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CONGRUENCES AND IDEALS IN LATTICE EFFECT ALGEBRAS AS BASIC ALGEBRAS

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Effect basic algebras (which correspond to lattice ordered effect algebras) are studied. Their ideals are characterized (in the language of basic algebras) and one-to-one correspondence between ideals and congruences is shown. Conditions under which the quotients are OMLs or MV-algebras are found.

Keywords: basic algebra, effect algebra, ideal, congruence

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1. INTRODUCTION

Effect algebras [6] (equivalently, D-posets, [10]) were introduced as abstract models of the set of quantum effects (self-adjoint operators between the zero and identity operator with respect to the usual ordering). Quantum effects represent sharp and unsharp properties of physical systems and play a basic role in the foundations of quantum mechanics. They contain the usual quantum logics (orthomodular posets and lattices) as special subclasses. Also MV-algebras, introduced by Chang [3] as algebraic bases for many-valued logic, are a special subclass of effect algebras. In this paper, we consider lattice ordered effect algebras, which are a common generalization of MV-algebras and orthomodular lattices.

Originally, effect algebras were introduced as partial algebraic structures. Several attempts have been made to represent them as structures with total operations (e. g., [4, 7, 8]). Recently, Chajda, Halaš and Kühr in [2] treated a special subclass of basic algebras, and showed that they are in one-to-one correspondence with lattice ordered effect algebras. In this paper, we refer to the special class of basic algebras as *effect basic algebras*. Making use of the axioms of the effect basic algebras, we find a simple characterization of their ideals. We show that these ideals generate congruences and are kernels of homomorphisms. We find conditions on the ideals under which the quotients are orthomodular lattices and MV-algebras. In the end we show that ideals in effect basic algebras coincide with d-ideals (equivalently, Riesz ideals) in the corresponding lattice effect algebras. Actually, the aim of the paper is to reformulate results that are known from the theory of (lattice) effect algebras into the language of basic algebras and to prove the most important results by the tools of the basic algebra axioms (e. g. Lemmas 2.11 – 2.14). We believe that this can be helpful to understand better the new approach (effect basic algebras) to previously studied structures (lattice effect algebras).

2. DEFINITIONS AND PRELIMINARY RESULTS

Definition 2.1. (Chajda et al. [2]) A *basic algebra* is an algebra $\mathbb{A} = (A; \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the following identities ($1 := \neg 0$):

(BA1) $x \oplus 0 = x$;

(BA2) $\neg\neg x = x$;

(BA3) $x \oplus 1 = 1 \oplus x = 1$;

(BA4) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$;

(BA5) $\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = 1$.

Remark 2.2. A basic algebra is an MV-algebra iff $\oplus$ is commutative and associative.

Lemma 2.3. (Chajda et al. [2, Lemma 3.4]) Every basic algebra satisfies the equations $0 \oplus x = x$ and $\neg x \oplus x = 1$.

Proposition 2.4. (Chajda et al. [2, Prop. 3.5]) Let $\mathbb{A} = (A; \oplus, \neg, 0)$ be a basic algebra. The relation $\leq$ defined by

$$x \leq y \;\Leftrightarrow\; \neg x \oplus y = 1$$

is a partial order on A such that 0 and 1 are the least and the greatest element of A, respectively. Moreover, for every $x, y, z \in A$ we have

(a) $x \leq y$ iff $\neg x \geq \neg y$;

(b) $x \leq y$ implies $x \oplus z \leq y \oplus z$ and $\neg x \oplus z \geq \neg y \oplus z$;

(c) $y \leq x \oplus y$.

Proposition 2.5. (Chajda et al. [2, Prop. 3.6]) For every basic algebra $\mathbb{A} = (A; \oplus, \neg, 0)$, the poset $(A; \leq)$ is a bounded lattice in which

$$x \vee y = \neg(\neg x \oplus y) \oplus y, \text{ and } x \wedge y = \neg(\neg x \vee \neg y).$$

Definition 2.6. An *effect algebra* is a system $\mathbb{E} = (E; +, 0, 1)$ where 0 and 1 are two special elements of E and $+$ is a partial binary operation on E, satisfying the following conditions:

(EA1) $a + b = b + a$ if $a + b$ is defined;

(EA2) $(a+b)+c = a+(b+c)$ if one side is defined;

(EA3) for every $a \in E$ there exists a unique $a' \in E$ such that $a + a' = 1$;

(EA4) if $a+1$ is defined then $a = 0$.

