KYBERNETIKA — VOLUME *45* (2009), NUMBER 6, PAGES 931 – 945

ON ESTIMATION OF INTRINSIC VOLUME DENSITIES OF STATIONARY RANDOM CLOSED SETS VIA PARALLEL SETS IN THE PLANE

Tomáš Mrkvička and Jan Rataj

A method of estimation of intrinsic volume densities for stationary random closed sets in $\mathbb{R}^d$ based on estimating volumes of tiny collars has been introduced in T. Mrkvička and J. Rataj, On estimation of intrinsic volume densities of stationary random closed sets, Stoch. Proc. Appl. *118* (2008), 2, 213–231. In this note, a stronger asymptotic consistency is proved in dimension 2. The implementation of the method is discussed in detail. An important step is the determination of dilation radii in the discrete approximation, which differs from the standard techniques used for measuring parallel sets in image analysis. A method of reducing the bias is proposed and tested on simulated data.

Keywords: random closed set, convex ring, curvature measure, intrinsic volume

AMS Subject Classification: 60D05, 62G05

1. INTRODUCTION

Let $\Xi \subseteq \mathbb{R}^d$ be a random closed set that takes values in the extended convex ring (i. e., Ξ is a locally finite union of convex bodies). For $k = 0, 1, \dots, d$, the kth curvature measure $C_k(\Xi;\cdot)$ is a random signed Radon measure (see [8]); hence, $C_k(\Xi; B)$ is defined and finite for any bounded Borel set $B \subseteq \mathbb{R}^d$.

We shall assume throughout the paper that Ξ is stationary (i. e., its distribution is invariant w.r.t. all shifts). Let $C^d = [0, 1]^d$ denote the unit d-cube. Under the condition

$$\mathrm{E}|C_k|(\Xi; C^d) < \infty, \tag{1}$$

where $|C_k|$ is the total variation measure of C_k, we can introduce the *intrinsic volume densities* of Ξ as

$$\overline{V}_k(\Xi) = \mathrm{E}C_k(\Xi; C^d), \quad k = 0, \dots, d,$$

cf. [2]. Note that we have by stationarity

$$\overline{V}_k(\Xi) = \frac{\mathrm{E}C_k(\Xi; B)}{\lambda^d(B)}$$

for any bounded Borel set B with positive Lebesgue measure $\lambda^d(B)$.

The following estimation procedure of the vector

$$\overline{V}(\Xi) = (\overline{V}_{d-1}(\Xi), \dots, \overline{V}_0(\Xi))^\mathsf{T}$$

was suggested in [2]. Consider for $\varepsilon > 0$ the ε-parallel set

$$\Xi_\varepsilon = \{z \in \mathbb{R}^d : \operatorname{dist}(z, \Xi) \le \varepsilon\}$$

and the closure to its complement

$$\Xi^*_\varepsilon = \overline{\mathbb{R}^d \setminus \Xi_\varepsilon}.$$

Since Ξ^*_ε has locally positive reach for any $\varepsilon > 0$ by [6, Theorem 2], its curvature measures are defined and the local Steiner formula holds for any bounded Borel set $B \subseteq \mathbb{R}^d$ and $\delta > 0$ small enough:

$$\lambda^d \left(((\Xi^*_\varepsilon)_\delta \setminus \Xi^*_\varepsilon) \cap \Pi^{-1}_{\Xi^*_\varepsilon}(B) \right) = \sum_{i=1}^{d} \delta^i \omega_i C_{d-i}(\Xi^*_\varepsilon; B); \tag{2}$$

here $\omega_i = \pi^{i/2}/\Gamma(1+i/2)$ is the volume of the unit i-ball and the mapping $\Pi_{\Xi^*_\varepsilon}$ assigns to a point z its unique nearest neighbour in Ξ^*_ε ($\Pi_{\Xi^*_\varepsilon}$ is defined almost everywhere in $\mathbb{R}^d$ and everywhere on a neighbourhood of $\Xi^*_\varepsilon \cap B$, see [2] for more details).

We choose as measured data the volume fractions

$$v(\Xi; \varepsilon, \delta, B) = \lambda^d \left(((\Xi^*_\varepsilon)_\delta \setminus \Xi^*_\varepsilon) \cap \Pi^{-1}_{\Xi^*_\varepsilon}(B) \right) / \lambda^d(B)$$

for $\delta > 0$. In particular, we choose for a fixed $\varepsilon > 0$ a set

$$\Delta = \{0 < \delta_1 < \dots < \delta_n < \varepsilon\}$$

of $n \ge d$ distances and estimate the vector of volume fractions

$$V(\Xi; \varepsilon, \Delta, B) = (v(\Xi; \varepsilon, \delta_1, B), \dots, v(\Xi; \varepsilon, \delta_n, B))^\mathsf{T}.$$

The local Steiner formula (2) with the values $\delta_1, \dots, \delta_n$ can be considered as a linear regression model with coefficient matrix

$$M_\Delta = \begin{pmatrix} \delta_1\omega_1 & , & \delta_2\omega_1 & , & \dots & , & \delta_n\omega_1 \\ \delta_1^2\omega_2 & , & \delta_2^2\omega_2 & , & \dots & , & \delta_n^2\omega_2 \\ \vdots & & \vdots & & & & \vdots \\ \delta_1^d\omega_d & , & \delta_2^d\omega_d & , & \dots & , & \delta_n^d\omega_d \end{pmatrix}.$$

Using the standard least-squares method, we find that

$$(M_\Delta M_\Delta^\mathsf{T})^{-1} M_\Delta V(\Xi; \varepsilon, \Delta, B)$$

agrees with

$$(\lambda^d(B))^{-1} (V_{d-1}(\Xi^*_\varepsilon; B), \dots, V_0(\Xi^*_\varepsilon; B))^\mathsf{T}$$

if $\max\Delta$ is small enough (more precisely, if the local reach of Ξ^*_ε is greater than $\max\Delta$ on B, see [2, p. 216]). As estimator of $\overline{V}(\Xi)$, we consider the vector

$$\widehat{V}(\Xi;\varepsilon,\Delta,B)=\Sigma_d(M_\Delta M_\Delta^\top)^{-1}M_\Delta V(\Xi;\varepsilon,\Delta,B),$$

where Σ_d is the diagonal $d\times d$ matrix with diagonal elements $\Sigma_d(i,i)=(-1)^{i-1}$, $i=1,\dots,d$. The estimation procedure is justified by the vague convergence result (see [6, Theorem 2])

$$\lim_{\varepsilon\to 0}(-1)^{d-1-k}C_k(\Xi^*_\varepsilon;\cdot)=C_k(\Xi;\cdot).$$

The following consistency results have been shown in [2]. Given a polyconvex set $Z\subseteq\mathbb{R}^d$, let $N(Z)$ denote the minimum number N of convex bodies $K_1,\dots,K_N$ such that $Z=K_1\cup\dots\cup K_N$.

