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CONSTRUCTING COPULAS BY MEANS OF PAIRS OF ORDER STATISTICS

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In this paper, we introduce two transformations on a given copula to construct new and recover already-existent families. The method is based on the choice of pairs of order statistics of the marginal distributions. Properties of such transformations and their effects on the dependence and symmetry structure of a copula are studied.

Keywords: copula, dependence ordering, FGM family, measure of association, symmetry, transformation

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1. INTRODUCTION

The construction of distributions with given marginals has been a problem of interest to statisticians for many years. Today, in view of Sklar's theorem [25, 26], this problem can be reduced to the construction of a copula. Copulas are a special type of aggregation functions, and nowadays are of interest in fuzzy set theory [5, 6, 12, 14, 16].

Nelsen [22] summarizes different methods of constructing copulas. Recently, various authors provide construction methods from the class of copulas to itself, or from a more general class of functions to another (see, for instance, [2, 8, 10, 11, 13, 18, 19, 20]). One of the purposes for such constructions is to increase the availability of copulas for modeling purposes.

In this paper we provide two new transformations of copulas, which recover some known families of copulas. The method is based on the choice of pairs of order statistics of the marginal distributions. Properties on dependence, symmetry, invariance, and random number generation of such transformations are also shown. Preliminary ideas can be found in [3].

2. PRELIMINARIES

A (bivariate) *copula* is the restriction to $[0,1]^2$ of a continuous bivariate distribution function whose margins are uniform on $[0,1]$. The importance of copulas as a tool for statistical analysis and modeling stems largely from the observation that the joint distribution H of the random vector (X,Y) with respective margins F and G can be expressed by $H(x,y) = C(F(x),G(y))$, $(x,y) \in [-\infty,\infty]^2$, where C is a copula that is uniquely determined on $\operatorname{Range} F \times \operatorname{Range} G$ (*Sklar's theorem*).

Equivalently, a copula is a function $C\colon [0,1]^2 \longrightarrow [0,1]$ which satisfies:

(C1) the *boundary conditions* $C(t,0) = C(0,t) = 0$ and $C(t,1) = C(1,t) = t$ for all $t \in [0,1]$,

(C2) the *2-increasing property*, i. e., $C(x_2,y_2) - C(x_2,y_1) - C(x_1,y_2) + C(x_1,y_1) \geq 0$ for all x_1,x_2,y_1,y_2 in $[0,1]$ such that $x_1 \leq x_2$ and $y_1 \leq y_2$.

Let Π denote the copula for independent random variables, i. e., $\Pi(x,y) = xy$ for all $(x,y) \in [0,1]^2$. For any copula C we have $W(x,y) = \max(0,x+y-1) \leq C(x,y) \leq \min(x,y) = M(x,y)$ for all (x,y) in $[0,1]^2$, where M and W are themselves copulas. For a complete survey on copulas, see [22].

Given a copula C, let $\overline{C}$ (respectively, $\hat{C}$) denote the *survival function* (respectively, *survival copula*) associated with C, i. e., $\overline{C}(x,y) = 1 - x - y + C(x,y)$ (respectively, $\hat{C}(x,y) = \overline{C}(1-x,1-y)$). Whereas $\hat{C}$ is always a copula, $\overline{C}$ never is since condition (C1) does not hold.

In the following, given bivariate functions A and B with a common domain K, let $A \leq B$ denote the pointwise inequality $A(x,y) \leq B(x,y)$ for every (x,y) in K (and similarly for "$\geq$" and "$=$").

3. THE TRANSFORMATIONS

3.1. Constructions and examples

Let (X_1,Y_1) and (X_2,Y_2) be two independent vectors of uniform $(0,1)$ random variables with common copula D. Let $X_{(1)},X_{(2)}$ and $Y_{(1)},Y_{(2)}$ be their corresponding order statistics. Consider the random vector

$$(Z_1,Z_2) = \begin{cases} (X_{(1)},Y_{(2)}), & \text{with probability } 1/2 \\ (X_{(2)},Y_{(1)}), & \text{with probability } 1/2. \end{cases}$$

