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A CLASS OF TESTS FOR EXPONENTIALITY BASED ON A CONTINUUM OF MOMENT CONDITIONS

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The empirical moment process is utilized to construct a family of tests for the null hypothesis that a random variable is exponentially distributed. The tests are consistent against the 'new better than used in expectation' (NBUE) class of alternatives. Consistency is shown and the limit null distribution of the test statistic is derived, while efficiency results are also provided. The finite-sample properties of the proposed procedure in comparison to more standard procedures are investigated via simulation.

Keywords: goodness-of-fit test, empirical moments, ageing distributions, Bahadur efficiency

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1. INTRODUCTION

A desirable property of goodness-of-fit statistics, is that of being consistent against all alternatives. Many such so-called omnibus tests for exponentiality exist. See for instance Henze and Meintanis [13] for a review. However, there exist situations that prior information may considerably reduce the spectrum of possible deviations from the null hypothesis. In view of this possibility, several tests for exponentiality have appeared lately that are not omnibus but consistent against a fairly wide class of alternatives. See for instance Epps and Pulley [11], Chaudhuri [6], Klar [14], and Henze and Klar [12]. Most of these classes of alternatives are defined by specifying a certain mode of ageing. Positive ageing for instance, whereby a component wears out with time has traditionally received a lot of attention, but negative ageing, whereby time has a beneficiary effect on the residual life, has also been considered. If the phenomenon of positive or negative ageing persists with time we talk about monotonic ageing. (No ageing of course means that time is irrelevant to the component's residual life.)

One of the most popular ageing properties is captured by the 'new better than used in expectation' (NBUE) class of life distributions. To fix notation, let X be a non-negative random variable with distribution function denoted by F, and finite mean μ. Then F is NBUE if

$$\int_0^\infty (1 - F(x+t))\,\mathrm{d}t \le \mu(1 - F(x)),\ x \ge 0.$$

Many authors have considered moment inequalities for NBUE distributions; see for instance Mugdadi and Ahmad [17], Ahmad [1], and Mitra and Basu [16]. One such inequality is that if $F \in$ NBUE, then

$$D(t) \ge 0,\ t \ge 1, \tag{1}$$

where $D(t) = \mu^t\,\Gamma(t+1) - M(t)$, and $M(t) = \mathrm{E}(X^t)$ is the moment process of F. In the present paper we develop a class of test statistics for testing exponentiality which is consistent within the class of NBUE distributions. Notice that the exponential distribution is a member of the NBUE class, and that in this case,

$$D(t) = 0,\ t \ge 1. \tag{2}$$

In view of (1) and (2), it is reasonable to test

$$H_0 : F \in \mathcal{EXP},$$

where $\mathcal{EXP}$ denotes the class of all exponential distributions, by constructing some empirical version, say $D_n(t)$, of $D(t)$, and reject H_0 in favor of

$$H_1 : F \in \text{NBUE},\ X \notin \mathcal{EXP},$$

for large values of $D_n(t),\ t \ge 1$. On the basis of independent observations $X_1, \ldots, X_n$, on X, the obvious candidate for $D_n(t)$ results by replacing $M(t)$ by $n^{-1}\sum_{j=1}^n X_j^t$, and μ by $\bar{X}_n = n^{-1}\sum_{j=1}^n X_j$, in $D(t)$. The moment process and its empirical counterpart have been recently utilized as tools for statistical inference by Carrasco and Florens [4, 5], Bening and Korolev [3] and Meintanis [15].

In order to obtain a scale-free test for H_0 against H_1 we propose to employ the standardized data $Y_j = X_j/\bar{X}_n,\ j = 1, 2, \ldots, n$, and reject the null hypothesis for large values of

$$T_{n,w} = \sqrt{n}\int_1^\infty D_n(t)w(t)\,\mathrm{d}t, \tag{3}$$

where $D_n(t) = \Gamma(t+1) - M_n(t),\ M_n(t) = n^{-1}\sum_{j=1}^n Y_j^t$, is the empirical moment of order t of $Y_j,\ j = 1, 2, \ldots, n$, and $w(t)$ denotes a non-negative weight function. Notice that in the proposed test, all moments of order $t \ge 1$ are taken into consideration. This is referred to in the econometric literature as 'a continuum of moment conditions'; see Carrasco and Florens [4, 5].