Theorem 1.1. Let Ξ be a stationary random closed set in $\mathbb{R}^d$ with values in the extended convex ring and assume that

$$\mathrm{E}2^{N(\Xi\cap C^d)}<\infty. \tag{3}$$

Then

$$\lim_{\varepsilon\to 0}\Sigma_d\overline{V}(\Xi^*_\varepsilon)=\overline{V}(\Xi). \tag{4}$$

Let further B be a bounded Borel set in $\mathbb{R}^d$ with positive Lebesgue measure and such that $\lambda^d(\partial B)=0$. Then

$$\lim_{\varepsilon\to 0}\lim_{\max\Delta\to 0}\mathrm{E}\widehat{V}(\Xi;\varepsilon,\Delta,B)=\overline{V}(\Xi). \tag{5}$$

If Ξ is, furthermore, ergodic and (B_j) is a sequence of convex bodies with inradii growing to infinity then

$$\lim_{\varepsilon\to 0}\lim_{j\to\infty}\lim_{\max\Delta\to 0}\widehat{V}(\Xi;\varepsilon,\Delta,B_j)=\overline{V}(\Xi)\quad \text{a.s. and in } L^1. \tag{6}$$

Note that in order to control the estimation error, it might be necessary to decrease the maximum radius $\max\Delta$ in dependence not only on $\varepsilon>0$, but also on the observation window B. This is not satisfactory for practical use since we usually plan the experiment (i. e., choose Δ in our case) and wish to get convergence with expanding window. The main task of this note is to prove a stronger consistency result overcoming this difficulty at least in $\mathbb{R}^2$.

Another slightly simpler estimator of $\overline{V}(\Xi)$ was introduced in [2], namely,

$$\widehat{W}(\Xi;\varepsilon,\Delta,B)=\Sigma_d(M_\Delta M_\Delta^\top)^{-1}M_\Delta W(\Xi;\varepsilon,\Delta,B),$$

where

$$W(\Xi;\varepsilon,\Delta,B)=(w(\Xi;\varepsilon,\delta_1,B),\dots,w(\Xi;\varepsilon,\delta_n,B))^\top$$

and

$$w(\Xi;\varepsilon,\delta,B)=\lambda^d\left(((\Xi^*_\varepsilon)_\delta\setminus\Xi^*_\varepsilon)\cap B\right)/\lambda^d(B),\quad \delta>0.$$

The asymptotic equivalence of $\widehat{V}$ and $\widehat{W}$ in $\mathbb{R}^2$ was shown in [2, Theorem 3].

2. ASYMPTOTIC CONSISTENCY IN THE PLANAR CASE

Let

$$X = \bigcup_i X^i \tag{7}$$

be a (deterministic) set from the extended convex ring in $\mathbb{R}^2$ with convex bodies X^i, $i \in \mathbb{N}$. We shall find an upper bound for the bias

$$\left| \widehat{V}(X;\varepsilon,\Delta,B) - \frac{C(X_\varepsilon^*;B)}{\lambda^2(B)} \right|,$$

where $C(X_\varepsilon^*;B)$ is the vector $(C_1(X_\varepsilon^*;B), C_0(X_\varepsilon^*;B))^\mathsf{T}$.

This will be expressed by means of the following quantity:

$$\mu_\varepsilon(X;A) = \sum_{i<j:\, X^i \cap X^j = \emptyset} \operatorname{card}\,(\partial X_\varepsilon^i \cap \partial X_\varepsilon^j \cap \partial(X_\varepsilon^i \cup X_\varepsilon^j) \cap A).$$

Note that $\mu_\varepsilon(X;\cdot)$ is a counting measure over certain intersection points of boundaries of the convex bodies X_ε^i. Of course, $\mu_\varepsilon(X;\cdot)$ depends not only on X but on the particular representation (7).

Proposition 2.1. Let X be a set from the extended convex ring in $\mathbb{R}^2$ with representation (7). Then for any $\varepsilon > 0$, $\Delta \subseteq (0,\varepsilon)$ and bounded Borel set $B \subseteq \mathbb{R}^2$ with positive Lebesgue measure we have

$$\left| \Sigma_2 \widehat{V}(X;\varepsilon,\Delta,B) - \frac{C(X_\varepsilon^*;B)}{\lambda^2(B)} \right| \le \sqrt{1 + \frac{\pi^2\varepsilon^2}{4}}\, q(\Delta)\, \frac{\mu_\varepsilon(X;B)}{\lambda^2(B)},$$

where

$$q(\Delta) = \frac{(\sum_i \delta_i^2)(\sum_i \delta_i^4) + (\sum_i \delta_i^3)^2}{(\sum_i \delta_i^2)(\sum_i \delta_i^4) - (\sum_i \delta_i^3)^2}.$$

Remark. The denominator in the definition of q is positive, which can be shown by using the Jensen inequality. Note that the function q of Δ is scaling invariant, i.e., $q(s\Delta) = q(\Delta)$ for any $s > 0$ and any finite set $\Delta \subseteq (0,\infty)$. For the “uniformly distributed” δ_i's, $\delta_i = i\delta$, $i = 1,\dots,n$, we obtain $q(\Delta) = 31 + \frac{15}{n-1} - \frac{15}{n+2} - \frac{15}{3n^2+3n+2}$.

Proof. The main problem with the approximation method is that the local Steiner formula (2) is valid only for sufficiently small δ. Here we use a generalization of (2) valid for all $\delta > 0$, where the volume of the collar is replaced with the integral of an index function, see [2, Lemma 1]. We thus have

$$C(X_\varepsilon^*;B) = \left(M_\Delta M_\Delta^\mathsf{T}\right)^{-1} M_\Delta \widetilde{V}(X;\varepsilon,\Delta,B)$$

with

$$\widetilde{V}(X;\varepsilon,\Delta,B) = (\tilde{v}(X;\varepsilon,\delta_1,B),\dots,\tilde{v}(X;\varepsilon,\delta_n,B))^\mathsf{T}$$

and

$$\tilde{v}(X;\varepsilon,\delta,B) = \frac{1}{\lambda^2(B)} \int_{\mathbb{R}^d} I_\delta(X_\varepsilon^*; z, B)\,\mathrm{d}z.$$

The index function is defined as

$$I_\delta(X_\varepsilon^*; z, B) = \mathrm{card}\,(\sigma_\delta(X_\varepsilon^*; z) \cap B)\,,$$

where $\sigma_\delta(X_\varepsilon^*; z)$ denotes the set of all points $x \in X_\varepsilon^*$ with $0 < |x-z| < \delta$ and such that $z-x$ is an outer normal vector to X_ε^* at x (for details, see [2, Lemma 1], where the sign in the definition of the index function is superfluous since it is always equal to $+1$).