The relation $\leq$ defined by

$$a \leq b \ \Leftrightarrow \ b = a + c$$

for some $c \in E$ is a partial order on E such that 0 and 1 are the least and the greatest element of E, respectively. Such element c is unique and we also write $c = b - a$. If $a+b$ exists, we sometimes write $a \perp b$. If the poset $(E; \leq)$ is a lattice then $\mathbb{E}$ is called a *lattice effect algebra*. The elements $a, b \in E$ are *compatible* iff $a \vee b - b = a - a \wedge b$.

Proposition 2.7. (Chajda et al. [2, Prop. 4.5]) Let $\mathbb{E} = (E; +, 0, 1)$ be a lattice effect algebra. Define

$$x \oplus y := (x \wedge y') + y \text{ and } \neg x := x'.$$

Then $(E; \oplus, \neg, 0)$ is a basic algebra (whose lattice order coincides with the original one).

Proposition 2.8. (Chajda et al. [2, Prop. 4.9, Lemma 4.10]) Let $\mathbb{E} = (E; +, 0, 1)$ be a lattice effect algebra. Then the derived basic algebra $\mathcal{A}(\mathbb{E}) = (E; \oplus, \neg, 0)$ satisfies the quasi-identity

$$x \leq \neg y \text{ and } x \oplus y \leq \neg z \quad \Longrightarrow \quad x \oplus (z \oplus y) = (x \oplus y) \oplus z. \tag{1}$$

The quasi-identity (1) is equivalent to the identity

$$(x \wedge \neg y) \oplus [(\neg(x \oplus y) \wedge z) \oplus y] = (x \oplus y) \oplus (\neg(x \oplus y) \wedge z). \tag{2}$$

Proposition 2.9. [2, Prop. 4.11] Let $\mathbb{A} = (A; \oplus, \neg, 0)$ be a basic algebra satisfying (1) (or (2)). Define a partial addition $+$ on A as follows: $a + b$ is defined iff $a \leq \neg b$ and in this case $a + b := a \oplus b$. Then $\mathcal{E}(\mathbb{A}) = (A; +, 0, 1)$ is a lattice effect algebra.

Moreover, $\mathcal{E}(\mathcal{A}(\mathbb{E})) \equiv \mathbb{E}$ and $\mathcal{A}(\mathcal{E}(\mathbb{A})) \equiv \mathbb{A}$.

In what follows, we will consider basic algebras satisfying (1) (or (2)) and we will call them *effect basic algebras* (eba's in short). If not said otherwise, we write A for $\mathbb{A} = (A; \oplus, \neg, 0)$. We will write $a \perp b$ if $a \leq \neg b$, and say that a and b are orthogonal. We will write $a \leftrightarrow b$ if $\neg(a \vee b) + b = \neg a + a \wedge b$, i. e., if a and b are compatible in the corresponding effect algebra.

Proposition 2.10. [2, Prop. 4.11] In A, if $a \leq \neg b$, then $a \oplus b = b \oplus a$.

If $a \leq \neg b$, we write $a + b$ instead of $a \oplus b$.

Lemma 2.11. If $a \leq b$, then there is $c \in A$ such that $c \perp a$ and $c + a = b$.