The distribution of (Z_1,Z_2) is then given by

$$\begin{aligned} H_1(x,y) &= \frac{1}{2}\mathrm{P}(X_{(1)} \leq x, Y_{(2)} \leq y) + \frac{1}{2}\mathrm{P}(X_{(2)} \leq x, Y_{(1)} \leq y) \\ &= \frac{1}{2}\{\mathrm{P}(Y_{(2)} \leq y) - \mathrm{P}(X_{(1)} > x, Y_{(2)} \leq y) + \mathrm{P}(X_{(2)} \leq x) \\ &\quad -\mathrm{P}(X_{(2)} \leq x, Y_{(1)} > y)\} \\ &= \frac{1}{2}\left\{y^2 - (y - D(x,y))^2 + x^2 - (x - D(x,y))^2\right\} \\ &= D(x,y)\{x + y - D(x,y)\} \\ &= D(x,y)\{1 - (1 - x - y + D(x,y))\} \\ &= D(x,y)\left\{1 - \overline{D}(x,y)\right\}. \end{aligned}$$

Now, for $\alpha \in [0,1]$, consider the random pair (T_1, T_2), defined by

$$(T_1, T_2) = \begin{cases} (Z_1, Z_2), & \text{with probability } \alpha \\ (X_1, Y_1), & \text{with probability } 1-\alpha. \end{cases}$$

Then the distribution of (T_1, T_2), denoted by $C_\alpha[D]$, is given by

$$\begin{aligned} C_\alpha[D](x,y) &= \alpha D(x,y)\left\{1-\overline{D}(x,y)\right\} + (1-\alpha)D(x,y) \\ &= D(x,y)\left\{1-\alpha\overline{D}(x,y)\right\}. \end{aligned}$$

Note that the bivariate distribution function $C_\alpha[D]$ satisfies the boundary conditions (C1), and hence it is a copula.

If we consider the random vector

$$(Z_1, Z_2) = \begin{cases} (X_{(1)}, Y_{(1)}), & \text{with probability } 1/2 \\ (X_{(2)}, Y_{(2)}), & \text{with probability } 1/2, \end{cases}$$

then the distribution function of (Z_1, Z_2) is given by

$$\begin{aligned} H_2(x,y) &= \frac{1}{2}\mathrm{P}(X_{(1)} \le x, Y_{(1)} \le y) + \frac{1}{2}\mathrm{P}(X_{(2)} \le x, Y_{(2)} \le y) \\ &= \frac{1}{2}\left\{1 - \mathrm{P}(X_{(1)} > x) - \mathrm{P}(Y_{(1)} > y) + \mathrm{P}(X_{(1)} > x, Y_{(1)} > y) + D^2(x,y)\right\} \\ &= \frac{1}{2}\left\{1-(1-x)^2-(1-y)^2+(1-x-y+D(x,y))^2+D^2(x,y)\right\} \\ &= \Pi(x,y) + D(x,y)\overline{D}(x,y). \end{aligned}$$

The distribution function of the pair

$$(T_1, T_2) = \begin{cases} (Z_1, Z_2), & \text{with probability } \alpha \\ (X_1, X_2), & \text{with probability } 1-\alpha, \end{cases}$$

denoted by $C_\alpha^*[D]$ – and which is copula as well – is then given by

$$\begin{aligned} C_\alpha^*[D](x,y) &= \alpha\left\{\Pi(x,y) + D(x,y)\overline{D}(x,y)\right\} + (1-\alpha)\Pi(x,y) \\ &= \Pi(x,y) + \alpha D(x,y)\overline{D}(x,y). \end{aligned}$$

In short, we have proved the following result.

Theorem 3.1. For any copula D, the functions defined from $[0,1]^2$ onto $[0,1]$ by

$$C_\alpha[D](x,y) = D(x,y)[1-\alpha\overline{D}(x,y)], \tag{1}$$

and

$$C_\alpha^*[D](x,y) = \Pi(x,y) + \alpha D(x,y)\overline{D}(x,y), \tag{2}$$

where $0 \le \alpha \le 1$, are copulas.

Each transformation (1) and (2) is "unique", in the sense that given a copula D, this generates a unique copula – or a family of copulas – (see examples below) under (1) and (2). We prove it in the following result.