We shall show later that for any $0 < \delta < \varepsilon$,

$$\lambda^2(B)|v(X;\varepsilon,\delta,B) - \tilde{v}(X;\varepsilon,\delta,B)| \le \pi\delta^2\mu_\varepsilon(X;B). \tag{8}$$

We have by the definition of M_Δ

$$M_\Delta(V-\widetilde{V}) = \left(2\sum_i \delta_i(v_i-\tilde{v}_i), \pi\sum_i \delta_i^2(v_i-\tilde{v}_i)\right)^{\mathsf{T}}$$

with $v_i = v(X;\varepsilon,\delta_i,B)$, $\tilde{v}_i = \tilde{v}(X;\varepsilon,\delta_i,B)$, and

$$\left(M_\Delta M_\Delta^{\mathsf{T}}\right)^{-1} = \frac{1}{4\pi^2((\sum_i\delta_i^2)(\sum_i\delta_i^4)-(\sum_i\delta_i^3)^2)}\begin{pmatrix} \pi^2\sum_i\delta_i^4, & -2\pi\sum_i\delta_i^3 \\ -2\pi\sum_i\delta_i^3, & 4\sum_i\delta_i^2 \end{pmatrix}.$$

Hence we get using (8)

$$\begin{aligned}
&\lambda^2(B)\left|\left(M_\Delta M_\Delta^{\mathsf{T}}\right)^{-1} M_\Delta(V(X;\varepsilon,\Delta,B)-\widetilde{V}(X;\varepsilon,\Delta,B))\right| \\
&= \frac{\left|\begin{pmatrix} \pi^2\sum_i\delta_i^4\cdot 2\sum_i\delta_i(v_i-\tilde{v}_i) - 2\pi\sum_i\delta_i^3\cdot\pi\sum_i\delta_i^2(v_i-\tilde{v}_i) \\ -2\pi\sum_i\delta_i^3\cdot 2\sum_i\delta_i(v_i-\tilde{v}_i) + 4\sum_i\delta_i^2\cdot\pi\sum_i\delta_i^2(v_i-\tilde{v}_i)\end{pmatrix}\right|}{4\pi^2((\sum_i\delta_i^2)(\sum_i\delta_i^4)-(\sum_i\delta_i^3)^2)} \\
&\le \frac{2\pi\cdot\pi\mu_\varepsilon(X;B)\left|\begin{pmatrix} \pi(\sum_i\delta_i^4\sum_i\delta_i^3 + \sum_i\delta_i^3\sum_i\delta_i^4) \\ 2(\sum_i\delta_i^3\sum_i\delta_i^3 + \sum_i\delta_i^2\sum_i\delta_i^4)\end{pmatrix}\right|}{4\pi^2((\sum_i\delta_i^2)(\sum_i\delta_i^4)-(\sum_i\delta_i^3)^2)} \\
&\le \frac{1}{2}\sqrt{\pi^2\varepsilon^2+4}\,q(\Delta)\mu_\varepsilon(X;B),
\end{aligned}$$

which yields the required inequality.

In order to prove (8), note that we have by definitions

$$\lambda^2(B)|\tilde{v}-v| \le \int \left|I_\delta(X_\varepsilon^*;z,B) - \mathbf{1}_{(X_\varepsilon^*)_\delta\setminus X_\varepsilon^*}(z)\mathbf{1}_B(\Pi_{X_\varepsilon^*}(z))\right|\mathrm{d}z.$$

We shall show that

$$\left|I_\delta(X_\varepsilon^*;z,B) - \mathbf{1}_{(X_\varepsilon^*)_\delta\setminus X_\varepsilon^*}(z)\mathbf{1}_B(\Pi_{X_\varepsilon^*}(z))\right| \le \mu_\varepsilon(X;B\cap B(z,\delta)). \tag{9}$$

Then, since

$$\begin{aligned}\int \mu_\varepsilon(X; B\cap B(z,\delta))\,\mathrm{d}z &= \int\int_{B(z,\delta)} \mathbf{1}_B(x)\mu_\varepsilon(X;\mathrm{d}x)\,\mathrm{d}z\\ &= \int \mathbf{1}_B(x)\int_{B(x,\delta)}\mathrm{d}z\,\mu_\varepsilon(X;\mathrm{d}x)\\ &= \pi\delta^2\mu_\varepsilon(X;B),\end{aligned}$$

we shall have

$$\int \left|I_\delta(X_\varepsilon^*; z, B) - \mathbf{1}_{(X_\varepsilon^*)_\delta\setminus X_\varepsilon^*}(z)\mathbf{1}_B(\Pi_{X_\varepsilon^*}(z))\right|\mathrm{d}z \le \pi\delta^2\mu_\varepsilon(X;B),$$

from which (8) follows and the proof will be complete.

In order to prove (9), denote $\iota(x) = \{i : x\in \partial X_\varepsilon^i\}$ and write

$$\sigma_\delta(X_\varepsilon^*; z) = \Gamma_1\cup\Gamma_2$$

with $\Gamma_1 = \{x\in\sigma_\delta(X_\varepsilon^*; z) : i,j\in\iota(x) \implies X^i\cap X^j \neq \emptyset\}$ and $\Gamma_2 = \sigma_\delta(X_\varepsilon^*; z)\setminus\Gamma_1$.

Claim 2.2. $\operatorname{card}\Gamma_1 \le 1$.

Proof. Take $x\in\Gamma_1$ such that

$$|x-z| = \sup\{|y-z| : y\in\Gamma_1\}.$$

Note that $\iota(x)$ is finite and let $n_i(x)$ be the (unique) unit outer normal vector to X_ε^i at x. Denote the closed convex cone

$$V = \left\{\sum_{i\in\iota(x)}\alpha_i n^i(x) : \alpha_i\ge 0\right\}.$$

Note that there is no pair $n_i(x), n_j(x)$ with $n_i(x)+n_j(x) = 0$ (since otherwise, X^i and X^j could not intersect). If $V=\mathbb{R}^2$ then x would be an interior point of X_ε which is impossible since $x\in\sigma_\delta(X_\varepsilon^*, z)$. Thus V is a cone of angle less than π, $\operatorname{Nor}(X_\varepsilon^*, x) = -V$ and, thus, $x-z\in V$. Hence, there exist $i,j\in\iota(x)$ (not necessarily different) such that $x-z = \alpha_i n_i(x)+\alpha_j n_j(x)$ with some $\alpha_i,\alpha_j\ge 0$. Since X_ε^i is an ε-neighbourhood, the ball $B(x-\varepsilon n_i(x),\varepsilon)$ is contained in X_ε^i and, similarly, $B(x-\varepsilon n_j(x),\varepsilon)$ is contained in X_ε^j. Let further y be a point of $X^i\cap X^j$ (clearly $x-y\in V$), so that $B(y,\varepsilon)$ is contained in both X_ε^i and X_ε^j. Then, by convexity,