2. ASYMPTOTIC PROPERTIES OF THE TEST STATISTIC

2.1. Consistency and limit distribution

Assume that the law of X belongs to the NBUE class and has finite mean μ, and write $M(t)$ for the moment process of X. In this section the behavior of $T_{n,w}$ is studied within this class of distributions. We begin with the following lemma. The proof is given in the Appendix.

Lemma 2.1. Assume that the weight function $w(\cdot)$ is such that $I_w(x) := \int_1^\infty x^t w(t)\,\mathrm{d}t$ can be differentiated w.r.t. x under the integral sign and that

$$\int_1^\infty t^2\Gamma(t)w(t)\,\mathrm{d}t < \infty. \tag{4}$$

Then

$$\lim_{\delta\to 0} \mathrm{E}\left[\sup_{h:|h|\le\delta} |XI'_w((X/\mu)+h)) - XI'_w(X/\mu)|\right] = 0,$$

where $I'_w(u)$ denotes the first derivative of $I_w(x)$ computed at $x = u$.

In the following theorem the consistency of the test that rejects the null hypothesis for large values of $T_{n,w}$ is shown.

Theorem 2.2. Let $X_1, X_2, \dots, X_n$ be independent copies of the random variable X. Then

$$\frac{T_{n,w}}{\sqrt{n}} \xrightarrow{P} \int_1^\infty \left(\Gamma(t+1) - \frac{M(t)}{\mu^t}\right) w(t)\,\mathrm{d}t := \Delta.$$

In particular $\Delta > 0$, under the alternative hypothesis H_1, which implies that the test is consistent against (non-exponential) NBUE alternatives.

Proof. Observe that,

$$\frac{T_{n,w}}{\sqrt{n}} = \mathcal{E}_w - \frac{1}{n}\sum_{j=1}^n I_w(Y_j),$$

where $\mathcal{E}_w = \int_1^\infty \Gamma(t+1)w(t)\,\mathrm{d}t$. Taking a linear Taylor expansion of $I_w(u)$ at $u = Y_j$ around $u_0 = X_j/\mu$ we have

$$\frac{T_{n,w}}{\sqrt{n}} = \frac{1}{n}\sum_{j=1}^n \left\{\mathcal{E}_w - I_w\left(\frac{X_j}{\mu}\right)\right\} - \left(\frac{1}{\bar{X}_n} - \frac{1}{\mu}\right)\frac{1}{n}\sum_{j=1}^n X_j I'_w\left(\frac{X_j}{\mu^*}\right), \tag{5}$$

where μ^* is such that $|\mu^* - \mu| \le |\bar{X}_n - \mu|$.

Now let

$$E_n = \frac{1}{n}\sum_{j=1}^n X_j I'_w\left(\frac{X_j}{\mu^*}\right) - \frac{1}{n}\sum_{j=1}^n X_j I'_w\left(\frac{X_j}{\mu}\right),$$

and choose $d > 0$ such that

$$\frac{X_j}{\mu^*} \in \left(\frac{X_j}{\mu} - d, \frac{X_j}{\mu} + d\right),\ j = 1, \dots, n.$$

Also define

$$\Delta(x,d) = \sup_{\{s:|s|\le d\}} \left| x I_w'\left(\frac{x}{\mu}+s\right) - x I_w'\left(\frac{x}{\mu}\right)\right|.$$