Theorem 3.2. Let D_1 and D_2 be two copulas such that $C_\alpha[D_1] = C_\alpha[D_2]$ (respectively, $C^*_\alpha[D_1] = C^*_\alpha[D_2]$) for every $\alpha \in [0,1]$. Then $D_1 = D_2$.

Proof. Let D_1 and D_2 be two copulas such that $C_\alpha[D_1] = C_\alpha[D_2]$ (in the case $C^*_\alpha[D_1] = C^*_\alpha[D_2]$ the result can be proved similarly). Suppose $\alpha \neq 0$ (the case $\alpha = 0$ is trivial). Then we have $D_1(1 - \alpha\overline{D_1}) = D_2(1 - \alpha\overline{D_2})$, which is equivalent to

$$[D_2(x,y) - D_1(x,y)]\,\{1 - \alpha(1 - x - y) - \alpha[D_2(x,y) + D_1(x,y)]\} = 0 \tag{3}$$

for every (x,y) in $[0,1]^2$. Suppose there exists a point (x_0,y_0) in $[0,1]^2$ such that – without loss of generality – $D_1(x_0,y_0) < D_2(x_0,y_0)$. Then the equality in (3) is equivalent to $1 - \alpha(1 - x_0 - y_0) - \alpha[D_2(x_0,y_0) + D_1(x_0,y_0)] = 0$, i.e., $D_2(x_0,y_0) + D_1(x_0,y_0) = x_0 + y_0 - 1 + 1/\alpha$. Since $1 \leq 1/\alpha$ and $D_1(x_0,y_0) < D_2(x_0,y_0) \leq y_0$, we have the following chain of inequalities: $D_2(x_0,y_0) \geq x_0 + y_0 - D_1(x_0,y_0) > x_0 + y_0 - y_0 = x_0$. This is absurd, and we conclude that $D_1 = D_2$. □

We now provide several examples.

Example 3.3. Consider the copula M. Since $\overline{M}(x,y) = M(1-x, 1-y)$ for every (x,y) in $[0,1]^2$, we have $M(x,y)\overline{M}(x,y) = M(x,y) - \Pi(x,y)$. The copulas generated by (1) and (2) are then given by

$$C_\alpha\,[M]\,(x,y) = \alpha\Pi(x,y) + (1-\alpha)M(x,y), \tag{4}$$

and

$$C^*_\alpha\,[M]\,(x,y) = \alpha M(x,y) + (1-\alpha)\Pi(x,y), \tag{5}$$

respectively. Copulas of the form (4) and (5) belong to the known Fréchet–Mardia family of copulas (see [22] for more details).

Example 3.4. Consider the copula W. Since $W(x,y)\overline{W}(x,y) = 0$ for every (x,y) in $[0,1]^2$, for each $\alpha \in [0,1]$ we have

$$C_\alpha\,[W]\,(x,y) = W(x,y),$$

and

$$C^*_\alpha\,[W]\,(x,y) = \Pi(x,y).$$

Example 3.5. Consider the product copula Π. Then we have

$$C_\alpha\,[\Pi]\,(x,y) = xy[1 - \alpha(1-x)(1-y)], \tag{6}$$

and

$$C^*_\alpha[\Pi](x,y) = xy[1+\alpha(1-x)(1-y)]. \tag{7}$$

Copulas given by (6) and (7) constitute the known Farlie–Gumbel–Morgenstern (FGM, for short) family of copulas, which is usually written as

$$\mathrm{E}(x,y) = xy[1+\alpha(1-x)(1-y)], \tag{8}$$

with $\alpha \in [-1,1]$ (see [9] and the references therein).

Remark 3.6. Let J be the family of copulas given by $J(x,y) = xy + \alpha f(x)g(y)$, with f and g two real functions and $\alpha \in \mathbb{R}\backslash\{0\}$. Conditions under f, g and α in order that J is a copula are given in [23]. If we compare J with $C^*_\alpha[D]$, i. e., $J = C^*_\alpha[D]$, for some copula D and appropriate α, we obtain $D(x,y)[1-x-y+D(x,y)] = f(x)g(y)$, whose solution for D is

$$D(x,y) = \frac{x+y-1+\sqrt{(1-x-y)^2+4f(x)g(y)}}{2}.$$

For example, taking $f(t) = g(t) = t(1-t)$ for all t in $[0,1]$, we obtain $D(x,y) = xy$ – recall Example 3.5.