$$\operatorname{conv}\left(B(x-\varepsilon n_i(x),\varepsilon)\cup B(y,\varepsilon)\right)\cup\operatorname{conv}\left(B(x-\varepsilon n_j(x),\varepsilon)\cup B(y,\varepsilon)\right)\subseteq X_\varepsilon^i\cup X_\varepsilon^j$$

and, hence, $B(z,|x-z|)\setminus\{x\}$ is contained in the interior of $X_\varepsilon^i\cup X_\varepsilon^j$, since $|x-y|<\varepsilon$. But then there can be no other point in Γ_1. □

It is easy to see that if $\operatorname{card}\Gamma_1 = 1$ then $z\in(X_\varepsilon^*)_\delta\setminus X_\varepsilon^*$. The inequality (9) thus follows from the fact $\operatorname{card}(\Gamma_2\cap B)\le\mu_\varepsilon(X; B(z,\delta)\cap B)$. □

We are now able to prove a stronger version of Theorem 1 in the planar case.

Theorem 2.3. Let Ξ be a stationary random closed set in $\mathbb{R}^2$ with values in the extended convex ring and fulfilling (3). Then we have for any $q > 0$ and any bounded Borel set B with positive Lebesgue measure

$$\lim_{\varepsilon\to 0} \sup_{\substack{\Delta\subseteq(0,\varepsilon)\\ q(\Delta)\le q}} \left|\mathrm{E}\widehat{V}(\Xi;\varepsilon,\Delta,B) - \overline{V}(\Xi)\right| = 0. \tag{10}$$

If Ξ is, furthermore, ergodic and (B_j) is a sequence of convex bodies with inradii growing to infinity then

$$\lim_{\varepsilon\to 0} \limsup_{j\to\infty} \sup_{\substack{\Delta\subseteq(0,\varepsilon)\\ q(\Delta)\le q}} \left|\widehat{V}(\Xi;\varepsilon,\Delta,B_j) - \overline{V}(\Xi)\right| = 0 \quad \text{a.s. and in } L^1. \tag{11}$$

Proof. As in [2], we can represent Ξ as the union set of a stationary point process Φ on the space of convex bodies in $\mathbb{R}^2$, see [8, § 4.4.2] (the representation is not unique, we simply choose one). Condition (3) can now be expressed in the form

$$\mathrm{E}2^{\Phi(\mathcal{K}_{C^2})} < \infty, \tag{12}$$

where $\mathcal{K}_{C^2}$ is the set of all convex bodies hitting the unit square C^2. Condition (12) implies, in particular, that the second order moment measure

$$\mathrm{E}\Phi^2 = \mathrm{E}(\Phi\otimes\Phi)$$

is locally finite in the sense that

$$\mathrm{E}\Phi^2(\mathcal{K}_B\times\mathcal{K}_B) < \infty$$

for any bounded Borel set B. Of course, Ξ is ergodic whenever Φ is.

In order to prove (10), we apply Proposition 1 and it suffices to show that

$$\lim_{\varepsilon\to 0} \mathrm{E}\mu_\varepsilon(\Xi,B) = 0. \tag{13}$$

We can express $\mathrm{E}\mu_\varepsilon(\Xi,B)$ as follows:

$$\mathrm{E}\mu_\varepsilon(\Xi,B) = \frac{1}{2}\mathrm{E}\int\int \mathbf{1}_{L\cap K=\emptyset}\mathrm{card}\,(\partial K_\varepsilon\cap\partial L_\varepsilon\cap\partial(K_\varepsilon\cup L_\varepsilon)\cap B)\Phi^2(\mathrm{d}(K,L)).$$

The integrated function clearly tends to zero as $\varepsilon\to 0$, for any fixed K, L. Further, assuming that $\varepsilon\le 1$ and, using Claim 2.4 below, we find that the integrated function is bounded by $2\cdot\mathbf{1}_{\mathcal{K}_{\tilde{B}}}(K)\mathbf{1}_{\mathcal{K}_{\tilde{B}}}(L)$, with $\tilde{B} = B_1$, which is integrable. Hence, (13) follows by the Lebesgue dominated theorem.

In order to prove (11) we apply the mean and individual ergodic theorems for spatial processes, see e. g. [4, Corollaries (4.9), (4.20)] to find that

$$\limsup_{j\to\infty} \sup_{\substack{\Delta\subseteq(0,\varepsilon)\\ q(\Delta)\le q}} \left|\widehat{V}(\Xi;\varepsilon,\Delta,B_j) - \overline{V}(\Xi)\right|$$

is bounded by $\mathrm{E}\mu_\varepsilon(\Xi,C^2)$ a.s. and in the mean. The assumptions of the above mentioned ergodic theorems are fulfilled since we have for any Borel subset $B\subseteq C^2$

$$\mu_\varepsilon(\Xi,B) \le \Phi^2(\mathcal{K}_{C^2_\varepsilon}\times\mathcal{K}_{C^2_\varepsilon}) \le \Phi^2(\mathcal{K}_{C^2_1}\times\mathcal{K}_{C^2_1})$$

for $\varepsilon\le 1$, and the random variable on the right hand side is integrable by (12). □

Claim 2.4. If $K \cap L = \emptyset$ then $\operatorname{card}(\partial K_\varepsilon \cap \partial L_\varepsilon \cap \partial(K_\varepsilon \cup L_\varepsilon)) \le 2$.

