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ATOMICITY OF LATTICE EFFECT ALGEBRAS AND THEIR SUB–LATTICE EFFECT ALGEBRAS

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We show some families of lattice effect algebras (a common generalization of orthomodular lattices and MV-effect algebras) each element E of which has atomic center $C(E)$ or the subset $S(E)$ of all sharp elements, resp. the center of compatibility $B(E)$ or every block M of E. The atomicity of E or its sub-lattice effect algebras $C(E)$, $S(E)$, $B(E)$ and blocks M of E is very useful equipment for the investigations of its algebraic and topological properties, the existence or smearing of states on E, questions about isomorphisms and so. Namely we touch the families of complete lattice effect algebras, or lattice effect algebras with finitely many blocks, or complete atomic lattice effect algebra E with Hausdorff interval topology.

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1. INTRODUCTION, BASIC DEFINITIONS AND FACTS

Lattice effect algebras generalize orthomodular lattices including non-compatible pairs of elements [10] and MV-algebras including unsharp elements [2]. Effect algebras were introduced by D. Foulis and M. K. Bennet [4] as a generalization of the Hilbert space effects, that means self-adjoint operators between zero and identity operator on a Hilbert space. They are providing an instrument for studying quantum effects that may be unsharp, and they may have importance in the investigation of the phenomenon of uncertainty.

Definition 1.1. A partial algebra $(E; \oplus, 0, 1)$ is called an *effect algebra* if 0, 1 are two distinct elements and $\oplus$ is a partially defined binary operation on E which satisfy the following conditions for any $x, y, z \in E$:

(Ei) $x \oplus y = y \oplus x$ if $x \oplus y$ is defined,

(Eii) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ if one side is defined,

(Eiii) for every $x \in E$ there exists a unique $y \in E$ such that $x \oplus y = 1$ (we put $x' = y$), (Eiv) if $1 \oplus x$ is defined then $x = 0$.

We often denote the effect algebra $(E; \oplus, 0, 1)$ briefly by E. On every effect algebra E the partial order $\leq$ and a partial binary operation $\ominus$ can be introduced as follows:

$$x \leq y \text{ and } y \ominus x = z \text{ iff } x \oplus z \text{ is defined and } x \oplus z = y\,.$$

If E with the defined partial order is a lattice (a complete lattice) then $(E; \oplus, 0, 1)$ is called a *lattice effect algebra (a complete lattice effect algebra)*.

Definition 1.2. Let E be an effect algebra. Then $Q \subseteq E$ is called a *sub-effect algebra* of E if

(i) $1 \in Q$

(ii) if out of elements $x, y, z \in E$ with $x \oplus y = z$ two are in Q, then $x, y, z \in Q$.

If E is a lattice effect algebra and Q is a sub-lattice and a sub-effect algebra of E then Q is called a *sub-lattice effect algebra* of E.

Note that a sub-effect algebra Q (sub-lattice effect algebra Q) of an effect algebra E (of a lattice effect algebra E) with inherited operation $\oplus$ is an effect algebra (lattice effect algebra) in its own right.

Important sub-lattice effect algebras of a lattice effect algebra E are

(i) $S(E) = \{x \in E \mid x \wedge x' = 0\}$ a *set of all sharp elements of* E (see [6, 7]), which is an orthomodular lattice (see [8]).

(ii) Maximal subsets of pairwise compatible elements of E called *blocks* of E (see [22]), which are in fact maximal sub-MV-algebras of E. Here, x, $y \in E$ are called *compatible* ($x \leftrightarrow y$ for short) if $x \vee y = x \oplus (y \ominus (x \wedge y))$ (see [12] and [3]).

(iii) The *center of compatibility* $B(E)$ of E, $B(E) = \bigcap\{M \subseteq E \mid M \text{ is a block of } E\} = \{x \in E \mid x \leftrightarrow y \text{ for every } y \in E\}$ which is in fact an MV-algebra (MV-effect algebra).

(iv) The *center* $C(E) = \{x \in E \mid y = (y \wedge x) \vee (y \wedge x') \text{ for all } y \in E\}$ of E which is a Boolean algebra (see [5]). In every lattice effect algebra it holds $C(E) = B(E) \cap S(E)$ (see [19] and [20]).