3.2. Some properties

In what follows, we provide several properties on symmetry, invariance, ordering, and measures of association of the copulas given by (1) and (2).

A copula D is C_α-*invariant* (respectively, C^*_α-*invariant*) if $C_\alpha[D] = D$ (respectively, $C^*_\alpha[D] = D$). The following result shows for which copulas D the transformations (1) and (2) are C_α-invariant and C^*_α-invariant, respectively.

Theorem 3.7. Given a copula D, for each $\alpha \in [0,1]$:

(i) D is C_α-invariant if, and only if, $D = W$.

(ii) D is C^*_α-invariant if, and only if, D is the Plackett family of copulas, which is given by

$$D(x,y) = \frac{1-\alpha(1-x-y)-\sqrt{(\alpha(1-x-y)-1)^2-4\alpha xy}}{2\alpha} \tag{9}$$

for $\alpha > 0$, and $D = \Pi$ if $\alpha = 0$.

Proof. By definition of $C_\alpha[D]$, we have $C_\alpha[D] = D$ if, and only if, $D(x,y)(1-x-y+D(x,y)) = 0$, or equivalently, $D(x,y) = 0$ or $D(x,y) = x+y-1$, i. e., $D = W$, which proves part (i).

To prove part (ii), note that if $\alpha > 0$ (the case $\alpha = 0$ is trivial and we omit it) $C^*_\alpha[D] = D$ is equivalent to the following equality: $\alpha D^2(x,y) - (1-\alpha(1-x-$

$y))D(x,y)+xy=0$, whose solution for D is given by (9) – see [22] for more details on this family of copulas. This completes the proof. □

Observe that if $\alpha = 1$ in the copula D given by (9), we obtain

$$D(x,y) = \frac{x+y-\sqrt{(x-y)^2}}{2} = \frac{x+y-|x-y|}{2},$$

i. e., $D = M$.

For each $\alpha \in [0,1]$, we have $C^*_\alpha[D] = D$ if, and only if, $C_\alpha[D] = \Pi$ (and $C_\alpha[D] = D$ if, and only if, $C^*_\alpha[D] = \Pi$). This fact and Theorem 3.7 lead us to the following

Corollary 3.8. Given a copula D, for each $\alpha \in [0,1]$ we have:

(i) $C_\alpha[D] = \Pi$ if, and only if, D is the family of copulas given by (9).

(ii) $C^*_\alpha[D] = \Pi$ if, and only if, $D = W$.

If the copula D is *symmetric*, i. e., $D(x,y) = D(y,x)$ for all $x, y \in [0,1]$, then (1) and (2) are symmetric as well. A copula C is *radially symmetric* if $C = \hat{C}$ [21, 22]. The following result shows that the transformations (1) and (2) preserve the radially symmetry property of a given copula.

Theorem 3.9. For every radially symmetric copula D, the transformations $C_\alpha[D]$ and $C^*_\alpha[D]$ are radially symmetric for each $\alpha \in [0,1]$.

Proof. By definition, we have

$$\begin{aligned}
\hat{C}_\alpha[D](x,y) &= x+y-1+C_\alpha[D](1-x,1-y) \\
&= x+y-1+D(1-x,1-y)\left\{1-\alpha\overline{D}(1-x,1-y)\right\} \\
&= x+y-1+D(1-x,1-y)-\alpha D(1-x,1-y)\overline{D}(1-x,1-y) \\
&= \hat{D}(x,y)\left\{1-\alpha\overline{\hat{D}}(x,y)\right\},
\end{aligned}$$

where the last line follows from the fact that $D(1-x,1-y) = \mathrm{P}(X \le 1-x, Y \le 1-y) = \mathrm{P}(1-X > x, 1-Y > y) = \overline{\hat{D}}(x,y)$ and $\overline{D}(1-x,1-y) = \hat{D}(x,y)$. Thus $\hat{C}_\alpha[D] = C_\alpha[\hat{D}] = C_\alpha[D]$, which completes the proof. □

If C_1 and C_2 are two copulas, we say that C_2 is *more concordant than* C_1 (written $C_1 \prec_c C_2$) if $C_1 \le C_2$. A copula C is *positively quadrant dependent* (written PQD) if $\Pi \prec_c C$, and *negatively quadrant dependence* (NQD) if $C \prec_c \Pi$. A totally ordered parametric family $\{C_\alpha\}$ of copulas is *positively ordered* if $C_{\alpha_1} \prec_c C_{\alpha_2}$ whenever $\alpha_1 \le \alpha_2$; and *negatively ordered* if $C_{\alpha_2} \prec_c C_{\alpha_1}$ whenever $\alpha_1 \le \alpha_2$ [15, 22]. For the families (1) and (2) of copulas we have the following results – the proof of Theorem 3.10 is simple, and we omit it.