Proof. Let H be a line strictly separating K and L and let there be three different points $x, y, z \in \partial K_\varepsilon \cap \partial L_\varepsilon$. Then, any of the three balls centred at x, y, z and with common radius ε hit both K and L, but their interiors are disjoint with $K \cup L$. Then the points x, y, z must line on a line. This shows that the set $\partial K_\varepsilon \cap \partial L_\varepsilon$ is contained in a line segment and only the two of them of greatest distance are boundary points of $K_\varepsilon \cup L_\varepsilon$. □

Combining Theorem 2.3 with [2, Theorem 3], we get:

Corollary 2.5. Under the assumptions of Theorem 2.3 we have,

$$\lim_{\varepsilon\to 0} \limsup_{j\to\infty} \sup_{\substack{\Delta\subseteq(0,\varepsilon)\\ q(\Delta)\le q}} \left| \widehat{W}(\Xi;\varepsilon,\Delta,B_j) - \overline{V}(\Xi) \right| = 0 \quad \text{a.s. and in } L^1. \tag{14}$$

3. DIGITIZATION IN THE PLANE

Assume that the random closed set Ξ is observed as a digital 0-1 image with square pixels. As a digitization model, we choose

$$\mathcal{G}_t\Xi := \Xi \cap t\,\mathbb{Z}^2, \quad t > 0,$$

which is called the *Gauss digitization* in [1]. In the following, we shall introduce estimators $\widehat{V}_t(\mathcal{G}_t\Xi;\varepsilon,\Delta,B)$ and $\widehat{W}_t(\mathcal{G}_t\Xi;\varepsilon,\Delta,B)$, which will be versions of $\widehat{V}(\Xi;\varepsilon,\Delta,B)$ and $\widehat{W}(\Xi;\varepsilon,\Delta,B)$, respectively, applied to the digital image $\mathcal{G}_t\Xi$.

The estimators $\widehat{V}(\Xi;\varepsilon,\Delta,B)$ and $\widehat{W}(\Xi;\varepsilon,\Delta,B)$ are obtained by measuring the collar set

$$\begin{aligned}(\Xi^*_\varepsilon)_\delta \setminus \Xi^*_\varepsilon &= \big((\Xi \oplus B_\varepsilon)^C \oplus B_\delta\big) \setminus (\Xi \oplus B_\varepsilon)^C \\ &= (\Xi \oplus B_\varepsilon) \setminus \big((\Xi \oplus B_\varepsilon) \ominus B_\delta\big),\end{aligned}$$

where $B_\varepsilon = B(0,\varepsilon)$ denotes the centred disc of radius ε. For the discretized version, we simply replace Ξ with its Gauss digitization $\mathcal{G}_t\Xi$ and B_ε, B_δ with appropriate digitizations $H_t(\varepsilon), H_t(\delta) \subset t\,\mathbb{Z}^2$ which will be specified later. We denote for $0 < \delta < \varepsilon$ two subsets of $t\,\mathbb{Z}^d$:

$$\Xi_{t,\varepsilon} := \mathcal{G}_t\Xi \oplus H_t(\varepsilon)$$

and

$$\Xi_{t,\varepsilon,\delta} := \big(\mathcal{G}_t\Xi \oplus H_t(\varepsilon)\big) \setminus \big((\mathcal{G}_t\Xi \oplus H_t(\varepsilon)) \ominus H_t(\delta)\big),$$

and define

$$w_t(\mathcal{G}_t\Xi;\varepsilon,\delta,B) := \lambda^2(B)^{-1} t^2 \#\left(\Xi_{t,\varepsilon,\delta} \cap B\right),$$

$$W_t(\mathcal{G}_t\Xi;\varepsilon,\Delta,B) := (w_t(\mathcal{G}_t\Xi;\varepsilon,\delta_1,B),\ldots,w_t(\mathcal{G}_t\Xi;\varepsilon,\delta_n,B))$$

and

$$\widehat{W}_t(\mathcal{G}_t\Xi;\varepsilon,\Delta,B) := \Sigma_d (M_\Delta M_\Delta^\top)^{-1} M_\Delta W_t(\mathcal{G}_t\Xi;\varepsilon,\Delta,B).$$

Before defining the discrete analogue of $\widehat{V}$, notice that, although the nearest point of Ξ^*_ε is uniquely determined for almost all points of $(\Xi^*_\varepsilon)_\delta \setminus \Xi^*_\varepsilon$ in the continuous case, this need not be the case in the discrete version. Hence, we need not be able to determine whether the nearest point lies in the window B, and we shall use an average count instead:

$$J_B(x) := \frac{\#(\xi(x) \cap B)}{\#\xi(x)},$$

$\xi(x)$ being the set of all points of $t\,\mathbb{Z}^d \setminus \Xi_{t,\varepsilon}$ with smallest distance from x. We set then

$$v_t(\mathcal{G}_t\Xi;\varepsilon,\delta,B) := \lambda^2(B)^{-1}t^2 \sum_{x\in\Xi_{t,\varepsilon,\delta}} J_B(x),$$

$$V_t(\mathcal{G}_t\Xi;\varepsilon,\Delta,B) := (v_t(\mathcal{G}_t\Xi;\varepsilon,\delta_1,B),\ldots,v_t(\mathcal{G}_t\Xi;\varepsilon,\delta_n,B))$$

and

$$\widehat{V}_t(\mathcal{G}_t\Xi;\varepsilon,\Delta,B) := \Sigma_d(M_\Delta M_\Delta^\mathsf{T})^{-1}M_\Delta V_t(\mathcal{G}_t\Xi;\varepsilon,\Delta,B).$$

Multigrid convergence

Let ϕ be a functional defined on a set class $\mathcal{M}$ and let $\widehat{\phi}_t(\mathcal{G}_tX)$ be an estimator of $\phi(X)$ determined from the discretization $\mathcal{G}_tX$ of $X \in \mathcal{M}$, $t>0$. An estimator $\widehat{\phi}_t$ of ϕ is called *multigrid convergent* for a set class $\mathcal{M}' \subseteq \mathcal{M}$ if

$$\lim_{t\to 0}\widehat{\phi}_t(\mathcal{G}_tX) = \phi(X), \quad X \in \mathcal{M}',$$

see [1].

We shall show below that given $\varepsilon > 0$, $\Delta \subseteq (0,\varepsilon)$ and convex body B with nonempty interior, the estimator $\widehat{W}_t(\mathcal{G}_tX;\varepsilon,\Delta,B)$ of $W(X;\varepsilon,\Delta,B)$ is multigrid convergent for X from the extended convex ring, provided that the digitizations $H_t(\varepsilon)$ satisfy

$$d_H(H_t(\varepsilon),B_\varepsilon) \le Ct, \quad t,\varepsilon > 0 \tag{15}$$

for some constant C (d_H stands for the Hausdorff metric). Note that (15) is satisfied e. g. for the Gauss digitization $H_t(\varepsilon) = \mathcal{G}_tB_\varepsilon$ with $C=\sqrt{2}$.

Let $\mathcal{N}$ denote the ring generated by convex compact sets, which is closed with respect to finite unions, intersections and set differences.

Lemma 3.1. For any set $K \in \mathcal{N}$ there exist constants C_K, D_K such that for any $t>0$ and any set $L \subseteq t\mathbb{Z}^2$,

$$|t^2\#L - \lambda^2(K)| \le C_K(\sigma + \tfrac{t}{2}) + D_K(\sigma+\tfrac{t}{2})^2,$$

where $\sigma = d_H(K,L)$.