For an element x of an effect algebra E we write $\text{ord}(x) = \infty$ if $nx = x \oplus x \oplus \cdots \oplus x$ (n-times) exists for every positive integer n and we write $\text{ord}(x) = n_x$ if n_x is the greatest positive integer such that $n_x x$ exists in E. An effect algebra E is *Archimedean* if $\text{ord}(x) < \infty$ for all $x \in E$, $x \neq 0$.

A minimal nonzero element of an effect algebra E is called an *atom* and E is called *atomic* if under every nonzero element of E there is an atom. Properties of the set of all atoms in a lattice effect algebra E are in several cases substantial for the algebraic structure of E. For instance, the “Isomorphism theorem based on atoms" for Archimedean atomic lattice effect algebras can be proved [15]. Further, the atomicity of the center $C(E)$ of E gives us the possibility to decompose E into subdirect product (resp. direct product for complete E) of irreducible effect algebras [27]. Moreover, in such case if $S(E) = C(E)$ then E is a direct product of horizontal sums of finite chains (see [30]) and the existence of completely additive states on E follows from that. If $S(E)$ is a Boolean algebra (not necessarily equal to $C(E)$, e. g., if there exists a pseudocomplementation on E) and E is complete then every (o)-continuous state ω on the Boolean algebra $S(E)$ can be extended onto E (see [32] and [33, Theorem 6.4]). Nevertheless, there exist atomic lattice effect algebras E with non-atomic $S(E) = C(E)$ (even for MV-effect algebra E, see [14]).

The aim of this paper is to show some families of atomic lattice effect algebras in which $S(E)$ or $C(E)$, resp. $B(E)$ or every block M of E is atomic for every E of these families.

2. SETS OF SHARP ELEMENTS OF ARCHIMEDEAN ATOMIC LATTICE EFFECT ALGEBRAS

For $x \in E$ and $Y \subseteq E$ we write $x \leftrightarrow Y$ iff $x \leftrightarrow y$ for all $y \in Y$. If every two elements are compatible then E is called an MV-effect algebra. In fact, every MV-effect algebra can be organized into an MV-algebra if we extend the partial $\oplus$ into total operation by setting $x \oplus y = x \oplus (x' \wedge y)$ for all $x, y \in E$ (also conversely, restricting total $\oplus$ into partial $\oplus$ for only $x, y \in E$ with $x \leq y'$ we obtain an MV-effect algebra).

In [22] it was proved that every lattice effect algebra is a set-theoretical union of MV-effect algebras called blocks. *Blocks* are maximal subsets of pairwise compatible elements of E, under which every subset of pairwise compatible elements is by Zorn's Lemma contained in a maximal one. Further, blocks are sub-lattices and sub-effect algebras of E and hence maximal sub-MV-effect algebras of E. Thus an MV-effect algebra is a lattice effect algebra with a unique block. A lattice effect algebra with finitely many blocks is called *block-finite*.

An orthomodular lattice L (see [10]) can be organized into a lattice effect algebra by setting $x \oplus y = x \vee y$ for every pair $x, y \in L$ such that $x \leq y^{\perp}$. This is the original idea of G. Boole, who supposed that $x + y$ denotes the logical disjunction of x and y when the logical conjunction $xy = 0$. Lattice effect algebras generalize orthomodular lattices (including Boolean algebras) if we assume existence of unsharp elements $x \in E$, meaning that $x \wedge x' \neq 0$. On the other hand the *set* $S(E) = \{x \in E \mid x \wedge x' = 0\}$ *of all sharp elements* of a lattice effect algebra E is an orthomodular lattice [8]. In this sense a lattice effect algebra is a "smeared" orthomodular lattice, while an MV-effect algebra is a "smeared" Boolean algebra.

Remark 2.1. Recall that elements x, y of an orthomodular lattice $(L; \vee, \wedge, ^{\perp}, 0, 1)$ are called *compatible* if $x = (x \wedge y) \vee (x \wedge y^{\perp})$ and maximal subsets of pairwise compatible elements of L called *blocks* are in fact maximal Boolean subalgebras of L (see [10, p. 23, 37]).

Lemma 2.2. If $(L; \oplus, 0, 1)$ is a lattice effect algebra derived from the orthomodular lattice L then elements x, y of the derived lattice effect algebra are compatible iff x, y are compatible in the orthomodular lattice $(L; \vee, \wedge, ^{\perp}, 0, 1)$.