Theorem 3.10. Given a copula D, we have:

(i) The parametric family $\{C_\alpha[D]\}$ (respectively, $\{C^*_\alpha[D]\}$) of copulas is negatively (respectively, positively) ordered.

(ii) $C_\alpha[D] \prec_c D$ and $\Pi \prec_c C^*_\alpha[D]$ for all $\alpha \in [0,1]$, i. e., $C^*_\alpha[D]$ is PQD for every $\alpha \in [0,1]$.

Theorem 3.11. Let D_1 and D_2 be two copulas such that $D_1 \prec_c D_2$. Then we have $C_\alpha[D_1] \prec_c C_\alpha[D_2]$ and $C^*_\alpha[D_1] \prec_c C^*_\alpha[D_2]$ for every $\alpha \in [0,1]$.

P r o o f. We first prove the result for the transformation (1). Note that, for a given copula D, we have $C_\alpha[D] = \alpha D(1-\overline{D}) + (1-\alpha)D$. So we only need to show that $D_1 \prec_c D_2$ implies that $D_1(1-\overline{D_1}) \le D_2(1-\overline{D_2})$; but this last inequality is equivalent to

$$D_1(x,y)\,[x+y-D_1(x,y)] - D_2(x,y)\,[x+y-D_2(x,y)] \le 0,$$

that is,

$$[D_2(x,y) - D_1(x,y)]\,[x - D_1(x,y) + y - D_2(x,y)] \ge 0.$$

Since $D_1(x,y) \le x$ and $D_2(x,y) \le y$, the result follows.

On the other hand, since $D_1 \prec_c D_2$ implies that $\overline{D_1} \prec_c \overline{D_2}$, and thus $D_1\overline{D_1} \le D_2\overline{D_2}$, it follows that $C^*_\alpha[D_1] \prec_c C^*_\alpha[D_2]$, which completes the proof. □

We note that, for a given copula D, the transformation $C_\alpha[D]$ may be PQD or NQD. For instance, from part (ii) of Theorem 3.10, if a copula D is NQD, then $C_\alpha[D]$ is NQD. And, for a given copula D, if $B \prec_c D$, where B is a copula given by (9), using Theorem 3.11 and part (i) of Corollary 3.8, we have that $\Pi = C_\alpha[B] \prec_c C_\alpha[D]$, i. e., $C_\alpha[D]$ is PQD.

As a consequence of our results, transformations of type (1) (respectively, (2)) decrease (respectively, increase) the degree of dependence of the copula D. A useful family for modeling could be a convex linear combination of $C_\alpha[D]$ and $C^*_\alpha[D]$ – which is a copula – namely

$$C(x,y) = \beta C_\alpha[D](x,y) + (1-\beta)C^*_\alpha[D](x,y) \;\text{ for all } (x,y) \in [0,1]^2,\; \beta \in [0,1]. \tag{10}$$

Observe that, for instance, taking $D = \Pi$ in (10), we easily obtain $C(x,y) = xy[1 + \alpha(1-2\beta)(1-x)(1-y)]$, i. e., the FGM family of copulas; which is PQD if, and only if, $\beta \le 1/2$, and NQD when $\beta \ge 1/2$.