P r o o f. If $a \leq b$, then $a \perp \neg b$, $\neg(\neg b + a) \perp a$ and $\neg(\neg b + a) + a = b \vee a = b$. Putting $c = \neg(\neg b + a)$ gives the desired result. □

Lemma 2.12. If $a, b, c \in A$ with $a \leq \neg c, b \leq \neg c$, then

$$c + a \wedge b = (c + a) \wedge (c + b). \tag{3}$$

P r o o f. First, it is clear by Prop. 2.4 and Prop. 2.10 that $c + a \wedge b \leq c + a, c + b$. To show that it is the greatest lower bound of $c + a$ and $c + b$, let w be another lower bound of $c + a$ and $c + b$. As we can write a as $a \wedge \neg c = \neg(\neg a \vee c) = \neg(\neg(c + a) + c)$ and also $b = b \wedge \neg c = \neg(\neg b \vee c) = \neg(\neg(c + b) + c)$, we have $a \geq \neg(\neg w \oplus c)$ and $b \geq \neg(\neg w \oplus c)$. Thus $a \wedge b \geq \neg(\neg w \oplus c)$. Now $c + a \wedge b \geq \neg(\neg w \oplus c) + c = w \vee c \geq w$ and therefore $c + a \wedge b = (c + a) \wedge (c + b)$. □

Lemma 2.13. If $a, b, c \in A$ with $a \leq \neg c, b \leq \neg c$, then

$$c + a \vee b = (c + a) \vee (c + b). \tag{4}$$

P r o o f. Similarly as in the last lemma, we have $c + a \vee b \geq c + a, c + b$. Let $w \geq c + a, c + b$ and let us write $a = \neg(\neg(a + c) + c) \leq \neg(\neg w \oplus c)$ and $b = \neg(\neg(b + c) + c) \leq \neg(\neg w \oplus c)$. Again, we have $a \vee b \leq \neg(\neg w \oplus c)$, so that $c + a \vee b \leq \neg(\neg w \oplus c) + c = w \vee c = w$ and $c + a \vee b = (c + a) \vee (c + b)$. □

Lemma 2.14. If $x, y, z \in A$ and if $x, y \perp z$ and $x + z = y + z$ then $x = y$.

P r o o f. If $x, y \perp z$ then $z \leq \neg x, \neg y$ and $\neg x = \neg x \vee z = \neg(x + z) + z = \neg(y + z) + z = \neg y \vee z = \neg y$. □

Lemma 2.15. For $a, b, c \in A$ such that $a \perp b$:

$$a + b = c \quad \Rightarrow \quad a = \neg(\neg c + b). \tag{5}$$

P r o o f. If $a + b = c$ then $c \geq b$ and $c = c \vee b = \neg(\neg c + b) + b$. Therefore $a + b = \neg(\neg c + b) + b$ and from cancellativity (Lemma 2.14) we get $a = \neg(\neg c + b)$. □

3. IDEALS AND CONGRUENCES IN EFFECT BASIC ALGEBRAS

Definition 3.1. A subset I of A is an *ideal* if (id1) $a, b \in I$ implies $a \oplus b \in I$ and (id2) $a \in I$, $b \in A$ implies $\neg(\neg a \oplus b) \in I$.

Lemma 3.2. If I is an ideal in A, then (i) if $a \in I$ and $b \leq a$ then $b \in I$. (ii) if $a, b \in I$, then $a \vee b \in I$.

P r o o f. (i) Let $a \in I$ and $b \leq a$. Then we have by (id2) $\neg(\neg a \oplus b) \in I$ and also $\neg(\neg a \oplus \neg(\neg a \oplus b)) \in I$. As $a \geq b$, by Prop. 2.10 $\neg(\neg a \oplus \neg(\neg a \oplus b)) = \neg(\neg a + \neg(\neg a + b)) = \neg(\neg(b + \neg a) + \neg a) \in I$. And by Prop. 2.5 $\neg(\neg(b + \neg a) + \neg a) = \neg(\neg b \vee \neg a) = b \wedge a = b$, so that $b \in I$.
(ii) If $a, b \in I$, then $\neg(\neg a \oplus b) \in I$ by (id2), and $a \vee b = \neg(\neg a \oplus b) \oplus b \in I$ by (id1). □

Definition 3.3. The *symmetric difference* of $a, b \in A$ is defined by

$$\Delta(a,b) = \neg(a \wedge b \oplus \neg a \wedge \neg b) = \neg(a \wedge b + \neg a \wedge \neg b). \tag{6}$$

Lemma 3.4. Properties of Δ:

(i) $\Delta(a,b) = \Delta(b,a)$;

(ii) $\Delta(a,b) = \Delta(\neg a, \neg b)$;

(iii) $\Delta(a,b) = 0$ iff $a = b$.