P r o o f. Assume first that K is a convex body and denote $L(t) = L \oplus [-t/2,t/2]^2$. Note that $t^2\#L = \lambda^2(L(t))$ and $d_H(K,L(t)) \le \sigma + t/2$. Hence, using [2, Lemma 6], we get

$$|t^2\#L - \lambda^2(K)| \le \lambda^2(K\triangle L(t)) \le 2(\sigma+\tfrac{t}{2})\mathcal{H}^1(\partial K) + \pi(\sigma+\tfrac{t}{2})^2,$$

where $\triangle$ denotes the symmetric difference of sets.

The second step of the proof is to show that the family of sets K satisfying the assertion of the lemma is closed under unions, intersections and set differences. We shall verify the closedness under unions only, since the other cases are similar.

Let thus $K = K_1 \cup K_2$, where $K_1, K_2 \in \mathcal{N}$ satisfy the assertion, and let L be a subset of the point grid $t\mathbb{Z}^2$ with $d_H(K, L) = \sigma$. Write $L = L_1 \cup L_2$ with $L_i = L \cap (K_i)_\sigma$, $L_i(t) = L_i \oplus [-\frac{t}{2}, \frac{t}{2}]^2$, $i = 1, 2$, $L(t) = L \oplus [-\frac{t}{2}, \frac{t}{2}]^2$. We have, as above,

$$|t^2 \# L - \lambda^2(K)| \le \lambda^2(K \triangle L(t)) \le \lambda^2(K_1 \triangle L_1(t)) + \lambda^2(K_2 \triangle L_2(t)),$$

since $K \triangle L(t) \subseteq (K_1 \triangle L_1(t)) \cup (K_2 \triangle L_2(t))$. But $\lambda^2(K_i \triangle L_i(t)) \le C_{K_i}(\sigma + \frac{t}{2}) + D_{K_i}(\sigma + \frac{t}{2})^2$ by assumption, and the assertion follows. □

Proposition 3.2. Let X be a set from the extended convex ring, $\varepsilon > 0$, $\Delta = \{\delta_1, \dots, \delta_n\} \subseteq (0, \varepsilon)$ and B a bounded Borel subset of $\mathbb{R}^2$ of positive Lebesgue measure. Assume that digitizations $H_t(\varepsilon) \subset t\,\mathbb{Z}^d$ of B_ε satisfy (15). Then

$$|\widehat{W}_t(\mathcal{G}_t X; \varepsilon, \Delta, B) - \widehat{W}(X; \varepsilon, \Delta, B)| = O(t), \quad t \to 0.$$

P r o o f. We consider first the set

$$X_{t,\varepsilon,\delta} = (\mathcal{G}_t(X) \oplus H_t(\varepsilon)) \setminus ((\mathcal{G}_t(X) \oplus H_t(\varepsilon)) \ominus H_t(\delta))$$

and its "continuous" counterpart

$$M = (X \oplus B_\varepsilon) \setminus ((X \oplus B_\varepsilon) \ominus B_\delta).$$

By definition of the Gauss digitization, $d_H(X, \mathcal{G}_t(X)) \le \sqrt{2}t$ for any $X \subseteq \mathbb{R}^2$. This implies, together with assumption (15),

$$\begin{aligned} d_H[X \oplus B_\varepsilon, \mathcal{G}_t(X) \oplus H_t(\varepsilon))] &\le (\sqrt{2} + C)t, \\ d_H[(X \oplus B_\varepsilon) \ominus B_\delta), (\mathcal{G}_t(X) \oplus H_t(\varepsilon)) \ominus H_t(\delta)] &\le (\sqrt{2} + 2C)t \end{aligned}$$

and, thus,

$$d_H[M, X_{t,\varepsilon,\delta}] \le (2\sqrt{2} + 3C)t.$$

It follows that also $d_H[M \cap B, X_{t,\varepsilon,\delta} \cap B] = O(t)$ and, applying Lemma 3.1, we get

$$|w_t(\mathcal{G}_t \Xi; \varepsilon, \delta_i, B) - w(\Xi; \varepsilon, \delta_i, B)| = O(t), t \to 0, \; i = 1, \dots, n,$$

and the statement follows. □

Theorem 3.3. Let Ξ be a stationary, ergodic random closed set in $\mathbb{R}^2$ with values in the extended convex ring and fulfilling (3). Let (B_j) be a sequence of convex bodies with inradii growing to infinity. Then we have for any $q > 0$

$$\lim_{\varepsilon \to 0} \limsup_{j \to \infty} \sup_{\substack{\Delta \subseteq (0,\varepsilon) \\ q(\Delta) \le q}} \lim_{t \to 0} \left| \widehat{W}_t(\mathcal{G}_t \Xi; \varepsilon, \Delta, B_j) - \overline{V}(\Xi) \right| = 0 \quad \text{a.s. and in } L^1. \tag{16}$$

P r o o f. We use the estimate

$$\left|\widehat{W}_t(\mathcal{G}_t\Xi;\varepsilon,\Delta,B_j)-\overline{V}(\Xi)\right| \le \left|\widehat{W}_t(\mathcal{G}_t\Xi;\varepsilon,\Delta,B_j)-\widehat{W}(\Xi;\varepsilon,\Delta,B_j)\right| + \left|\widehat{W}(\Xi;\varepsilon,\Delta,B_j)-\overline{V}(\Xi)\right|$$

The first summand tends to zero due to the Proposition 3.2, whereas the second one by Corollary 2.5. □

Comments on Theorem 3.3

First, it should be noted that an analogous version of Proposition 3.2 and Theorem 3.3 could be shown for $\widehat{V}_t$, but the proof would be slightly more technically complicated.

Second, the consistency result (16) requires that first the resolution of the image (t) is sent to zero and then ε (which measures the approximation of Ξ by Ξ^*_ε) is sent to zero. In our implementation below we are, however, limited by the linear dependence $\varepsilon = r_n t$, where $r_n = 4.327$. In this case the error

$$|v_{\varepsilon/r_n}(X_{\varepsilon/r_n};\varepsilon,\delta,B_j)-(v(X;\varepsilon,\delta,B_j))| = O(\delta),$$

but (as one can see from the proof of Proposition 1) it is necessary to have error of order $o(\delta^2)$ to cancel out the limit behaviour of the linear transformation $(M_\Delta M_\Delta^\top)^{-1}M_\Delta$. Thus it is not possible to prove multigrid convergence of the proposed estimator for the implementation given below.

Implementation of the estimator

The crucial step is to determine the digital sets $H_t(\delta)$ which approximate discs B_δ of small radii δ. The standard and natural way is to use the Gauss digitization $\mathcal{G}_t B_\delta$ again, which would, however, lead to bad results. In fact, if δ is small (comparable with the resolution t) then there are only few discrete sets in $t\,\mathbb{Z}^d$ with comparably small radii.