Note that from Lemma 2.1 we have $\lim_{d\to 0} \mathrm{E}[\Delta(X,d)] = 0$ and notice that

$$|E_n| \le \frac{1}{n}\sum_{j=1}^{n} \Delta(X_j,d) \xrightarrow{P} \mathrm{E}[\Delta(X,d)],$$

by the Law of Large Numbers. This shows that $E_n = o_P(1)$; for more details the reader is referred to Prakasa Rao [19], Proposition 3.5.6. Hence the last sum in equation (5) may be approximated by $n^{-1}\sum_{j=1}^{n} X_j I_w'(X_j/\mu)$ and finally by $\mathrm{E}[X I_w'(X/\mu)]$ (which is finite by assumption (4)), so that

$$\frac{T_{n,w}}{\sqrt{n}} \approx \frac{1}{n}\sum_{j=1}^{n}\left\{\mathcal{E}_w - I_w\left(\frac{X_j}{\mu}\right)\right\} - \left(\frac{1}{\bar{X}_n} - \frac{1}{\mu}\right)\mathrm{E}[X I_w'(X/\mu)], \tag{6}$$

where $\approx$ indicates two random variables sharing the same probability limit. The last term however in (6) is a product of a $o_P(1)$ and a $O_P(1)$ term, and therefore the Law of Large Numbers implies that,

$$\frac{T_{n,w}}{\sqrt{n}} \xrightarrow{P} \mathcal{E}_w - \mathrm{E}\left[I_w\left(\frac{X}{\mu}\right)\right], \tag{7}$$

which coincides with Δ in the statement of the theorem. In view of (1) and (2), Δ is zero under the null hypothesis and positive under H_1, which finishes the proof of the theorem. □

To obtain the asymptotic null distribution of $T_{n,w}$ write $\approx$ for random variables sharing the same limit law, and notice that if X is standard exponential then from expansion (6) one has

$$T_{n,w} \approx \frac{1}{\sqrt{n}}\sum_{j=1}^{n} V_w(X_j), \tag{8}$$

where $V_w(x) = \mathcal{E}_w - I_w(x) - (1-x)\mathrm{E}[X I_w'(X)]$. Consequently an application of the Central Limit Theorem yields that under H_0,

$$T_{n,w} \xrightarrow{D} N(0,\sigma_w^2), \tag{9}$$

where

$$\sigma_w^2 = \mathrm{E}[I_w^2(X)] + \left(\mathrm{E}[X I_w'(X)]\right)^2 + 2\mathrm{E}[X I_w'(X)]\left(\mathcal{E}_w - \mathrm{E}[X I_w(X)]\right) - \mathcal{E}_w^2$$

by straightforward computation.

2.2. Computation of asymptotic efficiency

Let $T_{n,a}$ denote the test statistic with $w(t) = e^{-at^2}$. Note that with this weight function, the assumptions imposed in Section 2.1 are satisfied. The efficiency of the sequence $\{T_{n,a}\}$ is addressed by using the simplest notion of approximate Bahadur efficiency. This type of efficiency is often used for the asymptotic comparison of test statistics when the limiting distribution is known. In contrast however to the exact Bahadur efficiency, which is bounded, there is no upper bound for approximate slopes. Nevertheless, it is well-known that for asymptotically normal statistics this type of efficiency usually coincides with the Pitman efficiency, and that the computation of approximate Bahadur efficiency is easier and requires less assumptions on the alternative. For details on the notion of test-efficiency the reader is referred to Bahadur [2], Nikitin [18] and Rublík [20, 21].

We are testing the null hypothesis H_0 according to which X follows an exponential law with unknown mean which can be assumed without loss of generality to be equal to unity. Consider an alternative with a distribution function F depending on a parameter θ, such that $\theta = 0$ corresponds to the null hypothesis. According to the Law of Large Numbers we have from (8) under the alternative F,

$$n^{-1/2}\frac{T_{n,a}}{\sigma_a} \xrightarrow{P} \frac{b_F(\theta)}{\sigma_a},$$

where in $b_F(\theta) = \mathrm{E}_F[V_a(X)]$, the expectation is taken with respect to F, and where $V_a(\cdot)$ and σ_a result from $V_w(\cdot)$ and σ_w, respectively, with $w(t)$ replaced by e^{-at^2}. It follows that $\{T_{n,a}\}$ is a so-called standard sequence of statistics (see Bahadur [2]) with the approximate slope

