KYBERNETIKA — VOLUME *45* (2009), NUMBER 6, PAGES 1052 – 1058

CONVERGENT REGULAR MEASURES ON MV–ALGEBRAS

Giuseppina Barbieri

We obtain for measures on MV-algebras the classical theorem of Dieudonné related to convergent sequences of regular maps.

Keywords: MV-algebras, measures, Dieudonné theorem, convergence, regular maps

AMS Subject Classification: 28E10, 06D35

1. INTRODUCTION

In 1933 Nikodým [7] proved the so-called Vitali–Hahn–Saks theorem, namely: "If a sequence of Borel measures converges pointwise to a map μ, then μ is a Borel measure".

In 1951 Dieudonné proved the following more general theorem: "If a sequence of regular measures defined on Borel sets of a compact metrizable space converges on every open set, then it converges on every Borel set. In this case, the sequence is uniformly regular". This theorem generalizes Nikodým's theorem if one substitutes the pointwise convergence on the Borel σ-algebra for the analogous condition on open sets provided a regularity assumption and a topological condition on the space are satisfied. Brooks in [2] generalizes this theorem to the case the space is either compact or the space is normal and the sequence is uniformly bounded.

In this note we prove a general version of Dieudonné's theorem valid for measures defined on MV-algebras with values in a topological group G. We use an abstract concept of regularity (see Definition 3) where $\mathcal{F}$ and $\mathcal{G}$ play the role of the compact and open sets, respectively.

The paper follows [4].

After notation and preliminaries we give the exhaustivity condition, the regularity condition and their relationship. Finally, we provide the main result (Theorem 13).

2. PRELIMINARIES

In this section we shall give some basic definitions and fix some notations.

Definition 1. An MV-algebra $(L, \oplus, ^\perp; 0, 1)$ is a commutative semigroup $(L, \oplus)$ with 0, 1 and a unary operation $^\perp : L \to L$ which satisfies the following axioms

(L1) $x \oplus 1 = 1$,

(L2) $x^{\perp\perp} = x$,

(L3) $0^\perp = 1$,

(L4) $(x^\perp \oplus y)^\perp \oplus y = (x \oplus y^\perp)^\perp \oplus x$ for every $x, y \in L$.

From now on, let L be a MV-algebra and let (G, τ) be a Hausdorff complete topological group. We may suppose that G is metrizable since every group can be embedded into a product of metrizable topological groups and our proofs can be reduced to this case.

We consider on L the relation $\leq$ defined by $x \leq y$ if and only if $x^\perp \oplus y = 1$. It can be shown that $\leq$ is a partial order on L and $(L, \leq)$ is a distributive lattice.

We say that a and b are *orthogonal* if $a \leq b^\perp$ and we write $a \perp b$. If $a_1, \cdots, a_n \in L$, we inductively define $a_1 \oplus \cdots \oplus a_n = (a_1 \oplus \cdots a_{n-1}) \oplus a_n$ provided that the right hand side exists. The definition is independent on permutations of the elements. We say that a finite subset $\{a_1, \cdots, a_n\}$, $n \in \mathbb{N}$, of L is orthogonal if $a_1 \oplus \cdots \oplus a_n$ exists. We define a sequence $(a_n)_{n\in\mathbb{N}}$ of L to be orthogonal if the set $\{a_1, \ldots a_n\}$ is orthogonal for every $n \in N$.

We define $x \gneqq y := ((x \wedge y) \oplus (x \vee y)^\perp)^\perp$ for $x, y \in L$.

The following lemma is straightforward:

Lemma 2. If $(b_n)_{n\in\mathbb{N}}$ is an increasing sequence in L, then $a_n := b_n \gneqq b_{n-1}$, with $b_0 := 0$, defines an orthogonal sequence.

In what follows we make use of the fact that each MV-algebra satisfies the following condition known as the Riesz decomposition property: (RDP) If x, x_1, $x_2 \in L$ with $x \leq x_1 \oplus x_2$, then there are $t_i \in L\,(i = 1, 2)$ such that $t_i \leq x_i$ and $x = t_1 \oplus t_2,\ t_1 \leq t_2^\perp$ (see [6, p. 432]).