We approach the problem from the other side. We start with choosing "small" symmetric (w.r.t. both axes) subsets $D_1,\dots,D_n$ of $\mathbb{Z}^2$ and determine their correct "radii" $r_1,\dots,r_n$, see Table 1 for $n=14$. Then, we define

$$H_t(tr_i) := tD_i, \quad i=1,\dots,n, \quad t>0.$$

Thus, $H_t(\delta)$ is not defined for all $\delta>0$ and $t>0$. Nevertheless, we can choose for a given resolution $t>0$ the parameters $\varepsilon = tr_n$ and $\delta_i = tr_i$, $i=1,\dots,n$ (so that ε is a multiple of t and we can not send t independently on ε to 0 as we did in the previous subsection in the proof of multigrid convergence).

The radii r_i are determined as follows. Since the boundary of a ball contains equally all possible tangent direction as the isotropic process does, the idea is that the Steiner formula for balls should be as exact as possible. Choose a ball B_R of large radius (we took $R=980$) and determine r_n from the equation

$$\#\big((\mathcal{G}_1B_R\oplus D_n)\setminus((\mathcal{G}_1B_R\oplus D_n)\ominus D_n)\big) = \lambda^2((B_R\oplus B_{r_n})\setminus B_R) = \pi(R+r_n)^2-\pi R^2.$$

Table 1. Discretized discs $D_1, \dots, D_{14}$ with their radii used in the estimating procedure. Each disc is described by the numbers of its pixels in consecutive horizontal lines.

number	discretized disc	radius
1	1,3,1	0.899417
2	3,3,3	1.27135
3	1,3,5,3,1	1.79737
4	3,5,5,5,3	2.16832
5	5,5,5,5,5	2.53977
6	1,3,5,7,5,3,1	2.69386
7	1,5,5,7,5,5,1	2.8389
8	3,5,7,7,7,5,3	3.06383
9	5,7,7,7,7,7,5	3.4343
10	1,5,7,7,9,7,7,5,1	3.73316
11	3,5,7,9,9,9,7,5,3	3.95788
12	1,7,7,7,9,7,7,7,1	4.01003
13	3,7,7,9,9,9,7,7,3	4.10273
14	5,7,9,9,9,9,9,7,5	4.32738

Consequently, we determine the radii r_i, $i < n$, by

$$\#\big((\mathcal{G}_1 B_R \oplus D_n) \setminus ((\mathcal{G}_1 B_R \oplus D_n) \ominus D_i)\big) = \pi(R + r_n)^2 - \pi(R + r_n - r_i)^2.$$

The resulting values are given in Table 1 for $n = 14$.

In order to be able to decide whether the nearest points lies in B, we have done minus sampling, i. e., we used a slightly smaller window, which was eroded by 5 pixels from all sides.

Remarks.

1. The discs $D_1, \dots, D_{14}$ satisfy condition (15) which means that when we would be able to increase the resolution sufficiently for each dilation radius ε, then this implementation will be multigrid convergent according to Theorem 3.3.

2. The radii r_i were determined in a way to minimize in the isotropic case the difference between the area of the continuous set $((\Xi^*_\varepsilon)_{\delta_i} \setminus \Xi^*_\varepsilon) \cap B$ and the t^2-multiple of the number of lattice points of $\Xi_{\varepsilon/r_n,\varepsilon,\delta_i} \cap B$.

3. Here the computation of the radii of discretized small discs is slightly different than in [2], nevertheless the radii of the chosen discs differ in the third decimal numbers only.

4. REDUCING BIAS OF $\widehat{V}(\Xi, \varepsilon, \Delta, B)$

It was shown in [2, Section 6, 7] that the deviation of $\widehat{V}(\Xi, \varepsilon, \Delta, B)$ is mainly caused by the bias (the variance is relatively small). When the considered model is Boolean, it is possible to calculate the theoretical bias $B_k(\varepsilon)$ of $\widehat{V}(\Xi,\varepsilon,\Delta,B)$ ([2, Section 6]) and eliminate the bias from the estimator.

But when the considered model is not Boolean, it is not easy to calculate the theoretical bias. The straightforward way of reducing bias is to decrease ε (see [2, Figure 2]). One way is by increasing the resolution, as it was shown in [2, Section 7]; it is time consuming and it is less robust. The other way is to break the assumption $\max\Delta \le \varepsilon$. Then the assumption of Steiner formula for Ξ^*_ε will be broken, namely that $\max\Delta \le \inf_{x\in B\cap\Xi^*_\varepsilon} \operatorname{reach}(\Xi^*_\varepsilon, x)$. Thus we will investigate the estimator $\widehat{V}(\Xi,\varepsilon,\Delta,B)$ in the case when $\max\Delta > \varepsilon$ in this section.

The following theorem says that the described estimator, in the case when $\max\Delta > \varepsilon$, is asymptotically consistent if the random closed set is built from smooth convex bodies.

Remark. The estimator with $\max\Delta$ greater than ε applied on convex bodies with vertices will be asymptotically biased. The bias depend on number of vertices with acute angle. The obtuse angle brings less bias than acute angle.

Theorem 4.4. Let Ξ, q and (B_j) be as in Theorem 3.3 and assume additionally that Ξ can be written as a locally finite union of convex bodies with curvature radii bounded from below by a constant $\gamma > 0$. Then we have for any $K > 0$,

$$\lim_{\varepsilon\to 0}\lim_{j\to\infty}\sup_{\substack{\Delta\subseteq(0,K\varepsilon]\\ q(\Delta)\le q}}\left|\widehat{V}(\Xi;\varepsilon,\Delta,B_j)-\overline{V}(\Xi)\right| = 0 \quad \text{a.s. and in } L^1.$$

Proof. By assumptions, the ball $\mathcal{B}_\gamma$ is a summand of each of the convex particles and, hence, of Ξ as well. Applying the estimations procedure to $\Xi_{-\gamma} = \Xi \ominus \mathcal{B}_\gamma$, we get

$$\widehat{V}(\Xi;\varepsilon,\Delta,B_j) = \widehat{V}(\Xi_{-\gamma};\varepsilon+\gamma,\Delta,B_j), \quad j\in\mathbb{N}.$$

Note that $\max\Delta \le K\varepsilon \le \varepsilon+\gamma$ for sufficiently small ε. We apply now Theorem 3.3 to $\Xi_{-\gamma}$, but with limit $\varepsilon \to \gamma_+$ instead of $\varepsilon \to 0_+$ (the proof would be slightly modified) and obtain

$$\lim_{\varepsilon\to 0}\lim_{j\to\infty}\sup_{\substack{\Delta\subseteq(0,\varepsilon+\gamma]\\ q(\Delta)\le q}}\left|\widehat{V}(\Xi_{-\gamma};\varepsilon+\gamma,\Delta,B_j)-\overline{V}((\Xi_{-\gamma})_\gamma))\right| = 0 \quad \text{a.s. and in } L^1,$$

and since $(\Xi_{-\gamma})_\gamma = \Xi$, the proof is finished. □

Remark. Theorem 3 for multigrid convergence of the digitized version of the proposed estimator $\widehat{V}(\Xi;\varepsilon,\Delta,B)$ will be valid in the case when $\max\Delta > \varepsilon$ too.