$$c_F(\theta) = \frac{b_F^2(\theta)}{\sigma_a^2}.$$

As only close hypotheses are relevant, we can look for the asymptotics of $c_F(\theta)$ as $\theta \to 0$. However, there is no upper bound for the approximate slopes, and therefore in most cases they are compared with the approximate Bahadur slopes of likelihood ratio tests. These slopes coincide under very general conditions with the approximate slopes $2Q(F,\theta)$, $\theta \to 0$, where $Q(F,\theta)$ is the corresponding Kullback–Leibler 'distance' between the alternative distribution and the set of all exponential distributions. If the quantity $Q(F,\theta)$ can be calculated (as $\theta \to 0$) for a concrete distribution F, then the local relative approximate Bahadur efficiency e_F can be calculated according to the formula

$$e_F = \lim_{\theta\to 0} \frac{c_F(\theta)}{2Q(F,\theta)}. \tag{10}$$

In the following result e_F is calculated for Gamma, Weibull and Linear Failure rate, alternatives.

Theorem 2.3. The relative approximate Bahadur efficiency of the sequence $\{T_{n,a}\}$ for Gamma alternatives is given by

$$\text{(i)} \quad e_{\mathcal{G}} = \frac{6\left(\Lambda_a^{(1)} - \Lambda_a^{(2)} - \gamma\mathcal{E}_a\right)^2}{(\pi^2-6)\sigma_a^2},$$

for Weibull alternatives by

$$\text{(ii)} \quad e_{\mathcal{W}} = \frac{6\left[(\gamma-1)\Lambda_a^{(1)} + \Lambda_a^{(3)} - \Lambda_a^{(2)} - \mathcal{E}_a\right]^2}{\pi^2\sigma_a^2},$$

and for Linear Failure rate alternatives by

$$\text{(iii)} \quad e_{\mathcal{L}} = \frac{\left(\Lambda_a^{(4)} - \Lambda_a^{(1)}\right)^2}{\sigma_a^2},$$

where

$$\begin{aligned}
\Lambda_a^{(1)} &= \int_1^\infty t\Gamma(1+t)e^{-at^2}\,\mathrm{d}t,\\
\Lambda_a^{(2)} &= \int_1^\infty \Gamma'(1+t)e^{-at^2}\,\mathrm{d}t,\\
\Lambda_a^{(3)} &= \int_1^\infty \Gamma'(2+t)e^{-at^2}\,\mathrm{d}t,\\
\Lambda_a^{(4)} &= \int_1^\infty \left[\frac{1}{2}\Gamma(3+t) - \Gamma(2+t)\right]e^{-at^2}\,\mathrm{d}t,
\end{aligned}$$

and $\gamma = 0.57721\ldots$, is Euler's constant.

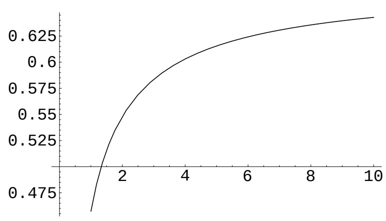

Fig. 1. Approximate Bahadur efficiency of $T_{n,a},\ a \in (1,10)$, for Gamma alternatives.

P r o o f. (i) For a Gamma alternative with density $f(x) = (x^\theta/\Gamma(1+\theta))e^{-x},\ \theta > -1$, we have

$$Q(F,\theta) = \theta^2\left(\frac{\pi^2}{6} - 1\right),\ \theta \to 0. \tag{11}$$

We use the approximation $f(x) \approx 1 + \theta \log(x),\ \theta \to 0$. Then by straightforward manipulation we have

$$b_{\mathcal{G}}(\theta) = \mathcal{E}_a \left(1 - \frac{1}{\Gamma(1+\theta)}\right) + \theta \left(\Lambda_a^{(1)} - \Lambda_a^{(2)}\right) + o(\theta), \tag{12}$$

as $\theta \to 0$. Use in (12) the approximation $\Gamma(1+\theta) \approx (1+\gamma\theta)^{-1}$. Then (i) follows from (11), (12), and by taking the limit in (10) as $\theta \to 0$.