P r o o f. (i) and (ii) are clear from the definition and Proposition 2.10. (iii): $\Delta(a,a) = \neg(a \oplus \neg a) = \neg 1 = 0$. Assume $\Delta(a,b) = 0$. Then $0 = \Delta(a,b) = \neg(a \wedge b + \neg a \wedge \neg b)$, which implies $\neg(a \vee b) + a \wedge b = 1$, this in turn implies $a \vee b \leq a \wedge b$, and from $a \wedge b \leq a, b \leq a \vee b$ we obtain $a = b$. □

Proposition 3.5. For any $a, b \in A$,

$$\Delta(a,b) = \neg((\neg a + a \wedge b) \wedge (\neg b + a \wedge b)) \tag{7}$$

P r o o f. Apply Lemma 2.12 with $\neg a$ replacing a, $\neg b$ replacing b and $a \wedge b$ replacing c. □

Proposition 3.6. Let I be an ideal of A. For any $a, b \in A$, the following are equivalent:

(i) $\Delta(a,b) \in I$;

(ii) there is $k \in A$, $k \perp \neg a, \neg b$ such that $\neg(\neg a + k) \in I, \neg(\neg b + k) \in I$;

(iii) there are $i, j \in I$, $i \perp \neg a, j \perp \neg b$ with $i + \neg a = j + \neg b$.

P r o o f. (i) ⇒ (ii): Assume (i) and put $k := a \wedge b$. Then from $k + \neg a \wedge \neg b \leq k + \neg a, k + \neg b$ we obtain $\neg(\neg a + k), \neg(\neg b + k) \leq \neg(\neg a \wedge \neg b + a \wedge b) \in I$, which implies the desired result.

(ii) ⇒ (iii): Assume (ii) and put $i := \neg(\neg a + k), j = \neg(\neg b + k)$. Then $i, j \in I$ and $i \perp \neg a, j \perp \neg b$. Moreover, $i + \neg a = \neg a + i = \neg a + \neg(\neg a + k) = \neg(k + \neg a) + \neg a = \neg k \vee \neg a = \neg k$. Similarly, $j + \neg b = \neg k$. Hence $i + \neg a = j + \neg b$.

(iii) ⇒ (i): Assume (iii). Put $k := \neg a + i = \neg b + j$. Then we have $k \geq \neg a, \neg b$, so that $a \wedge b \geq \neg k$. Moreover, $\neg a + \neg k = \neg a + \neg(\neg a + i) = \neg(i + \neg a) + \neg a = \neg i \vee \neg a = \neg i$. Similarly, $\neg b + \neg k = \neg j$. Then by Lemma 3.2

$$\begin{aligned} & \neg(\neg a \wedge \neg b + a \wedge b) \\ \leq\ & \neg(\neg a \wedge \neg b + \neg k) \\ =\ & \neg((\neg a + \neg k) \wedge (\neg b + \neg k)) \\ =\ & \neg(\neg a + \neg k) \vee \neg(\neg b + \neg k) = i \vee j \in I. \end{aligned}$$

□

Definition 3.7. Let I be an ideal of A. For $a, b \in A$, define

$$a \sim_I b \;\Leftrightarrow\; \Delta(a,b) \in I. \tag{8}$$

If there is no danger of confusion, we write $a \sim b$ instead of $a \sim_I b$.

Theorem 3.8. For every ideal I, the relation $\sim_I$ is an equivalence.