The population version of three of the most common nonparametric measures of association between the components of a continuous random pair (X,Y) are *Kendall's tau* (τ), *Spearman's rho* (ρ), and *Gini's gamma* (γ). Such measures are called *measures of concordance* since they satisfy a set of axioms due to Scarsini [24]. The coefficients ρ and γ are based on *average quadrant dependence* and τ on *expected quadrant dependence* [4], and depend only on the copula C of the pair (X,Y), and are given by

$$\tau(C) = 4\int_0^1\!\!\int_0^1 C(x,y)\,\mathrm{d}C(x,y) - 1, \tag{11}$$

$$\rho(C) = 12 \int_0^1 \int_0^1 [C(x,y) - xy] \, \mathrm{d}x\mathrm{d}y, \tag{12}$$

and

$$\gamma(C) = 8 \int_0^1 \int_0^1 [C(x,y) - xy] \, \mathrm{d}A(x,y), \tag{13}$$

where A denotes the copula $(M+W)/2$ (for a complete study on measures of association, see [22] and the references therein). The direct computation of these measures for the copulas in (1) and (2) does not provide too much information; however, we have the following result, in which we find relationships among these measures for the copulas C_α and C_α^*.

Theorem 3.12. Let τ be the measure given by (11), and let κ denote the set of the measures given by (12) and (13), i. e., $\kappa \in \{\rho, \gamma\}$. Then, for a given copula D and for each $\alpha \in [0,1]$, we have:

(i) $-1 \leq \tau(C_\alpha[D]) \leq [(1-\alpha)(3-\alpha)]/3$ and $-1 \leq \kappa(C_\alpha[D]) \leq 1-\alpha$;

(ii) $0 \leq \tau(C_\alpha^*[D]) \leq [\alpha(\alpha+2)]/3$ and $0 \leq \kappa(C_\alpha^*[D]) \leq \alpha$;

(iii) $\kappa(C_\alpha[D]) + \kappa(C_\alpha^*[D]) = \kappa(D)$.

Proof. First we recall that given two copulas C_1 and C_2 such that $C_1 \prec_c C_2$, we have $\tau(C_1) \leq \tau(C_2)$, $\rho(C_1) \leq \tau(C_2)$, and $\gamma(C_1) \leq \gamma(C_2)$ [22].

Since any copula D satisfies that $W \prec_c D \prec_c M$, from Theorem 3.11 and Examples 3.3 and 3.4, we have $W \prec_c C_\alpha[D] \prec_c (\alpha\Pi + (1-\alpha)M)$ and $\Pi \prec_c C_\alpha^*[D] \prec_c (\alpha M + (1-\alpha)\Pi)$. Thus, the lower bound for the transformation (1) for the three measures is -1, and easy computations give us the upper bounds, which proves part (i). A similar argument permits to prove part (ii). Finally, part (iii) follows from the fact that $(C_\alpha[D]-\Pi)+(C_\alpha^*[D]-\Pi) = (D-\Pi)$, then apply it to the expressions in (12) and (13). This completes the proof. □

We finish this section providing a procedure for generating random value (T_1, T_2), from copulas of types (1) and (2). For the first type, consider the following algorithm:

Step 1. Generate a uniform $(0,1)$ random variable S_1;

Step 2. generate independent random pairs (X_1, Y_1) and (X_2, Y_2), from the given copula D and sort them into $X_{(1)} \leq X_{(2)}$ and $Y_{(1)} \leq Y_{(2)}$.

Step 3. If $S_1 > \alpha$, take $(T_1, T_2) = (X_1, Y_1)$;

Step 4. else, generate a uniform $(0,1)$ random variable S_2 and take $(T_1, T_2) = (X_{(1)}, Y_{(2)})$ if $S_2 < 1/2$, otherwise take $(T_1, T_2) = (X_{(2)}, Y_{(1)})$.

A similar algorithm can be used for the second transformation.

3.3. Additional transformations

A known limitation of the FGM family of copulas given by (8) is that the range of Kendall's tau and Spearman's rho are limited to $[-2/9, 2/9]$ and $[-1/3, 1/3]$, respectively; so that it does not allow for modeling of strong dependence [15]. Applying the transformations (1) and (2) to the FGM family of copulas $D_\theta(x,y) = xy[1+\theta(1-x)(1-y)]$ with $\theta \in [-1,1]$, we obtain two new *iterated FGM distributions* of the form

$$C_\alpha\left[D_\theta\right](x,y) = xy[1+\theta(1-x)(1-y)][1-\alpha(1-x)(1-y)(1+\theta xy)], \tag{14}$$