(ii) For a Weibull alternative with density $f(x) = (\theta+1)x^{\theta} e^{-x^{\theta+1}},\ \theta > -1$, we have

$$Q(F,\theta) = \frac{\theta^2\pi^2}{6},\ \theta \to 0. \tag{13}$$

We use the approximation $f(x) \approx e^{-x}(1+\theta(1-x)\log(x)),\ \theta \to 0$ Then by straightforward manipulation we have

$$b_{\mathcal{W}}(\theta) = \left(\frac{\Gamma(1/(1+\theta))}{1+\theta} - 1\right) \Lambda_a^{(1)} - \theta \left(\mathcal{E}_a + \Lambda_a^{(2)} - \Lambda_a^{(3)}\right) + o(\theta), \tag{14}$$

as $\theta \to 0$. Use in (14) the approximation $\Gamma(1/(1+\theta)) \approx 1+\gamma\theta,\ \theta \to 0$. Then (ii) follows from (13), (14), and by taking the limit in (10) as $\theta \to 0$.

(iii) For a Linear Failure rate alternative with density $f(x) = (1+\theta x)e^{-x-(1/2)\theta x^2}$, $\theta > 0$, we have

$$Q(F,\theta) = \theta^2,\ \theta \to 0. \tag{15}$$

We use the approximation $f(x) \approx e^{-x}(1+\theta(x-(x^2/2))),\ \theta \to 0$ Then by straightforward manipulation we have

$$b_{\mathcal{L}}(\theta) = \theta \left(\Lambda_a^{(4)} - \Lambda_a^{(1)}\right) + o(\theta), \tag{16}$$

as $\theta \to 0$. Then (iii) follows from (15), (16), and by taking the limit in (10) as $\theta \to 0$. □

In Figure 1, the approximate Bahadur efficiency of $\{T_{n,a}\}$ is depicted as a function of $a,\ a \in (1,10)$, for Gamma alternatives. Corresponding graphs for Weibull and Linear Failure rate alternatives are shown in Figures 2 and 3, respectively. The figures suggest that a satisfactory value of efficiency may be achieved for values of a away from the origin.

3. SPECIFICATION OF THE TEST AND LIMIT STATISTIC

Although, consistency and asymptotic null distribution of $T_{n,w}$ remain qualitatively invariant with respect to the choice of the weight function, particular appeal lies with those functions which render the test statistic in a nice formula suitable for

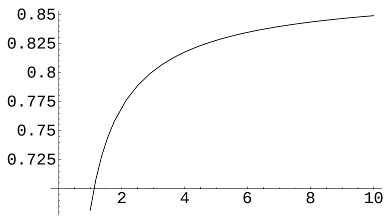

Fig. 2. Approximate Bahadur efficiency of $T_{n,a},\ a \in (1, 10)$, for Weibull alternatives.

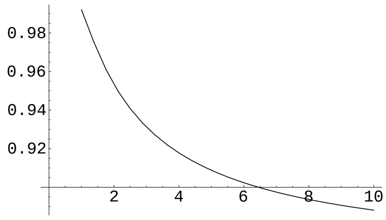

Fig. 3. Approximate Bahadur efficiency of $T_{n,a},\ a \in (1, 10)$, for Linear Failure rate alternatives.

computer implementation. The choice, $w(t) = \exp(-at^2),\ a > 0$, for example, leads to the test statistic,

$$T_{n,a} = \sqrt{n}\,\mathcal{E}_a - \frac{1}{\sqrt{n}} \sum_{j=1}^{n} I_a(Y_j),$$

where

$$\mathcal{E}_a = \int_1^\infty \Gamma(t+1) \exp(-at^2)\,\mathrm{d}t,$$

and

$$I_a(x) = \frac{1}{2}\sqrt{\frac{\pi}{a}} e^{\frac{\log^2 x}{4a}} \left[1 - \mathrm{Erf}\left(\sqrt{a} - \frac{\log x}{2\sqrt{a}}\right)\right],$$

with $\mathrm{Erf}(z) = (2/\sqrt{\pi}) \int_0^z e^{-t^2}\,\mathrm{d}t$ being the error function. Other weight functions could in principle be used, but it seems that their employment will lead to a non-analytic expression for the resulting test statistic. An additional interesting feature is that the class $\{T_{n,a}, a > 0\}$, when properly scaled, is closed at the boundary $a = \infty$. To see this replace $w(t)$ by $\exp(-at^2)$ in (3), and make the transformation $t \mapsto t^2 - 1$. Then the test statistic may be written as

$$T_{n,a} = \frac{1}{2} e^{-a} \int_0^\infty g(t) e^{-at}\,\mathrm{d}t,$$

Table 1. Asymptotic standard deviation of the test statistic.