Proof. Reflexivity and symmetry are ensured by Lemma 3.4 (iii) and (i). To show transitivity, we use Proposition 3.6 (iii). Let $a \sim b$ and $b \sim c$. Then there are $i, j, k, l \in I$ such that $\neg a + i = \neg b + j$ and $\neg b + k = \neg c + l$. As $\neg b \leq \neg j$ and $\neg b \leq \neg k$, we have $\neg b \leq \neg k \wedge \neg j$ and $b \geq k \vee j$. Now $\neg b + k \vee j \geq \neg b + j$, so there is a d such that $\neg b + k \vee j = \neg b + j + d$. From (5) we get $d = \neg(\neg(\neg b + k \vee j) + \neg b + j)$. But $\neg(\neg b + k \vee j) + \neg b = \neg(k \vee j) \vee \neg b = \neg(k \vee j)$. Thus by (id2) we obtain $d = \neg(\neg(k \vee j) + j) \in I$. Similarly $\neg b + k \vee j = \neg c + l + e$ where $e \in I$. The equality $\neg a + i + d = \neg b + j + d = \neg b + k \vee j = \neg c + l + e$ along with $i + d, l + e \in I$ then implies the result $a \sim c$. □

Theorem 3.9. Let I be an ideal of A. For any $a, b, c, d \in A$, $a \sim c$, $b \sim d$ and $a \perp b$, $c \perp d$ implies $a + b \sim c + d$.

Proof. By Prop. 3.6 $a \sim c \;\Rightarrow\; i + \neg a = m + \neg c$, $i, m \in I$ and similarly $b \sim d \;\Rightarrow\; j + \neg b = n + \neg d$, $j, n \in I$. It suffices to show that $\neg(\neg(\neg a + i) + \neg(\neg b + j)) = \neg(a+b) + i + j$, because then we will have $\neg(a+b) + i + j = \neg(\neg(\neg a + i) + \neg(\neg b + j)) = \neg(\neg(\neg c + m) + \neg(\neg d + n)) = \neg(c + d) + m + n$ where $i + j, m + n \in I$.

But this is clear while $a = a \vee i$ (because $\neg a \perp i \;\Rightarrow\; i \leq a$) and $a \vee i = \neg(\neg a + i) + i$ ($\neg(\neg a + i) \perp i$ because $\neg a + i \geq i$). The fact that $\neg(\neg a + i) \perp \neg(\neg b + j)$ is provided by $a \perp b$ and $\neg a + i \geq \neg a$ ($\Rightarrow\; \neg(\neg a + i) \leq a$). Thus $a + b = \neg(\neg a + i) + i + \neg(\neg b + j) + j = \neg(\neg a + i) + \neg(\neg b + j) + i + j$ and by (5) we get the desired result. □

Theorem 3.10. Let I be an ideal of A. For any $a, b, c, d \in A$, if $a \sim b$, $c \sim d$, then $a \vee c \sim b \vee d$.

Proof. It suffices to prove that $a \sim b$ implies $a \vee c \sim b \vee c$ for any $a, b, c \in A$. First we prove the following: $a \sim 0$ implies $a \vee c \sim c$ for any $c \in A$. Indeed, $a \sim 0$ implies $a \in I$, hence $\neg(\neg a \oplus c) \in I$ and $\Delta(a \vee c, c) = \neg(\neg(a \vee c) + c) = \neg(\neg(\neg(\neg a \oplus c) + c) + c) = \neg(\neg a \oplus c) \in I$ (by [2], Lemma 3.8), which entails $a \vee c \sim c$.

Now assume $a \sim b$. Then $r := \neg(\neg a \wedge \neg b + a) \sim 0$. Clearly, $a \leq \neg r$ and $r + a = \neg(\neg(a \vee b) + a) + a = a \vee b \vee a = a \vee b$. Further, $d := \neg(\neg a \wedge \neg c + a) \sim d \vee r$. Since $\neg a \wedge \neg c + a \geq a$, we have $d \leq \neg a$. Moreover, $d + a = \neg(\neg a \wedge \neg c + a) + a = a \vee c \vee a = a \vee c$. So $a \vee c = a + d \sim a + d \vee r = (a + d) \vee (a + r) = a \vee b \vee c$ (Lemma 2.13). Similarly we prove that $b \vee c \sim a \vee b \vee c$, hence $a \vee c \sim b \vee c$. □

Notice, that Theorem 3.9 and Theorem 3.10 together with an obvious fact that if $a \sim b$ then $\neg a \sim \neg b$ (cf. Lemma 3.4 (ii)) give the result: $a \sim b, c \sim d \;\Rightarrow\; a \oplus c \sim b \oplus d$ (Prop. 2.7).