A function μ on an MV-algebra with values in G is called a *measure* if for every $a, b \in L$, with $a \perp b$,

$$\mu(a \oplus b) = \mu(a) + \mu(b).$$

A *modular function* is a map which satisfies the modular law, that is for all $a, b \in L$

$$\mu(a \vee b) + \mu(a \wedge b) = \mu(a) + \mu(b).$$

Observe that every measure on an MV-algebra is a modular function.

According to [1] we denote by $\mathcal{LUA}(L)$ the system of all uniformities on L which make $\oplus$ and $^\perp$ uniformly continuous. If $\sqcap \in \mathcal{LUA}(L)$, we call $(L, \sqcap)$ a *uniform MV-algebra.* If $\eta : L \to [0, +\infty]$ is a submeasure, then (L, d_η) is a uniform MV-algebra, where $d_\eta(x, y) = \eta(x \gneqq y)$ for $x, y \in L$ and a submeasure η is a monotone subadditive (i. e. $\eta(x \oplus y) \leq \eta(x) + \eta(y)$ for orthogonal $x, y \in L$) map with $\eta(0) = 0$.

Notation 3. If a is an element of a lattice T, we write $T_a := \{x \in T : x \leq a\}$ and we put $\tilde{\mu}(a) := \sup\{|\mu(b)| : b \in T_a\}$.

It is easy to check that $\tilde{\mu}$ is a submeasure whenever μ is a measure.

From now on let $\mathcal{F}, \mathcal{G} \subseteq L$ be two sublattices. Moreover, suppose that

- *for each $f \in \mathcal{F}$, $f^{\perp}$ belongs to $\mathcal{G}$;*
- *for each $f \in \mathcal{F}$ and $a \in \mathcal{G}$, $a \gneqq (f \wedge a)$ belongs to $\mathcal{G}$.*

3. EXHAUSTIVITY

Definition 4. A measure μ on L is called $\mathcal{G}$-exhaustive if for every orthogonal sequence $(g_n)_{n\in\mathbb{N}}$ in $\mathcal{G}$ we have $\lim_n \mu(g_n) = 0$. A sequence $(\mu_i)_{i\in\mathbb{N}}$ is called uniformly $\mathcal{G}$-exhaustive if for every orthogonal sequence $(g_n)_{n\in\mathbb{N}}$ in $\mathcal{G}$ we have $\lim_n \mu_i(g_n) = 0$ uniformly with respect to $i \in \mathbb{N}$. If $\mathcal{G} = L$ we use the terms exhaustive and uniformly exhaustive.

Theorem 5. Suppose that $\mathcal{G}$ is closed under countable sums. Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of pointwise convergent $\mathcal{G}$-exhaustive measures. Then $(\mu_n)_{n\in\mathbb{N}}$ is uniformly $\mathcal{G}$-exhaustive.

Proof. By way of contradiction there exist $\varepsilon > 0$, a strictly increasing sequence $(i_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$ and an orthogonal sequence $(g_n)_{n\in\mathbb{N}}$ in $\mathcal{G}$ such that $|\mu_{i_n}(g_n)| \geq \varepsilon$ for each $n \in \mathbb{N}$. Define $\nu_n(A) := \mu_{i_n}(\bigoplus_{h\in A} g_h)$. One can check that they form a sequence of finitely additive measures on the power set of $\mathbb{N}$.

We can apply the classical Vitali–Hahn–Saks theorem. So these restrictions form a uniformly exhaustive sequence. This contradicts the assumptions and completes the proof. □

4. REGULARITY

We now extend the notion of regularity to our context.