$a \to$	1.0	1.5	2.0	3.0	5.0
σ_a	0.055991	0.0138322	0.00457451	0.733811×10^{-3}	0.354947×10^{-4}

where $g(t) = \sqrt{n} D_n(\sqrt{1+t}\,)(1+t)^{-1/2}$. By straightforward algebra we have

$$g(t) = \sqrt{n}\left(\Gamma(\sqrt{1+t}) - \frac{1}{\sqrt{1+t}} - \frac{t}{2\sqrt{1+t}}\frac{1}{n}\sum_{j=1}^{n} Y_j \log Y_j\right) + o(t), \qquad (17)$$

as $t \to 0$. Taking the limit in (17) yields, $\lim_{t\to 0} g(t)/t = (1/2)\sqrt{n}((1-\gamma) - n^{-1}\sum_{j=1}^{n} Y_j \log Y_j)$. Then an Abelian theorem for Laplace transforms (see Zayed, [22, § 5.11] yields,

$$\lim_{a\to\infty} 4a^2 e^a T_{n,a} = \sqrt{n}\left((1-\gamma) - n^{-1}\sum_{j=1}^{n} Y_j \log Y_j\right) := T_{n,\infty}.$$

Hence the test statistic when properly standardized and apart from irrelevant scaling factors, possesses a limit value $T_{n,\infty}$, as $a \to \infty$. This limit statistic measures the normalized ($\times\sqrt{n}$) distance between the sample mean of $Y_j \log Y_j$, and $1-\gamma$, and it is interesting to notice that as $n \to \infty$, this distance vanishes under the null hypothesis.

4. SIMULATIONS

This section presents the results of a Monte Carlo study conducted to assess the finite-sample behavior of the new test in comparison with the Kolmogorov–Smirnov (KS) and the Cramér–von Mises (CM) test that utilize the empirical distribution function (EDF). Write $Y_{(1)} \le Y_{(2)} \le \ldots \le Y_{(n)}$ for the order statistics of $Y_1, Y_2, \ldots, Y_n$. Then the KS-statistic is

$$\text{KS} \;=\; \max\{D^+, D^-\},$$

with

$$D^+ = \max_{j=1,2,\ldots,n} \left(j/n - F(Y_{(j)})\right) \text{ and } D^- = \max_{j=1,2,\ldots,n} \left(F(Y_{(j)}) - (j-1)/n\right),$$

while the Cramér–von Mises statistic is given by

$$\text{CM} \;=\; \frac{1}{12n} + \sum_{j=1}^{n}\left(F(Y_{(j)}) - \frac{2j-1}{2n}\right)^2,$$

where $F(x) = 1 - e^{-x}$.

The EDF-tests possess non-standard asymptotic behavior. Therefore they are typically implemented by using approximate critical values. (See for instance D'Agostino and Stephens [8] or Doksum and Yandell [10]). These values for the KS test are,

$$p_\alpha = \frac{0.2}{n} + \frac{d_\alpha}{\sqrt{n} + 0.26 + (0.5/\sqrt{n})},$$

with $d_{0.05} = 1.094$ and $d_{0.10} = 0.990$. The corresponding points for the CM test are,

$$p_\alpha = \frac{d_\alpha}{1 + (0.16/n)},$$

with $d_{0.05} = 0.222$ and $d_{0.10} = 0.175$.

On the other hand the new test was implemented as an asymptotic test by using figures for the asymptotic standard deviation $\sigma_a,\ a = 1, 1.5, 2, 3, 5$, given in Table 1. Then according to (9), the $(1-\alpha) \times 100\%$ large-sample rejection region is $T_{n,a}/\sigma_a > z_\alpha$, where z_α denotes the $(1-\alpha)\times 100\%$ quantile of the standard normal distribution. For simplicity we write T_a for the new test statistic.