Corollary 3.11. For every ideal I, the relation $\sim_I$ is basic algebra congruence.

Lemma 3.12. If $a \in A$ is such that $\neg a \sim_I 1$ and $\neg a + x \sim_I 1$, then $x \sim_I 0$.

Proof. If $\neg a + x$ exists, then $x \leq a$ and while $a \sim_I 0$, we have $x \in I \Rightarrow x \sim_I 0$. □

Corollary 3.13. Ideals and congruences in effect basic algebras are in one-to-one correspondence.

Proof. By Corollary 3.11 every ideal defines a congruence. We need to prove the opposite. Let $\sim$ be a basic algebra congruence and let $[0]_\sim = \{a; a \sim 0\}$ be its zero class. We show that $I := [0]_\sim$ is an ideal and that for congruence defined by this ideal holds $\sim_I = \sim$.

For $a, b \in I$ we have $a, b \sim 0$ and $a \oplus b \sim 0 \oplus 0 = 0$ and $a \oplus b \in I$, since $\sim$ is a basic algebra congruence. If $a \sim 0$ and $b \in A$ is an arbitrary element, then $\neg a \sim 1$ and $\neg a \oplus b \sim 1 \oplus b = 1$, thus $\neg(\neg a \oplus b) \sim 0$. This proves that I is an ideal.

If $a \sim b$, then $a \wedge b \sim b \wedge b = b$ and $\neg a \sim \neg b$. Also $\neg a \wedge \neg b \sim \neg b$. Thus $a \wedge b + \neg a \wedge \neg b \sim b + \neg b = 1$ and $\Delta(a, b) \sim 0 \Rightarrow \Delta(a, b) \in I \Rightarrow a \sim_I b$. On the other hand, if $a \sim_I b$, then $\neg a + a \wedge b, \neg b + a \wedge b \sim_I 1$. From $\neg a + a \wedge b \leq 1$ we obtain $\neg a + a \wedge b + x = 1$ for some x and by Lemma 3.12 $x \sim_I 0$. Similarly $\neg b + a \wedge b + y = 1$ for some $y \sim_I 0$. Moreover $\neg a + a \wedge b + x = 1 = \neg b + a \wedge b + y$ and by cancellativity $\neg a + x = \neg b + y$ where $x, y \sim_I 0$. So that $\neg a \sim \neg b \Rightarrow a \sim b$. □

As in every variety there is a one-to-one correspondence between homomorphisms and congruences, we wanted to see how this correspondence works on the structure of eba. Define a homomorphism of basic algebras as usual, that is $f : A \to B$ is a homomorphism from the basic algebra A to the basic algebra B iff $f(0) = 0$, $f(a \oplus b) = f(a) \oplus f(b)$ and $\neg f(a) = f(\neg a)$. Notice, that this implies that also other operations are preserved ($+, \wedge$ and $\vee$).

Let us have a homomorphism f of basic algebras and let $\ker f := [0]_f = \{a \in A; f(a) = 0\}$. Then $\ker f$ is an ideal in A. Indeed $a, b \in [0]_f \Rightarrow a \oplus b \in [0]_f$ and $a \in [0]_f \Rightarrow f(\neg a) = 1 \Rightarrow f(\neg a \oplus b) = 1 \oplus f(b) = 1$. Thus $\neg(\neg a \oplus b) \in \ker f$.

Now let $a \sim b \;:\Leftrightarrow\; f(a) = f(b)$ for a homomorphism f. Then clearly $\sim$ is a congruence ($a \sim b \Leftrightarrow \neg a \sim \neg b$ and $a \sim b, c \sim d \Rightarrow a \oplus c \sim b \oplus d$) and $[0]_f = [0]_\sim$. Conversely, if we have a congruence $\sim$, we will define a mapping f such that $f(a) = [a]_\sim$ for every $a \in A$. Then f is a homomorphism of basic algebras A and B, where $B := A/I$ and I is the ideal defined by $\sim$. Indeed, we have $f(a) \oplus f(b) = [a]_\sim \oplus [b]_\sim = [a \oplus b]_\sim = f(a \oplus b)$ and $f(\neg a) = [\neg a]_\sim = \neg[a]_\sim = \neg f(a)$.