Definition 6. We say that a measure μ is $(\mathcal{F}, \mathcal{G})$-*regular* or simply *regular* if for every $\varepsilon > 0$ and

- for every $a \in L$, there exist $f \in \mathcal{F}$ and $g \in \mathcal{G}$ such that
$$f \leq a \leq g \quad \text{and} \quad \tilde{\mu}(g \gneqq f) < \varepsilon$$
- for every $f \in \mathcal{F}$, there exist $e \in \mathcal{G}$, $h \in \mathcal{F}$, $g \in \mathcal{G}$ such that
$$f \leq e \leq h \leq g \quad \text{and} \quad \tilde{\mu}(g \gneqq f) < \varepsilon.$$

Definition 7. Let $\mu_n : L \to G$ be a sequence of measures. Then it is uniformly regular if it satisfies the following properties:

(i) For every $a \in L$ and $\varepsilon > 0$, there exist $g \in \mathcal{G}$ and $f \in \mathcal{F}$ with

$$f \leq a \leq g \quad \text{and} \quad \sup_i \tilde{\mu}_i(g \gneqq f) < \varepsilon$$

(ii) For every $f \in \mathcal{F}$ and $\varepsilon > 0$, there exist $e, g \in \mathcal{G}$ and $h \in \mathcal{F}$ with

$$f \leq e \leq h \leq g \quad \text{and} \quad \sup_i \tilde{\mu}_i(g \gneqq f) < \varepsilon.$$

Lemma 8. Let μ be a regular measure and let $\varepsilon > 0$, $h \in \mathcal{F}$ and $a \in \mathcal{G}$ with $h \leq a$. Then there exist $a_1, a_2 \in \mathcal{G}$ and $h_2 \in \mathcal{F}$ such that $h \leq a_2 \leq h_2 \leq a_1 \leq a$ with $\tilde{\mu}(a_1 \gneqq h) < \varepsilon$ and $\tilde{\mu}(a_2 \gneqq h) < \varepsilon/2$.

Proof. By regularity we can choose $a_1, a_1' \in \mathcal{G}$, $a \geq a_1, a_1' \geq h$ such that $\tilde{\mu}(a_1 \gneqq h) < \varepsilon$ and $\tilde{\mu}(a_1' \gneqq h) < \varepsilon/2$. Then h and $a_1^{\perp}$ are two orthogonal elements in $\mathcal{F}$. Let a' and a'' be orthogonal elements in $\mathcal{G}$ with $a' \geq h$ and $a'' \geq a_1'^{\perp}$.

Put $a_2 := a_1' \wedge a'$ and $h_2 := a''^{\perp}$, then we have $a_2 \leq a_1'$ and $a_2 \leq h_2 \leq a_1$ and these elements satisfy the required properties. □

Lemma 9. Let μ be a regular measure. Then for every $a \in \mathcal{G}$,

$$\mu(L_a) \subseteq \overline{\mu(\mathcal{G}_a) - \mu(\mathcal{G}_a)},$$

where $\overline{\mu(\mathcal{G}_a) - \mu(\mathcal{G}_a)}$ denotes the closure of $\mu(\mathcal{G}_a) - \mu(\mathcal{G}_a)$.

Proof. Let $x \leq a$. For every $\varepsilon > 0$ there exist $g \in \mathcal{G}$ and $h \in \mathcal{F}$ such that $h \leq x \leq g \leq a$ and $\tilde{\mu}(g \gneqq h) < \varepsilon$, then we have $|\mu(x) - (\mu(g) - \mu(g \gneqq h))| = |\mu(x \gneqq h)| < \varepsilon$ with $\mu(g), \mu(g \gneqq h) \in \mu(\mathcal{G}_a)$. □

5. REGULARITY AND EXHAUSTIVITY

Proposition 10. Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of uniformly $\mathcal{G}$-exhaustive regular measures. Then for every $h \in \mathcal{F}$ and every $\varepsilon > 0$ there exists $a \in \mathcal{G}$ such that $h \leq a$ and $\sup_i \tilde{\mu}_i(a \gneqq h) < \varepsilon$.

Proof. Let $\varepsilon > 0$ and $h \in \mathcal{F}$. We can pick $a_i \in \mathcal{G}$ and $h_i \in \mathcal{F}$ such that

(i) $h \leq a_{i+1} \leq h_{i+1} \leq a_i$ for every $i \in \{1, \dots, n-1\}$,

(ii) $\sup_{j=1}^{i} \tilde{\mu}_j(a_i \gneqq h) < \frac{\varepsilon}{2^i}$ for every $i \in \{1, \dots, n\}$.

