1$. Furthermore, we have $\sum_{t=1}^{\infty} r_t = -1/2$ and $\sum_{k=0}^{\infty} a_k = 0$ which illustrates the conclusion of Theorem 2.5.

Finally, let us remark that in view of the well-known Wold's decomposition theorem, see e. g. [3, p. 187], Theorem 2.4 claims that a process with the autocorrelation structure (1) is always purely non-deterministic.

3. SUFFICIENT CONDITIONS

In this section we give some sufficient conditions under which the resulting process have non-positive autocorrelations, i. e. it satisfies (1). These conditions are formulated for a construction in the time domain as well as in the spectral domain.

Let us start with the construction in the time domain. Theorem 2.5 shows that the class of weakly stationary processes with non-positive autocorrelations is a subset of a class of all linear processes of the form $X_t = \sum_{k=0}^{\infty} a_k \varepsilon_{t-k}$, where $\{a_k\}_0^{\infty}$ is a sequence of real numbers such that $a_0 = 1$, $\sum_{k=0}^{\infty} a_k^2 < \infty$, and $\{\varepsilon_t\}$ is a white noise with $\mathsf{var}\, \varepsilon_t = \sigma^2 > 0$. Moreover, Theorem 2.5 claims that except the trivial case ($r_t = 0$ for all $t \ge 1$) the coefficients $\{a_k\}$ always satisfy

$$-1 \le \sum_{k=1}^{\infty} a_k < 0. \tag{10}$$

The first inequality in (10) is due to the fact that we always choose the representation such that the zeros of $\sum_{k=0}^{\infty} a_k z^k$ do not lie in the interior of the unit disc. The second inequality follows from the properties of a spectral density of a process satisfying (1). Hence, these constraints are necessary for $\{a_k\}_{k=0}^{\infty}$ to define $\sum_{k=0}^{\infty} a_k \varepsilon_{t-k}$ a process with non-positive autocorrelations. The following theorem provides some sufficient conditions for $\{a_k\}_{k=0}^{\infty}$.

Theorem 3.1. Let $a_0 = 1$ and let $\{a_k\}_{k=1}^{\infty}$ be a non-decreasing sequence of real numbers such that $a_k \le 0$ for all $k \ge 1$ and $\sum_{k=1}^{\infty} a_k \ge -1$ holds. Then the autocorrelation function $\{r_t\}$ of the process $X_t = \sum_{k=0}^{\infty} a_k \varepsilon_{t-k}$ satisfies $r_t \le 0$ for all $t \ge 1$. Moreover, $\sum_{t=1}^{\infty} r_t = -1/2$ holds if and only if $\sum_{k=1}^{\infty} a_k = -1$.

Proof. The autocovariance function of $\{X_t\}$ is given as $R_t = \sigma^2 \sum_{k=0}^{\infty} a_k a_{t+k}$. Hence, $R_t \le 0$ for all $t \ge 1$ if and only if

$$-a_t \ge \sum_{k=1}^{\infty} a_{t+k} a_k \qquad \text{for all } t \ge 1. \tag{11}$$

According to the assumptions we have $-a_k = |a_k|$, and the sequence $\{|a_k|\}_{k=1}^{\infty}$ is non-increasing, i. e. $|a_k| \ge |a_{t+k}|$ holds for all $k \ge 1$. We get $\sum_{k=1}^{\infty} a_k a_{t+k} = \sum_{k=1}^{\infty} |a_k||a_{t+k}| \le |a_t| \sum_{k=1}^{\infty} |a_k| \le |a_t|$ for all $t \ge 1$. Hence, (11) holds. Since $\sum_{k=0}^{\infty} a_k = 0$ if and only if $\sum_{k=1}^{\infty} a_k = -1$, the rest of the assertion follows from Theorem 2.5. □

The assumptions of Theorem 3.1 can be very easily verified, and we are able to construct linear processes with non-positive correlations.

Let us briefly discuss the assumptions of Theorem 3.1. Recall that we always consider only linear processes $\sum_{k=0}^{\infty} a_k \varepsilon_{t-k}$ such that the zeros of the function $\sum_{k=0}^{\infty} a_k z^k$ do not lie in the interior of the unit disc. Under this condition it is shown in Theorem 2.5, see also (10), that $\sum_{k=1}^{\infty} a_k \geq -1$ holds. Furthermore, $\sum_{k=1}^{\infty} a_k \leq 0$ holds for all processes with non-positive correlations, see (10). In comparison to this Theorem 3.1 works with a stronger assumption $a_k \leq 0$ for all $k \geq 1$. Finally, the monotonicity of the sequence $\{a_k\}_{k=1}^{\infty}$ is only a sufficient condition. We have already presented an example where $a_k = 0$ for all odd k and $a_k \neq 0$ otherwise.