4. QUOTIENTS AS MV-ALGEBRAS AND OML'S

Definition 4.1. An ideal I is called *prime*, if $a \wedge b \in I \Rightarrow a \in I$ or $b \in I$.

Proposition 4.2. I is a prime ideal of A iff A/I is linearly ordered.

P r o o f. Let I be a prime ideal in A. By Lemma 2.13 $(\neg a + a \wedge b) \vee (\neg b + a \wedge b) = \neg a \vee \neg b + a \wedge b = \neg(a \wedge b) + a \wedge b = 1$ and therefore $\neg(\neg a + a \wedge b) \wedge \neg(\neg b + a \wedge b) = 0$. So we have $\neg(\neg a + a \wedge b) \in I$ or $\neg(\neg b + a \wedge b) \in I$. As $\neg(a \wedge b + \neg a) = \neg(a \wedge a \wedge b + \neg a \wedge (\neg a \vee \neg b)) = \Delta(a, a \wedge b)$, we have that either $a \sim a \wedge b$ or $b \sim a \wedge b$. But then either $[a] = [a] \wedge [b] \Rightarrow [a] \leq [b]$ or $[b] = [a] \wedge [b] \Rightarrow [b] \leq [a]$.

On the other hand, let A/I be linearly ordered and assume that $a \wedge b \in I$. Then $[a] \leq [b]$ implies $[a] = [a] \wedge [b] = [a \wedge b]$ while $[b] \leq [a]$ implies $[b] = [a] \wedge [b] = [a \wedge b]$. So we have $a \sim a \wedge b$ or $b \sim a \wedge b$ which means that either a or b is in I. □

We also note that if A is linearly ordered, than A is an MV-algebra. Indeed, A is an MV-algebra iff the operation $\oplus$ is commutative [2, Corollary 4.7]. So that if A is linearly ordered and $a, b \in A$, then either $a \leq \neg b$ or $\neg b \leq a$. In either case $a \leftrightarrow b$ in the corresponding effect algebra, and so $a \oplus b = b \oplus a$ ([2, Theorem 4.6]).

Definition 4.3. *Commutator* of elements $a, b \in A$ is the element $\mathrm{com}(a, b) := \Delta(a \oplus b, b \oplus a)$.

Theorem 4.4. *If I is an ideal of A then A/I is an MV-algebra iff $\mathrm{com}(a, b) \in I\ \forall\, a, b \in A$.*

P r o o f. Assume $\mathrm{com}(a, b) \in I$. Then $a \oplus b \sim b \oplus a$. Therefore $[a] \oplus [b] = [a \oplus b] = [b \oplus a] = [b] \oplus [a]$.
Conversely, if A/I is an MV-algebra then for every $a, b \in A$: $[a] \oplus [b] = [b] \oplus [a]$. So $[a \oplus b] = [b \oplus a]$ and $a \oplus b \sim b \oplus a \Rightarrow \Delta(a \oplus b, b \oplus a) \in I$. □

Corollary 4.5. *If I is a prime ideal, then $\mathrm{com}(a, b) \in I\ \forall\, a, b \in A$.*

According to [9, Prop. 3.4] and a fact, that ideals of ebas coincide with Riesz ideals (d-ideals) in corresponding lattice effect algebras (see section 5), we have also the next result.

Theorem 4.6. *A/I is an MV-algebra iff $I = \bigcap I_\alpha$, where I_α are prime ideals for all α.*

Recall that an effect basic algebra is an orthomodular lattice (OML) iff the operation $\oplus$ is idempotent [2, Corollary 5.5, Remark 5.6 c]. This implies the following.