and

$$C^*_\alpha\left[D_\theta\right](x,y) = xy+\alpha xy[1+\theta(1-x)(1-y)]\{(1-x-y+xy[1+\theta(1-x)(1-y)]\}, \tag{15}$$

respectively, where $\theta \in [-1,1]$ and $\alpha \in [0,1]$ – see [9] for other iterated FGM Distributions. After some algebra, we obtain

$$\begin{aligned}
\rho(C_\alpha\left[D_\theta\right]) &= \frac{\theta}{3} - \left(\frac{1}{3}+\frac{\theta}{6}+\frac{\theta^2}{75}\right)\alpha, \\
\rho(C^*_\alpha\left[D_\theta\right]) &= \left(\frac{1}{3}+\frac{\theta}{6}+\frac{\theta^2}{75}\right)\alpha, \\
\tau(C_\alpha\left[D_\theta\right]) &= \frac{2\theta}{9} + \left(\frac{4\theta^2}{11025}+\frac{\theta}{450}+\frac{1}{225}\right)\alpha^2\theta - \left(\frac{\theta^2}{75}+\frac{\theta}{9}+\frac{2}{9}\right)\alpha, \\
\tau(C^*_\alpha\left[D_\theta\right]) &= \left(\frac{4\theta^2}{11025}+\frac{\theta}{450}+\frac{1}{225}\right)\alpha^2\theta + \left(\frac{2\theta^2}{225}+\frac{\theta}{9}+\frac{2}{9}\right)\alpha.
\end{aligned}$$

We have the following ranges: $\rho(C_\alpha\left[D_\theta\right]) \in [-0.51\widehat{3}, 0.\widehat{3}]$, $\rho(C^*_\alpha[D_\theta]) \in [0, 0.51\widehat{3}]$, $\tau(C_\alpha[D_\theta]) \in [-0.3492517, 0.\widehat{2}]$, and $\tau(C^*_\alpha\left[D_\theta\right]) \in [0, 0.3492517]$. So that, for instance, using (14), Spearman's rho can be decreased down to $-0.51\widehat{3}$ at $(\alpha,\theta) = (1,-1)$; and using (15), Spearman's rho can be increased up to $0.51\widehat{3}$ at $\alpha = \theta = 1$.

4. DISCUSSION

In this paper, we have introduced and studied two new transformations of copulas. Other transformations of copulas into copulas can be defined. For example, for a given copula D, we define the transformations

$$H_\beta(x,y)[D] = \frac{D(x,y)}{1-\beta\overline{D}(x,y)} \tag{16}$$

and

$$N_\delta(x,y)[D] = D(x,y)\exp[\delta\overline{D}(x,y)] \tag{17}$$

for every (x,y) in $[0,1]^2$ with $\beta,\delta \in \mathbb{R}$. Observe that

$$H_\beta(x,y)[\Pi] = \frac{xy}{1-\beta(1-x)(1-y)},$$

with $\beta \in [-1,1]$, is the *Ali–Mikhail–Haq* family of copulas [1]; and

$$N_\delta(x,y)[\Pi] = xy\,\exp[\delta(1-x)(1-y)],$$

with $\delta \in [-1,1]$, is a copula studied in [7]. We also note that $N_\delta[M]$ is not a copula.

Observe that the transformations (1), (2), (16), and (17) provide a (single) copula or a one-parametric family of copulas. We can also define transformations of copulas which provide multi-parametric families of copulas. For instance

$$\mathrm{P}_{\alpha,\beta,\theta_1,\theta_2}(x,y)[D] = D(x,y)[1+\alpha\overline{D}(x,y)]^{\theta_1}[1+\beta\overline{D}(x,y)]^{\theta_2}, \tag{18}$$

with $\alpha, \beta, \theta_1, \theta_2$ real numbers. In particular, we have

$$\mathrm{P}_{\alpha,\beta,\theta_1,\theta_2}(x,y)[\Pi] = xy[1+\alpha(1-x)(1-y)]^{\theta_1}[1+\beta(1-x)(1-y)]^{\theta_2},$$

with $-1 \le \alpha, \beta \le 1$ and $|\theta_1| + |\theta_2| = 1$, is a copula studied in [7].

Conditions on the parameters or on the copula D so that the transformations (16)–(18) are copulas is the subject of further work.

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