Let us remark that the condition $a_k \leq 0$ for all $k \geq 1$ alone in general does not imply $r_t \leq 0$ for all $t \geq 1$. Consider an invertible MA(3) process X_t with $a_1 = -5/66$, $a_2 = -13/22$ and $a_3 = -5/33$ as an example (here $R_1 = 85\sigma^2/1452 > 0$). However, for invertible MA(2) processes the condition $r_t \leq 0$ for $t = 1, 2$ is equivalent to $a_t \leq 0$ for $t = 1, 2$. Indeed, for MA(2) we have $R_2 = a_2$ and $R_1 = a_1 + a_1 a_2 = a_1(1 + a_2)$. The process is invertible and therefore, $a(z) = 1 + a_1 z + a_2 z^2 \neq 0$ for all $|z| \leq 1$. In particular, $a(1) = 1 + a_1 + a_2 > 0$ and $a(-1) = 1 - a_1 + a_2 > 0$. It follows that $1 + a_2 > 0$, and therefore $R_t \leq 0$ if and only if $a_t \leq 0$, $t = 1, 2$.

The requirement $r_t \leq 0$ for all $t \geq 1$ can be expressed in terms of the spectral density of the process $\{X_t\}$ as well. Recall that we have shown in Proposition 2.1 that there always exists a continuous spectral density. Let us first summarize some of its general properties.

Theorem 3.2. Let $\{r_t\}_0^{\infty}$ be a sequence of real numbers such that $r_0 = 1$, $r_t \leq 0$ for all $t \geq 1$ and $\sum_{t=1}^{\infty} r_t \geq -1/2$. Let $f(\lambda) = [1/(2\pi)]\,[1 + 2\sum_{t=1}^{\infty} r_t \cos(t\lambda)]$ for $\lambda \in [-\pi, \pi]$.

1. The function f is continuous, non-negative and has a global minimum at the point $\lambda = 0$. If $\sum_{t=1}^{\infty} r_t \neq 0$ then 0 is also a strong local minimum.
2. If $f(0) = f(\pi) = 0$ and $f(\lambda) > 0$ for all $\lambda \in (0, \pi)$ then f is symmetric around the point $\pi/2$, i.e. $f(\lambda) = f(\pi - \lambda)$ for all $\lambda \in [0, \pi]$.
3. If $r_k = \mathcal{O}(k^{-2-\varepsilon})$ for some $\varepsilon > 0$ then f is differentiable, its derivative f' is continuous on $[-\pi, \pi]$ and is given as $f'(\lambda) = -\sum_{k=1}^{\infty} k r_k \sin(k\lambda)$. In particular, $f'(0) = 0$.

P r o o f. 1. The inequality $f(\lambda) \geq f(0)$ holds for all $\lambda \in [-\pi, \pi]$, see (5). If there exists $t_0 \geq 1$ such $r_{t_0} \neq 0$ then there exists $\varepsilon > 0$ such that $-|r_{t_0}| \cos(t_0 \lambda) > -|r_{t_0}|$ for all $\lambda \in (-\varepsilon, \varepsilon) \smallsetminus \{0\}$. Hence, it follows from (5) that $f(\lambda) > f(0)$ holds for $\lambda \in (-\varepsilon, \varepsilon) \smallsetminus \{0\}$.

2. It follows from Remark 2.3 that $r_t = 0$ for all odd $t \in \mathbb{Z}$, and thus $f(\lambda) = [1/(2\pi)]\,[1 + \sum_{k=1}^{\infty} r_{2k} \cos(2k\lambda)]$. The assertion follows from the equality $\cos[2k(\pi - \lambda)] = \cos(2k\lambda)$.

3. If $r_k = \mathcal{O}(k^{-2-\varepsilon})$ then $\sum_{k=1}^{\infty} k r_k$ converges, and $\sum_{k=1}^{\infty} k r_k \sin(k\lambda)$ converges uniformly for all $\lambda \in [-\pi, \pi]$. It follows from the theory of Fourier series, see for instance [7, p. 40], that $f'(\lambda) = -\sum_{k=1}^{\infty} k r_k \sin(k\lambda)$ for $\lambda \in [-\pi, \pi]$. □

In the following we provide some sufficient conditions for the spectral density f such that the corresponding autocorrelations satisfy $r_t \le 0$ for all $t \ge 1$. These conditions are formulated in Theorem 3.6.

Lemma 3.3. Let g be a measurable non-negative and non-increasing function on $[0,\pi]$ and let $n \in \mathbb{N}$, $n \neq 0$. Then $\int_0^\pi g(x)\sin(nx)\,\mathrm{d}x \ge 0$.