Proof. If the lattice effect algebra $(L; \oplus, 0, 1)$ is derived from the orthomodular lattice L then for every $x \in L$ we have $x \wedge x' = 0$. If moreover $x \leftrightarrow y$ in the effect algebra then $y \ominus (x \wedge y) = (x \vee y) \ominus x \leq (x \vee y) \wedge x' = (x' \wedge x) \vee (x' \wedge y) = x' \wedge y$ since $x' \leftrightarrow x, y$ (see [8]). Thus $y \leq (x \wedge y) \oplus (x' \wedge y) = (x \wedge y) \vee (x' \wedge y) \leq y$. It follows that $x \leftrightarrow y$ in the orthomodular lattice L.

Conversely, if $y = (x \wedge y) \vee (x' \wedge y)$ then $y = (x \wedge y) \oplus (x' \wedge y)$ and hence $y \ominus (x \wedge y) = x' \wedge y$. Moreover, $x \vee y = x \vee (x \wedge y) \vee (x' \wedge y) = x \vee (x' \wedge y) = x \oplus (x' \wedge y)$, which gives that $(x \vee y) \ominus x = x' \wedge y$. This proves that $y \ominus (x \wedge y) = (x \vee y) \ominus x$ and hence $x \leftrightarrow y$ in the derived lattice effect algebra $(L; \oplus, 0, 1)$. □

It follows that blocks in the orthomodular lattice $(L; \vee, \wedge, ^{\perp}, 0, 1)$ coincide with blocks of the derived lattice effect algebra $(L; \oplus, 0, 1)$.

Lemma 2.3. Let $(E; \oplus, 0, 1)$ be an Archimedean atomic lattice effect algebra. Then

(i) [24, Theorem 3.3] To every nonzero element $x \in E$ there are mutually distinct atoms $a_\alpha \in E$ and positive integers k_α, $\alpha \in \mathcal{E}$ such that

$$x = \bigoplus \{k_\alpha a_\alpha : \alpha \in \mathcal{E}\} = \bigvee \{k_\alpha a_\alpha : \alpha \in \mathcal{E}\},$$

and $x \in S(E)$ iff $k_\alpha = n_{a_\alpha} = \mathrm{ord}(a_\alpha)$ for all $\alpha \in \mathcal{E}$.

(ii) [13, Theorem 8] A block M of E is atomic iff there exists a maximal pairwise compatible set A of atoms of E such that $A \subseteq M$ and if M_1 is a block of E with $A \subseteq M_1$ then $M = M_1$. Moreover, for $x \in E$ it holds $x \in M$ iff $x \leftrightarrow a$ for all $a \in A$.

Proposition 2.4. Let E be an Archimedean atomic lattice effect algebra. Then

(i) If $a \in E$ is an atom of E then $n_a a \in S(E)$ and $ka \notin S(E)$ for all k, $0 < k < n_a$.

(ii) If $p \in S(E)$ is an atom of $S(E)$ then there exists an atom a of E such that $p = n_a a$.

(iii) If $a, b \in E$, $a \neq b$ are atoms of E then $ka \leq lb$, $1 \leq k \leq n_a, 1 \leq l \leq n_b$ implies $a \not\leftrightarrow b$ (a and b are non-compatible).

Proof. (i), (ii): It follows easily by Lemma 2.3 (i).
(iii): It follows from [31, Theorem 2.4 (ii)]. □

Example 2.5. The next example shows that the converse assertion of Proposition 2.4 (ii) is not true in general. Really, assume $E = \{0, a, a', b, 1 = a \oplus a' = 2b\}$ i. e. E is a horizontal sum of a 4-element Boolean algebra and a 3-element chain. Here the element b is an atom of E, but $2b = n_b b = 1$ is not an atom of $S(E) = \{0, a, a', 1\}$.

The well known fact is that there exists an atomic orthomodular lattice (hence its derived lattice effect algebra is Archimedean and atomic) with a non-atomic block (see [1]). Nevertheless, if a lattice effect algebra E has only finitely many blocks, we can prove the following statement.

Theorem 2.6. Let E be an Archimedean atomic block-finite lattice effect algebra. Then

(i) $S(E)$ is a block-finite atomic orthomodular lattice.

(ii) Every block of $S(E)$ is atomic.

Proof. (i): By [8, Theorem 3.3, (c)] in every lattice effect algebra E the set $S(E)$ is an orthomodular lattice.