By Lemma 8 there exist $a_{n+1} \in \mathcal{G}$ and $h_{n+1} \in \mathcal{F}$ such that

(i) $h \leq a_{n+1} \leq h_{n+1} \leq a_n$,

(ii) $\sup_{j=1}^{n+1} \tilde{\mu}_j(a_{n+1} \gneqq h) < \frac{\varepsilon}{2^{n+1}}$.

Therefore we can construct, by induction, a sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathcal{G}$ and a sequence $(h_n)_{n\in\mathbb{N}}$ in $\mathcal{F}$ such that

(i) $h \leq a_{n+1} \leq h_{n+1} \leq a_n$,

(ii) $\sup_{j=1}^{n} \tilde{\mu}_j(a_n \gneqq h) < \frac{\varepsilon}{2^n}$ for $n \in \mathbb{N}$.

To complete the proof it is enough to see that for every $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that $\sup_i \tilde{\mu}_i(a_m \gneqq h) < \varepsilon$.

Suppose the contrary. Then there is $\sigma > 0$ such that for every $p \in \mathbb{N}$ there exist $y \in \mathcal{G}_{a_p \gneqq h}$ and $i \in \mathbb{N}$ with $|\mu_i(y)| > \sigma$. For every $n, i \in \mathbb{N}$ we get

$$\mu_i(y) = \mu_i(y \gneqq (y \wedge h_n)) + \mu_i(y \wedge h_n).$$

Moreover, for every $n > i$ we have $|\mu_i(y \wedge h_n)| < \frac{\varepsilon}{2^{n-1}}$. Then

$$\lim_{n\to\infty} \mu_i(y \gneqq (y \wedge h_n)) = \mu_i(y)$$

for every $i \in \mathbb{N}$. Therefore there exists $q_i \in \mathbb{N}$ such that $|\mu_i(y \gneqq (h_r \wedge y))| > \sigma$ for every $r \geq q_i$.

We can now construct, by induction, two sequences of natural numbers, namely, $(i_r)_{r\in\mathbb{N}}$ and $(q_r)_{r\in\mathbb{N}}$, and a sequence $(y_r)_{r\in\mathbb{N}}$ in $\mathcal{G}$ such that $y_r \leq a_{q_{r-1}} \gneqq h$ and

$$|\mu_{i_r}(y_r \gneqq (h_{q_r} \wedge y_r))| > \sigma$$

for every $r \in \mathbb{N} \setminus \{1\}$. Since $y_r \gneqq (h_{q_r} \wedge y_r) \leq (a_{q_{r-1}} \gneqq h) \gneqq (h_{q_r} \wedge y_r)$ and the last sequence is orthogonal, as so is $(a_{q_{r-1}} \gneqq h_{q_r})_{r\in\mathbb{N}}$, we have that $(y_r \gneqq (h_{q_r} \wedge y_r))_{r\in\mathbb{N}}$ is orthogonal, too. This contradicts the uniform $\mathcal{G}$-exhaustivity and completes the proof. □

Proposition 11. If $(\mu_n)_{n\in\mathbb{N}}$ is a sequence of uniformly $\mathcal{G}$-exhaustive regular measures, then it is uniformly $\mathcal{F}$-exhaustive.

Proof. By way of contradiction, suppose that there exist $\varepsilon > 0$, an orthogonal sequence $(h_k)_{k\in\mathbb{N}}$ in $\mathcal{F}$ and a subsequence μ_{k_i} such that $|\mu_{k_i}(h_i)| > \varepsilon$ for each $i \in \mathbb{N}$. We may write

$$|\mu_i(h_i)| > \varepsilon. \qquad (*)$$

Put $b_n := \bigoplus_{k=1}^{n} h_k$ for every $n \in \mathbb{N}$. By Proposition 10 choose $\sigma > 0$, $a_1 \in \mathcal{G}$ and $f_1 \in \mathcal{F}$ such that $b_1 \leq a_1 \leq f_1$ and