The distributions considered are:

- the Gamma with density $[\Gamma(1+\theta)]^{-1} x^\theta e^{-x}$, denoted by $\mathcal{G}(\theta)$,
- the Weibull with density $(1+\theta)x^\theta e^{-x^{\theta+1}}$, denoted by $\mathcal{W}(\theta)$,
- the Log-Normal with density $(x\theta\sqrt{2\pi})^{-1} e^{-\log^2 x/(2\theta^2)}$, denoted by $\mathcal{LN}(\theta)$,
- the Inverse Gaussian with density $(\theta/2\pi)^{1/2} x^{-3/2} \exp[-\theta(x-1)^2/2x]$, denoted by $\mathcal{IG}(\theta)$,
- the Half-Normal with density $(2/\pi)^{1/2} \exp(-x^2/2)$, denoted by $\mathcal{HN}$,
- the Linear Failure rate with density $(1+\theta x)\exp(-x - \theta x^2/2)$, denoted by $\mathcal{LF}(\theta)$
- the Generalized Exponential with distribution function $(1-e^{-x})^\vartheta$, denoted by $\mathcal{GE}(\theta)$
- Dhillon's [9] model with distribution function $1-\exp(-\log^{\theta+1}(x+1))$, denoted by $\mathcal{DL}(\theta)$.

The Gamma, the Weibull, and the Linear Failure rate, with $\theta > 0$, as well as the Half-Normal are NBUE, but we have also included in the simulation other families which are not in the NBUE-class.

Each figure mentioned below corresponds to 10,000 replications and it is based on the aforementioned critical points. Table 2 reports rejection rates for the new test statistic (percentage of rejection rounded to the nearest integer), with sample size $n = 50$ and $n = 100$. Corresponding results for the EDF tests are also shown.

The results of Table 2 indicate that the proposed procedure captures the nominal size to a satisfactory degree, being a bit conservative though at nominal level 5 %.

Table 2. Observed percentage of rejection at nominal level 5 % (left entry) and 10 % (right entry), with sample size $n = 50$ (upper entry) and $n = 100$ (lower entry).

↓Model Test →	$T_{1.0}$	$T_{1.5}$	$T_{2.0}$	$T_{3.0}$	$T_{5.0}$	KS	CM
$\mathcal{G}(0.0)$	3 8	3 9	4 10	4 10	4 10	5 10	5 10
	3 9	4 10	4 10	4 10	4 10	5 10	5 10
$\mathcal{G}(0.5)$	36 61	44 66	47 69	50 71	53 73	36 50	44 57
	68 84	74 87	77 89	79 90	81 91	66 79	75 85
$\mathcal{G}(0.75)$	62 82	70 86	73 88	76 89	78 91	63 76	72 82
	92 97	95 98	96 99	97 99	98 99	92 97	96 99
$\mathcal{W}(0.25)$	37 63	44 67	47 69	50 71	52 72	32 46	39 53
	73 88	78 90	80 91	82 92	83 93	59 72	70 81
$\mathcal{W}(0.40)$	73 91	80 93	82 94	84 94	86 95	63 76	74 84
	98 100	99 100	99 100	99 100	99 100	93 97	97 99
$\mathcal{LN}(0.80)$	28 44	35 50	38 54	42 57	46 61	70 84	76 87
	40 53	48 60	52 63	57 68	61 72	97 99	98 100
$\mathcal{LN}(0.70)$	63 78	71 83	75 85	78 87	81 89	96 99	97 99
	82 89	88 93	91 94	93 96	94 97	100 100	100 100
$\mathcal{DL}(1.0)$	27 47	33 52	37 55	40 58	44 60	45 59	52 66
	45 60	52 66	56 69	60 72	63 75	79 89	85 92
$\mathcal{DL}(1.2)$	56 76	64 80	68 82	72 85	75 86	73 84	80 90
	82 90	87 93	89 95	92 96	93 97	97 99	99 100
$\mathcal{HN}$	54 80	60 82	62 82	64 82	65 83	38 53	47 62
	92 98	93 98	93 98	93 98	93 98	68 81	80 89
$\mathcal{IG}(1.0)$	11 23	15 28	18 31	20 34	23 37	53 71	55 73
	15 27	20 33	24 37	28 41	32 45	94 98	93 98
$\mathcal{IG}(1.5)$	51 70	61 76	65 79	70 82	74 85	95 98	95 98
	76 86	83 91	87 93	90 95	92 96	100 100	100 100
$\mathcal{LF}(1.0)$	45 71	51 74	53 75	55 75	56 76	32 46	40 54
	84 95	86 95	87 95	87 95	87 95	59 73	71 82
$\mathcal{LF}(0.8)$	37 64	43 67	45 68	47 68	48 69	27 40	33 47
	76 91	79 91	80 91	80 91	80 91	50 64	61 74
$\mathcal{GE}(1.5)$	29 53	36 58	39 60	42 62	46 64	31 44	37 51
	57 76	64 80	67 82	70 84	74 85	58 72	68 79
$\mathcal{GE}(2.0)$	68 86	76 89	79 91	82 92	84 93	73 84	82 90
	94 98	96 99	97 99	98 99	99 100	97 99	99 100