Let $B \subseteq S(E)$ be a block of $S(E)$. Then there exists a block M of E with $B \subseteq M$, which implies $B = M \cap S(E)$. This proves that $S(E)$ is a block-finite.

By Lemma 2.3 (i), for every atom $a \in E$ we have $n_a a \in S(E)$. Assume that there exists an infinite set $\{a_1, a_2, a_3, \dots\}$ of atoms of E such that $n_1 a_1 > n_2 a_2 > n_3 a_3 > \dots$. Then by Proposition 2.4 (iii), for every $i \neq k$ we have that $a_i \not\leftrightarrow a_k$ and consequently for blocks M_i, M_k of E such that $a_i \in M_i$ and $a_k \in M_k$ we must have $M_i \neq M_k$, which contradicts to finite number of blocks of E. This proves, according to Proposition 2.4 (ii) that for every $x \in S(E)$, $x \neq 0$ there exists an atom $a \in E$ such that $n_a a \leq x$ and $n_a a$ is an atom of $S(E)$.

(ii): By [17, Theorem 1.2] every block of atomic block-finite orthomodular lattice is atomic. Thus the statement follows from (i). □

In [25, Theorem 2.2] and [28] it was proved that every Archimedean atomic distributive (including MV-effect) lattice effect algebra has $S(E) = C(E)$ being an atomic Boolean algebra. Using this we obtain

Theorem 2.7. Let E be an Archimedean atomic lattice effect algebra such that every block of E is atomic. Then $S(E)$ is an atomic orthomodular lattice and every block of $S(E)$ is atomic.

Proof. Assume that $B \subseteq S(E)$ be a block of $S(E)$. In view of Remark 2.1 there exists a block M of E such that $B \subseteq M$, as B is a pairwise compatible subset of elements in E. It follows that $B = B \cap S(E) \subseteq M \cap S(E)$ and hence $B = M \cap S(E)$ by maximality of B in $S(E)$, as $M \cap S(E)$ is a pairwise compatible subset of $S(E)$.

By the assumption M is an atomic Archimedean MV-effect algebra, which implies that the centrum $C(M)$ of M is atomic. We conclude that $B = M \cap S(E) = S(M) = C(M)$ is an atomic Boolean algebra. This proves that $S(E)$ is atomic because every atom of its blocks is also an atom of $S(E)$. □

Recall that the *interval topology* τ_i on a bounded lattice L is a topology for which complements of finite unions of closed intervals generate an open base. Hence τ_i is the coarsest topology on L in which every closed interval is a closed set.

It is well known that a complete Boolean algebra B is atomic iff the interval topology τ_i on B is Hausdorff (see [11, 34]).

Theorem 2.8. The set $S(E)$ of every complete atomic lattice effect algebra E with Hausdorff interval topology is an atomic orthomodular lattice.

Proof. If E is a complete atomic lattice effect algebra then $S(E)$ is a complete orthomodular lattice [8] and every block B of $S(E)$ is a complete Boolean algebra. Moreover, the interval topology $\tau_i^B = \tau_i^E \cap B$. Since τ_i^E and hence also τ_i^B is Hausdorff, we obtain that every block B of $S(E)$ is atomic. It follows that $S(E)$ is atomic because every atom of a block of $S(E)$ is also an atom of $S(E)$. □

Note that a complete lattice effect algebra E with Hausdorff interval topology (even MV-effect algebra) need not be atomic. For example, standard effect algebra: interval of real numbers $E = [0,1] \subseteq \mathbb{R}$ with $a \oplus b = a + b$ iff $a + b \leq 1$, $a, b \in E$ is an MV-effect algebra with $S(E) = \{0, 1\}$. Clearly, the interval topology τ_i is Hausdorff. Nevertheless, E is not atomic.

In [18] it was proved that the interval topology τ_i on a complete lattice L is Hausdorff iff L has separated intervals, i.e., if given any two disjoint intervals $[a, b], [c, d] \subseteq L$, the lattice L can be covered by a finite number of closed intervals each of which is disjoint with at least one of the intervals $[a, b]$ and $[c, d]$.