$$\sup_i \tilde{\mu}_i(f_1 \gneqq b_1) < \frac{\sigma}{2}.$$

Analogously, there exist $a_2 \in \mathcal{G}$ and $f_2 \in \mathcal{F}$ such that $b_2 \vee f_1 \leq a_2 \leq f_2$ and

$$\sup_i \tilde{\mu}_i(f_2 \gneqq (b_2 \vee f_1)) < \frac{\sigma}{4}.$$

Observe that $f_2 \gneqq b_2 \leq (f_2 \gneqq (b_2 \vee f_1)) \bigoplus (f_1 \gneqq b_1)$.

Then if $y \leq f_2 \gneqq b_2$ we can choose $y_1 \leq f_2 \gneqq (b_2 \vee f_1)$ and $y_2 \leq f_1 \gneqq b_1$ such that $y = y_1 \oplus y_2$. So $\tilde{\mu}_i(f_2 \gneqq b_2) < \frac{\sigma}{2} + \frac{\sigma}{4}$.

We continue choosing $a_p \in \mathcal{G}$ and $f_p \in \mathcal{F}$ for $p = 1, \ldots, n$ such that

$$b_p \vee f_{p-1} \leq a_p \leq f_p$$

and

$$\sup_i \tilde{\mu}_i(f_p \gneqq b_p) < \sum_{q=1}^{p} \frac{\sigma}{2^q}.$$

Let $a_{n+1} \in \mathcal{G}$ and $f_{n+1} \in \mathcal{F}$ such that $b_n \vee f_n \leq a_{n+1} \leq f_{n+1}$ and

$$\sup_i \tilde{\mu}_i(f_{n+1} \gneqq (b_{n+1} \vee f_n)) < \frac{\sigma}{2^{n+1}}.$$

Observe that $f_{n+1} \gneqq b_{n+1} \leq f_{n+1} \gneqq (b_{n+1} \vee f_n) \oplus (f_n \gneqq b_n)$.

So, for $y \leq f_{n+1} \gneqq b_{n+1}$ we can choose $y_1 \leq f_n \gneqq b_n$ and $y_2 \leq f_{n+1} \gneqq (b_{n+1} \vee f_n)$ with $y = y_1 \oplus y_2$, hence $\sup_i \tilde{\mu}_i(f_{n+1} \gneqq b_{n+1}) < \sum_{q=1}^{n+1} \frac{\sigma}{2^q}$.

In this way we can construct, by induction, two sequences $(a_n)_{n\in\mathbb{N}}$ in $\mathcal{G}$ and $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{F}$ such that $b_n \leq a_n \leq f_n \leq a_{n+1}$ and $\sup_i \tilde{\mu}_i(f_n \gneqq b_n) < \sum_{q=1}^{n} \frac{\sigma}{2^q}$.

For every $n \in \mathbb{N}$

$$h_{n+1} \leq a_{n+1} \gneqq b_n \leq (a_{n+1} \gneqq f_n) \oplus (f_n \gneqq b_n).$$

Then there exists an element $y_n \leq a_{n+1} \gneqq f_n$ with

$$|\mu_i(h_{n+1})| \leq |\mu_i(y_n)| + \sum_{q=1}^{n} \frac{\sigma}{2^q} < |\mu_i(y_n)| + \sigma. \qquad (**)$$

By Lemma 9 we can also find $a'_n,\ a''_n \in \mathcal{G}_{a_{n+1} \gneqq f_n}$ such that

$$|\mu_i(y_n)| \leq |\mu_i(a'_n)| + |\mu_i(a''_n)| + \sigma.$$

As the sequence $(a_{n+1} \gneqq f_n)_{n\in\mathbb{N}}$ is orthogonal by Lemma 2, the sequences $(a'_n)_{n\in\mathbb{N}}$ and $(a''_n)_{n\in\mathbb{N}}$ are orthogonal, too. By the $\mathcal{G}$-uniform exhaustivity of $(\mu_n)_{n\in\mathbb{N}}$ there exists $m \in \mathbb{N}$ such that $\sup_i |\mu_i(y_n)| \leq 3\sigma$ for $n \geq m$.