Theorem 4.7. *A/I is an OML iff $\Delta(a \oplus a, a) \in I\ \ \forall\, a \in A$.*

Remark 4.8. We may look properly at $\Delta(a \oplus a, a)$ and see that it is in fact $a \wedge \neg a$. Indeed, $\Delta(a \oplus a, a) = \neg(a \wedge (a \oplus a) \oplus \neg a \wedge \neg(a \oplus a)) = \neg(a \oplus \neg(a \oplus a)) = \neg(\neg(a \oplus a) \oplus a) = \neg(\neg(\neg\neg a \oplus a) \oplus a) = \neg(a \vee \neg a) = a \wedge \neg a$. It follows that A/I is an OML iff all classes are sharp (i. e., $[a] \wedge \neg[a] = 0$ for all $a \in A$).

5. CONCLUDING REMARKS

In this section, we show that ideals in effect basic algebras coincide with d-ideals in lattice ordered effect algebras.

Recall that a subset I of an effect algebra E is an ideal if $a+b \in I \Leftrightarrow a \in I$ and $b \in I$ whenever $a, b \in E$ are such that $a+b$ is defined. An ideal I in lattice ordered effect algebra E is a d-ideal if $a \vee c - c \in I$ whenever $a \in I$ and $c \in E$ [1]. Notice that the notion of a d-ideal coincides with the notion of a Riesz ideal considered in [9].

A binary relation $\sim$ on an effect algebra E is a congruence if (i) $\sim$ is an equivalence, (ii) $a \perp b$, $a_1 \perp b_1$, $a \sim a_1$, $b \sim b_1$ implies $a+b \sim a_1+b_1$, (iii) $a \sim b$, $c \perp a$ implies $\exists d$ with $d \sim c$ and $d \perp b$. In the quotient $E/\sim$ we define $[a] \perp [b]$ if there are representatives $a_1 \in [a], b_1 \in [b]$ such that $a_1 \perp b_1$, and put $[a]+[b] = [a_1+b_1]$. If $\sim$ is a congruence, then $E/\sim$ is an effect algebra.

If I is an ideal of an effect algebra E, define a binary relation $\sim_I$ by $a \sim_I b$ if there are $i, j \in E$ such that $i \leq a, j \leq b$ and $a - i = b - j$. The relation $\sim_I$ is a congruence iff I is a Riesz ideal [5]. In particular, if I is a d-ideal (equivalently, Riesz ideal) of a lattice ordered effect algebra E, then the quotient with respect to $\sim_I$ is again a lattice ordered effect algebra.

Let E be a lattice effect algebra and A the corresponding effect basic algebra. Let I be a d-ideal of E. Then $a \in I, c \in E$ implies that $a \vee c - c \in I$. Now $a \vee c - c = (a' \wedge c' + c)' = \neg(\neg a \oplus c)$ in A. Moreover, if $a, b \in I$, then $a \wedge b', b \in I$, whence $a \wedge b' + b \in I$, but $a \wedge b' + b = a \oplus b$ in A. It follows that I is an ideal of A. Conversely, if I is an ideal of A, then $a, b \in I$ and $a \perp b$ implies $a+b = a \oplus b \in I$, and if $a+b \in I$, then $a, b \leq a \oplus b = b \oplus a \in I$ implies $a, b \in I$, hence I is a d-ideal of E. By Proposition 3.6 the relations $\sim_I$ in both structures are equivalent. From this we easily derive the following statement.

Theorem 5.1. Let $\mathbb{E}$ be a lattice effect algebra, and $\mathcal{A}(\mathbb{E})$ the corresponding effect basic algebra. If I is a d-ideal of $\mathbb{E}$, then $\mathcal{A}(\mathbb{E}/I) \cong \mathcal{A}(\mathbb{E})/I$.

Conversely, let $\mathbb{A}$ be an effect basic algebra and $\mathcal{E}(\mathbb{A})$ the corresponding lattice effect algebra. Let I be an ideal of $\mathbb{A}$. Then $\mathcal{E}(\mathbb{A}/I) \cong \mathcal{E}(\mathbb{A})/I$.

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