P r o o f. Assume first that n is even. Then $\int_0^\pi g(x)\sin(nx)\,\mathrm{d}x = \sum_{i=0}^{n/2-1} I_i$, where $I_i = \int_{2i\pi/n}^{(2i+2)\pi/n} g(x)\sin(nx)\,\mathrm{d}y$. The substitution $y = nx$ gives

$$\begin{aligned} I_i &= \frac{1}{n}\int_{2i\pi}^{(2i+1)\pi} g(y/n)\sin(y)\,\mathrm{d}y + \frac{1}{n}\int_{(2i+1)\pi}^{(2i+2)\pi} g(y/n)\sin(y)\,\mathrm{d}y \\ &= \frac{1}{n}\int_{2i\pi}^{(2i+1)\pi} \Big[g\Big(\frac{y}{n}\Big) - g\Big(\frac{y+\pi}{n}\Big)\Big]\sin(y)\,\mathrm{d}y. \end{aligned}$$

Since $\sin(y) \ge 0$ for $y \in (0,\pi)$ and g is non-increasing, the inequality $I_i \ge 0$ holds for all $i = 0,\dots,n/2-1$. Hence, $\int_0^\pi g(x)\sin(nx)\,\mathrm{d}x \ge 0$.

Let n be odd. Then we get $\int_0^\pi g(x)\sin(nx)\,\mathrm{d}x = \sum_{i=0}^{(n-3)/2} I_i + J$, where $J = \int_{(n-1)\pi/n}^{\pi} g(x)\sin(nx)\,\mathrm{d}x$ and I_i are defined as before. We have shown that $I_i \ge 0$ holds. The inequality $J \ge 0$ follows from the non-negativeness of g and due to the positive sign of $\sin(nx)$ on the interval $\big((n-1)\pi/n, \pi\big)$. □

Corollary 3.4. Let f be a differentiable function on $(0,\pi]$ such that its derivative f' is non-negative and non-increasing on $(0,\pi)$. Then for any $n \in \mathbb{N}$, $n \neq 0$, the inequality $\int_0^\pi f(x)\cos(nx)\,\mathrm{d}x \le 0$ holds.

P r o o f. Integration by parts gives

$$\int_0^\pi f(x)\cos(nx)\,\mathrm{d}x = \frac{1}{n} f(x)\sin(nx)\Big|_0^\pi - \frac{1}{n}\int_0^\pi f'(x)\sin(nx)\,\mathrm{d}x.$$

The first term is equal to 0, and the assertion follows from Lemma 3.3 □

Corollary 3.5. Let f be a measurable function on $(0,\pi)$ such $f(\lambda) = f(\pi-\lambda)$ for all $\lambda \in [0,\pi]$. Let f be differentiable on $(0,\pi/2)$ such that the derivative f' is non-negative and non-increasing on $(0,\pi/2)$. Then for any $n \in \mathbb{N}$, $n \neq 0$, the inequality $\int_0^\pi f(x)\cos(nx)\,\mathrm{d}x \le 0$ holds.

P r o o f. The property $f(\lambda) = f(\pi-\lambda)$ implies that $\int_0^\pi f(x)\cos(nx)\,\mathrm{d}x = 0$ for all odd k and $\int_0^\pi f(x)\cos(nx) = 2\int_0^{\pi/2} f(x)\cos(nx)\,\mathrm{d}x$ for even k. The proof that $c_k \le 0$ for even k is then a simple analogy to the proof of Corollary 3.4. □

Theorem 3.6. Let $\tilde{f} \geq 0$, $\tilde{f} \not\equiv 0$, satisfy the assumptions of Corollary 3.4 or 3.5. Then the function f defined as $f(\lambda) = \tilde{f}(|\lambda|)$ for $\lambda \in [-\pi, \pi]$ is a spectral density of a process with non-positive autocorrelations.

Proof. Since $f \geq 0$ on $[-\pi, \pi]$, the function f is a spectral density of a weakly stationary process $\{X_t\}$. It follows from Corollary 3.4 and 3.5 that the autocovariances R_t of $\{X_t\}$ satisfy

$$R_0 = \int_{-\pi}^{\pi} f(\lambda)\,\mathrm{d}\lambda = 2\int_0^{\pi} \tilde{f}(\lambda)\,\mathrm{d}\lambda > 0,$$

$$R_t = \int_{-\pi}^{\pi} f(\lambda)\mathrm{e}^{it\lambda}\,\mathrm{d}\lambda = 2\int_0^{\pi} \tilde{f}(\lambda)\cos(t\lambda)\,\mathrm{d}\lambda \leq 0 \quad \text{for } t \geq 1.$$

The autocorrelations of $\{X_t\}$ are well-defined, and they satisfy the condition (1). □

Theorem 3.6 gives a clue how a stationary process with the desirable property $r_t \leq 0$ for all $t \geq 1$ can be constructed in a spectral domain. The choice of $\tilde{f}$ such that $\tilde{f}(0) = 0$ further ensures that $\sum_{t=1}^{\infty} r_t = -1/2$.