By Theorem 2.8, *every complete lattice effect algebra E with separated intervals has atomic $S(E)$.*

3. CENTERS OF COMPATIBILITY OF ATOMIC LATTICE EFFECT ALGEBRAS AND ATOMIC BLOCKS

In every lattice effect algebra $(E; \oplus, 0, 1)$ the *center of compatibility* $B(E) = \bigcap\{M \subseteq E \mid M$ is a block of $E\}$ is an MV-effect algebra which is a a sub-lattice and sub-effect algebra of E. Since every block M of a complete lattice effect algebra E is a complete sub-lattice of E (all infima and suprema of subsets of M coincide with those in E, [22]), also $B(E)$ is a complete sub-lattice effect algebra of E.

Moreover, for the center [5] $C(E) = \{x \in E \mid y = (y \wedge x) \vee (y \wedge x')$ for all $y \in E\}$ of a lattice effect algebra E we have $C(E) = B(E) \cap S(E)$, as $x \in C(E)$ iff $x \leftrightarrow E$ and $x \wedge x' = 0$ ([19, Theorem 2.5, (iv)]. Thus $C(S(E)) = B(S(E)) \cap S(E) = B(S(E)) = \bigcap\{B \subseteq S(E) \mid B$ is a block of $S(E)\}$. Since by [8], in a complete lattice effect algebra E, $S(E)$ is a complete sub-lattice effect algebra of E, we obtain that so is $C(E)$.

In [9] it was proved that $C(E)$ of any orthocomplete (meaning that every orthogonal set of elements has the sum which is the supremum of sums of all finite subsets) atomic effect algebra E is atomic. Clearly every complete lattice effect algebra is also orthocomplete. It follows that in every complete atomic lattice effect algebra E the center $C(E)$ is atomic and hence E is isomorphic to a direct product $\prod\{[0, p_\kappa] \mid p_\kappa \in E$ is an atom of $C(E), \kappa \in H\}$, where $[0, p_\kappa]$ are complete atomic *irreducible* lattice effect algebras, meaning that $C([0, p_\kappa]) = \{0, p_\kappa = 1_\kappa\}$ (see [26, Lemma 4.3]).

Lemma 3.1. [13] Let E be an Archimedean atomic lattice effect algebra. Then

(i) $E = \bigcup\{M \subseteq E \mid M$ is an atomic block of $E\}$.

(ii) $B(E) = \bigcap\{M \subseteq E \mid M$ is an atomic block of $E\}$.

Theorem 3.2. Let E be an irreducible complete atomic lattice effect algebra. Then either $B(E) = C(E) = \{0,1\}$ or $E = B(E) = \{0, a, 2a, \ldots, 1 = n_a a\}$ is a finite chain.

Proof. Since E is a complete atomic lattice effect algebra, by "Basic decomposition of elements" (shortly "BDE", see [29, Theorem 3.3]) for every $x \in E$, $x \neq 0$ there exists a unique $w_x \in S(E)$ and a unique set $\{a_\alpha \mid \alpha \in \mathcal{H}\}$ of atoms of E and unique positive integers $k_\alpha \neq \mathrm{ord}(a_\alpha)$, $\alpha \in \mathcal{H}$ such that $x = w_x \oplus (\bigoplus\{k_\alpha a_\alpha \mid \alpha \in \mathcal{H}\})$.

Let $x \in B(E) \setminus \{0,1\}$. Since, by Lemma 3.1 (ii), $B(E) = \bigcap\{M \subseteq E \mid M$ is an atomic block of $E\}$ and every atomic block M of E is a complete and hence Archimedean atomic lattice effect algebra, we obtain that $w_x \in M$ and $\{a_\alpha \mid \alpha \in \mathcal{H}\} \subseteq M$ by "BDE" in M. It follows that $w_x \in B(E) \cap S(E) = C(E)$ and therefore $w_x = 0$. Since $a_\alpha \in B(E)$ for all $\alpha \in \mathcal{H}$, we have that $n_{a_\alpha} a_\alpha \in B(E) \cap C(E)$. Hence $n_{a_\alpha} a_\alpha = 1 = n_{a_\beta} a_\beta$ for all $\alpha, \beta \in \mathcal{H}$. Since $a_\alpha \leftrightarrow a_\beta$, we have by Lemma 2.4 (iii) that $a_\alpha = a_\beta$ i.e. $\mathcal{H} = \{\alpha_0\}$ for a suitable index α_0. Hence $x = k_{\alpha_0} a_{\alpha_0}$. Now, let $y \in B(E) \setminus \{0,1\}$. By the same arguments we have that $y = lc$ for some atom $c \in B(E)$ and a positive integer $l < n_c$. Hence $1 = n_{a_{\alpha_0}} a_{\alpha_0} = n_c c$ and therefore $a_{\alpha_0} = c$. This proves that either $B(E) = \{0,1\}$ or $B(E) = \{0, a, 2a, \ldots, 1 = n_a a\}$ for a suitable atom a of E.