From $(**)$ we derive $\sup_i |\mu_i(h_n)| < 4\sigma$ for $n \geq m$, a contradiction with $(*)$ if we choose $\sigma < \frac{\varepsilon}{4}$. □

Proposition 12. Let $\mu_n : L \to G$ be a sequence of uniformly exhaustive regular measures. Then it is uniformly regular.

P r o o f. Apply [8, 6.2] for $\rho = \sup_n \tilde{\mu}_n$-topology. □

6. THE THEOREM

Theorem 13. Let G be a Hausdorff complete topological group, let L be an MV-algebra and let $\mathcal{G}$ be a closed sublattice of L closed under countable sums. Let $\mu_n : L \to G$ be a sequence of $\mathcal{G}$-exhaustive regular measures converging on every element of $\mathcal{G}$. Then the sequence is converging on every element of L and it is uniformly exhaustive and uniformly regular. Therefore the limit of $(\mu_n)_{n\in\mathbb{N}}$ is exhaustive and regular.

P r o o f. We may suppose that G is seminormed.

Thanks to Theorem 5, the sequence is uniformly $\mathcal{G}$-exhaustive.

We shall show that for each element $a \in L$, the sequence $(\mu_n(a))_{n\in N}$ is a Cauchy sequence; that's enough: apply Proposition 11 for obtaining uniform exhaustivity and apply Proposition 12 for obtaining uniform regularity.

Let $\varepsilon > 0$; let $f \in \mathcal{F}$ and $g \in \mathcal{G}$ satisfying $f \le a \le g$ and $\sup_i \tilde{\mu}_i(g \gneqq f) < \frac{\varepsilon}{3}$. Since $(\mu_n(g))_{n\in\mathbb{N}}$ is a Cauchy sequence, there exists $k \in \mathbb{N}$ such that $|\mu_i(g) - \mu_j(g)| < \frac{\varepsilon}{3}$ for every i, $j \ge k$. Then for such integers

$$|\mu_i(a) - \mu_j(a)| = |\mu_i(g) - \mu_j(g) + \mu_j(g \gneqq a) - \mu_i(g \gneqq a)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

This proves that $(\mu_n(a))_{n\in\mathbb{N}}$ is a Cauchy sequence and this completes the proof. □

(Received November 12, 2008.)

REFERENCES

[1] G. Barbieri and H. Weber: Measures on clans and on MV-algebras. In: Handbook of Measure Theory, North-Holland, Amsterdam 2002, pp. 911–945.

[2] J. K. Brooks: On a theorem of Dieudonné. Adv. in Math. *36* (1980), 2, 165–168.

[3] D. Butnariu and E. P. Klement: Triangular norms and some applications to measure and game theory. In: Fuzzy Approach to Reasoning and Decision-Making (Bechyně, 1990), pp. 89–105; Theory Decis. Lib. Ser. D System Theory, Knowledge Engineering and Problem Solving, 8, Kluwer Academic Publishers, Dordrecht 1992.

[4] P. de Lucia and E. Pap: Cafiero approach to the Dieudonné type theorems. J. Math. Anal. Appl. *337* (2008), 2, 1151–1157.

[5] J. Dieudonné: Sur la convergence des suites de mesures de Radon (in French). Anais Acad. Brasil. Ci. *23* (1951), 21–38.

[6] A. Dvurečenskij and S. Pulmannová: New trends in quantum structures. (Mathematics and its Applications 516.) Kluwer Academic Publishers, Dordrecht; Ister Science, Bratislava 2000.

[7] O. Nikodým: Sur les suites convergentes de fonctions parfaitement additives d'ensemble abstrait (in French). Monatsh. Math. Phys. *40* (1933), 1, 427–432.

[8] H. Weber: On modular functions. Funct. Approx., Comment. Math. *24* (1996), 35–52.

Giuseppina Barbieri, Dipartimento di Matematica e Informatica, Università di Udine, Via delle Scienze 206, 33100 Udine. Italy.
e-mail: giuseppina.barbieri@dimi.uniud.it