A simple example of a spectral density f constructed using Theorem 3.6 (with $\tilde{f}$ satisfying the assumptions of Corollary 3.4) is the function $f(\lambda) = k|\lambda|^{\alpha}$ for $k > 0$ and $\alpha \in (0, 1]$. For instance, if $f(\lambda) = |\lambda|$ then $R(0) = \pi^2$ and $r_k = -4/(\pi^2 k^2)$ for odd k and $r_k = 0$ for even k. An example of f constructed from $\tilde{f}$, which satisfies the assumptions of Corollary 3.5, is the function $f(\lambda) = k|\sin\lambda|$, $k > 0$. The case $k = 1$ has been already discussed at the end of Section 2.

Let $\tilde{f} \geq 0$ satisfy the assumptions of Corollary 3.4. This means that $\tilde{f}$ is differentiable, non-decreasing and concave on $(0, \pi]$. If g is a non-decreasing concave function such that $g \circ \tilde{f}$ is differentiable on $(0, \pi]$ then $g \circ \tilde{f}$ satisfies the assumptions of Corollary 3.4 as well. This is the case for example for $g = \log$ if $\tilde{f}(\lambda) \neq 0$ for all $\lambda \in (0, \pi]$. In particular, if $\tilde{f}(\lambda) \geq 1$ for all $\lambda \in [0, \pi]$ then $\log(\tilde{f}(|\lambda|))$ is a spectral density of a process with non-positive autocorrelations as well.

Corollary 3.5 considers a function symmetric around the point $\pi/2$ and non-decreasing on $(0, \pi/2)$. It is shown in the proof that $\int_0^{\pi} f(\lambda)\cos(n\lambda) \neq 0$ if and only if t is even. One could proceed even further and divide the interval $[0, \pi]$ into more pieces. For instance consider a function $\tilde{f}$ defined on $[0, \pi/3]$ and differentiable on $(0, \pi/3)$ with $\tilde{f}'$ non-negative and non-increasing. Take $f(\lambda) = \tilde{f}(\lambda)$ for $\lambda \in [0, \pi/3]$, $f(\lambda) = \tilde{f}(|\lambda - 2\pi/3|)$ for $\lambda \in (\pi/3, \pi]$. Then $\int_0^{\pi} f(\lambda)\cos(n\lambda) \leq 0$ for all $n \geq 1$, and $\int_0^{\pi} f(\lambda)\cos(n\lambda) \neq 0$ if and only if $n = 3k$ for some $k \in \mathbb{Z}$. Similarly, we could consider $\tilde{f}$ defined on $[0, \pi/4]$ etc. Using this approach we are able to construct for a fixed $j \in \mathbb{N}$ a process $\{X_t\}$ with autocorrelations $r_t \leq 0$ for all $t \geq 1$ such that $r_t \neq 0$ only for $t = kj$, $k \in \mathbb{Z}$.

Finally, let us remark that if $\tilde{f}$ satisfies the assumptions of Corollary 3.4 then f is concave and increasing on $(0, \pi)$ and concave and decreasing on $(-\pi, 0)$. Hence, $f'(0)$ does not exist. It then follows from Theorem 3.2 that $r_k = \mathcal{O}(k^{-\beta})$ for some $1 < \beta \leq 2$. The same conclusion is obtained for $\tilde{f}$ satisfying the assumptions of Corollary 3.5. Hence, if one wants to obtain a process with autocorrelations $\{r_k\}$ such that $r_k \leq 0$ for all $k \geq 1$ and $r_k = \mathcal{O}(k^{-\beta})$ for $\beta > 2$ then the generating procedures cannot be based on Theorem 3.6.

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Šárka Došlá, Department of Probability and Mathematical Statistics, Faculty of Mathematics and Physics, Charles University in Prague, Sokolovská 83, 186 75 Praha 8. Czech Republic

e-mail: dosla@karlin.mff.cuni.cz

Jiří Anděl, Department of Probability and Mathematical Statistics, Faculty of Mathematics and Physics, Charles University in Prague, Sokolovská 83, 186 75 Praha 8. Czech Republic

e-mail: jiri.andel@mff.cuni.cz