Assume now that $B(E) \neq E$ and $B(E) = \{0, a, 2a, \ldots, 1 = n_a a\}$ for a suitable atom a of E. Since E is atomic there exists an atom b of E such that $b \neq a$. Hence, by Proposition 2.4 (iii), $a \wedge b = 0$, $b \leq 1 = n_a a$ implies $a \not\leftrightarrow b$, a contradiction with $a \in B(E)$. This proves that if $B(E) \neq \{0,1\}$ then $B(E) = E = \{0, a, 2a, \ldots, 1 = n_a a\}$ for a suitable atom a of E. □

Theorem 3.3. In every complete atomic effect algebra E the center of compatibility $B(E)$ is atomic.

Proof. By ([26, Lemma 4.3] E is isomorphic to a direct product of irreducible complete atomic lattice effect algebras $E_\kappa, \kappa \in H$ (written $E \cong \prod\{E_\kappa \mid \kappa \in H\}$). Moreover, because operations on the cartesian product are defined "componentwise" we have that $B(E) \cong \prod\{B(E_\kappa) \mid \kappa \in H\})$ (see [27]). In view of Theorem 3.2, every $B(E_\kappa)$ is a finite chain. It follows that $B(E)$ is a complete atomic MV-effect algebra. □

The next example shows that if a complete atomic lattice effect algebra E is irreducible then $S(E)$ need not be an irreducible orthomodular lattice, hence $C(E) \neq C(S(E))$ in general.

Example 3.4. Let E be the complete atomic lattice effect algebra from Example 2.5. Then $S(E) = \{0, a, a', 1\} = C(S(E))$ since $S(E)$ is a Boolean algebra and $C(E) = S(E) \cap B(E) = \{0,1\}$.

4. COMPACTLY GENERATED LATTICE EFFECT ALGEBRAS

Definition 4.1. (1) An element a of a lattice L is called *compact* iff, for any $D \subseteq L$ with $\bigvee D \in L$, if $a \leq \bigvee D$ then $a \leq \bigvee F$ for some finite $F \subseteq D$.

(2) A lattice L is called *compactly generated* iff every element of L is a join of compact elements.

It was proved in [16, Theorem 6] that every compactly generated lattice effect algebra is atomic. If moreover E is Archimedean then every compact element $u \in E$ is finite, meaning that $u = a_1 \oplus a_2 \oplus \cdots \oplus a_n$ for some finite sequence $a_1, a_2, \ldots, a_n$ of atoms of E ([16, Lemma 4]). Compact elements are important in the semantic approach in computer science as primitive elements representing information which cannot be approximated by elements strictly below them. Moreover, desirable cases are those when all elements can be obtained as directed suprema of compact elements because usually they form a smaller subset than original poset. In [16, Theorem 7] it was proved that an Archimedean lattice effect algebra E is compactly generated iff E is atomic and (o)-continuous. The *(o)-continuity* of E means that for any net $(x_\alpha)_{\alpha \in \mathcal{E}}$ of elements of E and any $y \in E$ the implication $x_\alpha \uparrow x \Longrightarrow y \wedge x_\alpha \uparrow y \wedge x$ holds. Here $x_\alpha \uparrow x$ means that $x_{\alpha_1} \leq x_{\alpha_2}$ for every $\alpha_1 \leq \alpha_2$, $\alpha_1, \alpha_2 \in \mathcal{E}$ and $x = \bigvee\{x_\alpha \mid \alpha \in \mathcal{E}\}$.

Lemma 4.2. Let L be a compactly generated lattice. If L is a complete lattice and $D \subseteq L$ is a complete sub-lattice of L then D is compactly generated.

Proof. Let $x \in D$. Since L is compactly generated and $D \subseteq L$, there exists $C \subseteq L$ such that every $u \in C$ is a compact element in L and $x = \bigvee_L C \in D$. Let $u \in C$ be arbitrary and let $u_D = \bigwedge\{z \in D \mid u \leq z\} \in D$. Clearly, $Z = \{z \in D \mid u \leq z\} \neq \emptyset$ since $x \in Z$ and so $u \leq u_D \leq x$.