5. SIMULATION STUDY OF BEHAVIOUR OF $\widehat{V}(\Xi;\varepsilon,\Delta,B)$ IN THE CASE WHEN $\max\Delta > \varepsilon$

The simulation study was done on the Boolean model with same parameters as in [2]. Therefore we can compare the statistical properties of both variations of the

Table 2. Theoretical and estimated values of specific intrinsic volumes estimated with resolution 2000 × 2000 and $\varepsilon = 0.899$.

p	0.2	0.5	0.8
$\overline{V}_0(\Xi) \times 10^4$	0.477816	0.39153	-0.605932
$\widetilde{V}_0 \times 10^4$	0.490184	0.407404	-0.587746
$\widetilde{V}_0^R \times 10^4$	0.497216	0.438647	-0.566375
$\delta_{\overline{V}_0,\widetilde{V}_0}$, %	2.59	3.94	-3.00
$\delta_{\overline{V}_0,\widetilde{V}_0^R}$, %	4.06	11.91	-6.53
StandardDeviation$(\widetilde{V}_0) \times 10^4$	0.046429	0.082124	0.123054
Mean square error$(\widetilde{V}_0) \times 10^8$	0.002308	0.006983	0.015473
$2\overline{V}_1(\Xi)$	0.011476	0.02228	0.020693
$2\widetilde{V}_1$	0.011537	0.022172	0.020201
$2\widetilde{V}_1^R$	0.011440	0.022291	0.020853
$\delta_{\overline{V}_1,\widetilde{V}_1}$, %	0.53	-0.48	-2.38
$\delta_{\overline{V}_1,\widetilde{V}_1^R}$, %	-0.31	0.05	0.77
StandardDeviation$(2 \times \widetilde{V}_1)$	0.000962	0.000728	0.000895
Mean square error$(2 \times \widetilde{V}_1)$	0.93×10^{-6}	0.54×10^{-6}	1.04×10^{-6}

estimator $\widehat{V}(X; \varepsilon, \Delta, B)$ and we can compare the reduction of the bias by implementing a theoretical bias and by the presented method. The resolution was set to 2000 × 2000. The best result was obtained with the smallest possible ε (i. e., $\varepsilon = 0.899$) and Δ as in Table 1 ($\max \Delta = 4.327$).

There were made 200 realizations of the planar Boolean model Ξ of discs in the observation window $B = [0, 2000]^2$. The disc radii are uniformly distributed in $[20, 40]$. The intensity of the underlying Poisson point process of the centers of the discs was chosen to fit the area fractions $p = 0.2, 0.5, 0.8$. The averages of the estimates over 200 realizations are denoted by $\widetilde{V}_i$ for the presented method and $\widetilde{V}_i^R$ for the reduction of the bias by the theoretical one (where the information that the process is Boolean is used to compute the reduction, for more details see [2]). Its values are compared with the theoretical counterparts $\overline{V}_0(\Xi), \overline{V}_1(\Xi)$ in Table 2. To compare the precision of all algorithms, the relative error $\delta_{A,B} = \frac{B-A}{A} \cdot 100\%$ of an estimated quantity B with respect to theoretical value A is given.

6. DISCUSSION

The simulations show that, for the model with no vertices, the presented bias reduction method is comparable with method using the reduction by theoretical bias, which is applied in the Boolean case only.

Furthermore the presented bias reduction method with the resolution 2000 × 2000 is comparable with the original method with the resolution 10000 × 10000 [2, Table 3] and it gives better results than the estimators proposed by Spodarev and Schmidt [7] and by Ohser and Mücklich [5], for the model with no vertices (for comparison see [2, Table 2]).

For models with vertices (as, e. g., Boolean models of triangles), we recommend to use the original method with $\max \Delta \leq \varepsilon$ which is applicable to any stationary locally finite union of convex bodies.

The open question remains, what happens with the presented bias reduction method in $\mathbb{R}^3$. Some simulation experiments are described in [3].

The computer programme prepared for public use can be downloaded from `http://www.pf.jcu.cz/~mrkvicka/math`.

ACKNOWLEDGEMENT

The first author was supported by the Grant Agency of Academy of Sciences of the Czech Republic, Project No. KJB101420801, and the second author by the Ministry of Education, Youth and Sport of the Czech Republic, Project MSM 0021620839, and the Czech Science Foundation, Grant 201/06/0302.

(Received July 31, 2008.)

REFERENCES

[1] R. Klette and A. Rosenfeld: Digital Geometry. Elsevier, New York 2004.

[2] T. Mrkvička and J. Rataj: On estimation of intrinsic volume densities of stationary random closed sets. Stoch. Proc. Appl. *118* (2008), 2, 213–231.

[3] T. Mrkvička: Estimation of intrinsic volume via parallel sets in plane and space. Inzynieria Materialowa *4* (2008), 392–395.

[4] X.-X. Nguyen, H. Zessin: Ergodic theorems for spatial processes. Z. Wahrsch. Verw. Gebiete *48* (1979), 133–158

[5] J. Ohser and F. Mücklich: Statistical Analysis of Microstructures in Materials Science. Wiley, Chichester 2000.

[6] J. Rataj: Estimation of intrinsic volumes from parallel neighbourhoods. Rend. Circ. Mat. Palermo, Ser. II, Suppl. *77* (2006), 553–563.

[7] V. Schmidt and E. Spodarev: Joint estimators for the specific intrinsic volumes of stationary random sets. Stoch. Proc. Appl. *115* (2005), 959–981.

[8] R. Schneider and W. Weil: Stochastische Geometrie. Teubner, Stuttgart 2000.

Tomáš Mrkvička, University of South Bohemia, Faculty of Science, Institute of Mathematics and Biomathematics, Branišovská 31, 37005 České Budějovice. Czech Republic.
mrkvicka@prf.jcu.cz

Jan Rataj, Charles University, Faculty of Mathematics and Physics, Sokolovská 83, 186 75 Praha 8. Czech Republic.
rataj@karlin.mff.cuni.cz