Moreover, the new test has good power characteristics. Specifically it is more powerful than the EDF tests for Gamma, Weibull, Half-Normal, $\mathcal{LF}$, and $\mathcal{GE}$ alternatives. On the other hand for samples from the Log-Normal, the Inverse Gaussian and Dhillon's family, the classical tests prevail. Hence we suggest the new test T_a with a away from the origin, say T_3 or T_5, as a procedure which performs competitively when compared to more established omnibus tests for the exponential distribution, having the extra advantage of a simple limit distribution.

APPENDIX

Proof of Lemma 2.1. By definition and under the first assumption we have for sufficiently small h,

$$\begin{aligned}
xI_w'\left((x/\mu)+h\right) &= x\left(\int_1^\infty u^t w(t)\,\mathrm{d}t\right)'_{u=(x/\mu)+h} \\
&= x\int_1^\infty t\left(\frac{x}{\mu}+h\right)^{t-1} w(t)\,\mathrm{d}t \\
&= x\int_1^\infty t\left(\frac{x}{\mu}\right)^{t-1}\left(1+\frac{h\mu}{x}\right)^{t-1} w(t)\,\mathrm{d}t \\
&\approx x\int_1^\infty t\left(\frac{x}{\mu}\right)^{t-1}\left(1+(t-1)\frac{h\mu}{x}\right) w(t)\,\mathrm{d}t \\
&= \int_1^\infty \frac{t}{\mu^{t-1}}x^t w(t)\,\mathrm{d}t + h\mu\int_1^\infty t(t-1)\left(\frac{x}{\mu}\right)^{t-1} w(t)\,\mathrm{d}t,\ h\to 0,
\end{aligned}$$

where we have used the approximation $(1+u)^t \approx 1+tu,\ u\to 0$. Since the first integral in the last equation coincides with $xI_w'(x/\mu)$ we have

$$\sup_{h:|h|\le\delta}|XI_w'((X/\mu)+h)) - XI_w'(X/\mu)| \le \delta\mu\int_1^\infty t(t-1)\left(\frac{x}{\mu}\right)^{t-1} w(t)\,\mathrm{d}t,$$

and hence Fubini's theorem yields

$$\begin{aligned}
&\mathrm{E}\left[\sup_{h:|h|\le\delta}|XI_w'((X/\mu)+h)) - XI_w'(X/\mu)|\right] \\
\le\ & \delta\mu\int_1^\infty \frac{t(t-1)}{\mu^{t-1}}\int_0^\infty x^{t-1}\,\mathrm{d}F(x)w(t)\,\mathrm{d}t \\
=\ & \delta\mu\int_1^\infty \frac{t(t-1)}{\mu^{t-1}}M(t-1)w(t)\,\mathrm{d}t \le \delta\mu\int_1^\infty t(t-1)\Gamma(t)w(t)\,\mathrm{d}t,
\end{aligned}$$

since $X\in$ NBUE. The last term however goes to zero as $\delta\to 0$, because of (4) and the proof is complete. □

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