Assume that $Q \subseteq D$ and $u_D \leq \bigvee_D Q$. Since $u \leq u_D \leq \bigvee_D Q = \bigvee_L Q$, where $Q \subseteq D \subseteq L$ and $u \in C$, we obtain that there is a finite subset $F \subseteq Q$ such that $u \leq \bigvee_E F = \bigvee_D F \in D$ as $F \subseteq Q \subseteq D$. It follows that $\bigvee_D F \in Z$ and hence $u_D \leq \bigvee_D F$, which proves that u_D is compact in D. Further, $x = \bigvee_L C = \bigvee_L \{u \in L \mid u \leq x, u \text{ is compact in } L\} \leq \bigvee_L \{u_D \in D \mid u \leq x, u \text{ is compact in } L\} = \bigvee_D \{u_D \in D \mid u \leq x, u \text{ is compact in } L\} \leq x$, which proves that D is compactly generated. □

Theorem 4.3. Let E be an (o)-continuous and atomic complete lattice effect algebra. Let $G \subseteq E$ be a complete sub-lattice effect algebra of E. Then

(i) G is an (o)-continuous and atomic complete lattice effect algebra.

(ii) $S(E)$, $C(E)$, $B(E)$ and every block of E are (o)-continuous and atomic complete lattice effect algebras.

Proof. (i): By [21, Theorem 3.3], every complete lattice effect algebra is Archimedean. Moreover, by [16, Theorem 7], an Archimedean lattice effect algebra E is compactly generated iff E is atomic and (o)-continuous. Thus E is compactly generated and by Lemma 4.2 G is again compactly generated which gives that G is (o)-continuous and atomic.

(ii): This follows by part (i) since $S(E)$, $C(E)$, $B(E)$ and all blocks of E are complete sub-lattices and sub-effect algebras of E (see [8, 23]). □

Note that examples of compactly generated (hence (o)-continuous and atomic) complete lattice effect algebras are e. g. all complete atomic Boolean algebras and also all complete atomic modular lattice effect algebras (including complete atomic

MV-effect algebras), see [26]. The well known fact is that all distributive lattice effect algebras (including MV-effect algebras and hence also Boolean algebras) are (o)-continuous, but if they are not atomic they cannot be compactly generated.

It is worth to note that, for $G \subseteq E$ in Theorem 4.3, if G and E are two (o)-continuous atomic complete lattice effect algebras such that, for the set $\mathcal{A}_G$ of all atoms of G and for the set $\mathcal{A}_E$ of all atoms of E, we have $\mathcal{A}_G \nsubseteq \mathcal{A}_E$ in many cases.

Example 4.4. Let E be an (o)-continuous atomic complete lattice effect algebra. Let $\mathcal{A}_E$ be the set of all atoms of E. If $G = S(E)$ is a set of all sharp elements then for the set $\mathcal{A}_{S(E)}$ of all atoms of G we have $\mathcal{A}_{S(E)} \subseteq \{n_a a \mid a \in \mathcal{A}_E\}$. If $E \neq S(E)$ then $\mathcal{A}_{S(E)} \nsubseteq \mathcal{A}_E$.

Theorem 4.5. Let E be a block-finite Archimedean atomic lattice effect algebra such that $\widehat{E} = \mathcal{MC}(E)$ is the MacNeille completion of E. Then

(i) Every block of E and of $\widehat{E}$ is atomic.

(ii) To every block $\widehat{M}$ of $\widehat{E}$ there is a block M of E and a maximal set A of atoms of A such that $A \subseteq M \subseteq \widehat{M}$ and $\widehat{M} = \mathcal{MC}(M)$ and M and $\widehat{M}$ are unique blocks of E resp. $\widehat{E}$ with the property $A \subseteq M \subseteq \widehat{M}$.

(iii) $S(\widehat{E})$ is atomic and $S(\widehat{E}) = \mathcal{MC}(S(E))$.

Proof. (i): By [21, Theorem 4.5], for a block-finite Archimedean atomic lattice effect algebra E, its MacNeille completion $\widehat{E} = \mathcal{MC}(E)$ is a complete lattice effect algebra containing E as a sub-lattice effect algebra. By Schmidt characterization of the MacNeille completion [35] E and $\widehat{E}$ have the same set of all atoms, hence $\mathcal{A} = \widehat{\mathcal{A}}$, where $\mathcal{A}$ resp. $\widehat{\mathcal{A}}$ are sets of all atoms of E resp. $\widehat{E}$. By [13] to every maximal set A of pairwise compatible atoms of E there exist a unique block M of E and hence a unique block $\widehat{M}$ of $\widehat{E}$ with $A \subseteq M$ and $A \subseteq \widehat{M}$. Since every atom a of E has $\mathrm{ord}(a) < \infty$, the effect algebra $\widehat{E}$ is also Archimedean and atomic lattice effect algebra.

Moreover, by [21, Theorem 4.4] and by [13], we have that $E = \bigcup^n{}_{k=1} M_k$, where M_k are atomic blocks of E with maximal set A_k of pairwise compatible atoms and $\widehat{E} = \bigcup^n{}_{k=1} \widehat{M}_k$ and again by [13] $\widehat{M}_k$ are atomic blocks of $\widehat{E}$. This yields $\mathcal{A} = \bigcup^n{}_{k=1} A_k$ and $A_k \subseteq M_k \subseteq \widehat{M}_k$.

Let $\widehat{M}$ be a block of $\widehat{E}$. Then $\widehat{M} = \bigcup^n{}_{k=1} \widehat{M}_k \cap \widehat{M}$. Since $\widehat{M}_k \cap \widehat{M} \subseteq \widehat{M}_k$, $\widehat{M}_k$ is a compactly generated complete effect algebra and $\widehat{M}_k \cap \widehat{M}$ is a complete sub-lattice of $\widehat{M}_k$, we obtain by Theorem 4.3 that $\widehat{M}_k \cap \widehat{M}$ is atomic. Let $\widehat{x} \in \widehat{M}$, $\widehat{x} \neq 0$, then there is k_1 such that $\widehat{x} \in \widehat{M}_{k_1} \cap \widehat{M}$ and there exists an atom $\widehat{q}_{k_1}$ of $\widehat{M}_{k_1} \cap \widehat{M}$ with $\widehat{q}_{k_1} \leq \widehat{x}$. Further, either $\widehat{q}_{k_1}$ is an atom of $\widehat{M}$ or there exists $0 \neq \widehat{y}_{k_1} < \widehat{q}_{k_1}$ and thus there exist k_2 such that $\widehat{y}_{k_1} \in \widehat{M}_{k_2} \cap \widehat{M}$ and an atom $\widehat{q}_{k_2}$ of $\widehat{M}_{k_2} \cap \widehat{M}$ with $\widehat{q}_{k_2} \leq \widehat{y}_{k_1} < \widehat{q}_{k_1} \leq \widehat{x}$. Proceeding by induction we obtain that there exists an atom $\widehat{q}$ of $\widehat{M}$ with $\widehat{q} \leq \widehat{x}$, because we have $k \in \{1, 2, \dots, n\}$ a finite set. Thus every block $\widehat{M}$ of $\widehat{E}$ is atomic.

If $M \subseteq E$ is a block of E then there exists a block $\widehat{M}$ of $\widehat{E}$ such that $M \subseteq \widehat{M} \cap E$. This gives that the set A of all atoms of $\widehat{M}$ is a subset of M since $A \subseteq E$. Thus $A \subseteq M$ implies that M is atomic by [13].

(ii): Since blocks M_k and $\widehat{M}_k$ have the same set A_k of all atoms we have that $\widehat{M}_k = \mathcal{MC}(M_k)$ by [35] and [24, Theorem 3.3].

(iii): It follows by Theorem 2.7 that $S(\widehat{E})$ is atomic. Let $\widehat{p} \in S(\widehat{E})$. Then there exists an atom a of $\widehat{E}$ and hence $a \in E$ such that $\widehat{p} = n_a a$. Clearly, $n_a a \in S(E)$. Moreover, $S(E) \subseteq S(\widehat{E})$ by the definition of a sharp element. It follows that $n_a a$ is an atom of $S(E)$ and also conversely by Schmidt characterization of $S(\widehat{E}) = \mathcal{MC}(S(E))$ (the equality follows by [33, Lemma 6.2]). It follows that $S(\widehat{E})$ and $S(E)$ have the same set of all atoms and hence also the same set of finite elements. □

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